On Formulas for Equivalent Potential Temperature

ROBERT DAVIES-JONES
NOAA/National Severe Storms Laboratory, Norman, Oklahoma

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ABSTRACT

Several new formulas for pseudoadiabatic equivalent potential temperature (EPT) are devised and compared to previous ones. The maximum errors of all the formulas are determined from calculations on a dense grid of points in the region of a thermodynamic diagram defined by wet-bulb potential temperature $\theta_e \leq 400$ K and pressure between 100 and 1050 mb. One of the new formulas has an accuracy of 0.015 K in the specified region. Finding the imitation first law of thermodynamics that they satisfy approximately reveals how the formulas work.

1. Introduction

There are several formulas for equivalent potential temperature (EPT) in the literature (Bolton 1980; Bryan 2008). Rossby (1932) devised the first such formula by dropping an intractable term from the equation governing pseudoadiabatic water-saturation processes. This term is part of the entropy of water vapor. Its omission is equivalent to setting the specific heat of water to zero. Owing to the missing entropy, the Rossby formula underestimates the EPT by several degrees in very warm and moist environments. By using an optimized constant latent heat instead of the actual one, which is a linear function of temperature, Bryan (2008) obtained a simple formula that is accurate to about half a degree. His formula is associated with a simple version of the first law of thermodynamics, which is useful in some theoretical studies. Except at very cold temperatures where latent heat release is small, Bryan’s constant latent heat is larger than the actual one. The upward adjustment in latent heat compensates to a large extent for the missing term. Bolton’s (1980) modification of the Betts and Dugan (1973) formula is similar to Bryan’s formula, and is also associated with a constant latent heat (Davies-Jones 2008).

The most accurate formula to date is from Bolton (1980). It computes EPT to accuracy, relative to numerical solutions of the governing equation, better than 0.02 K. Note that accuracies quoted in the literature are based on a limited number of calculations (e.g., Bolton’s Table 3 and Bryan’s Table 1). Bolton’s formula has three adjustable parameters and is hard to interpret physically.

In this paper, the gap between Bryan’s formula with one adjustable parameter and Bolton’s very accurate formula with three is bridged by constructing formulas that have two degrees of freedom and intermediate accuracies. These new formulas are obtained by optimally adjusting the line in a graph of latent heat versus temperature instead of constraining it to have zero slope. Also, understanding of how the formulas work is gained by working backward from the formulas to an imitation first law of thermodynamics that they satisfy. This procedure identifies the adjustments needed to compensate for setting the specific heat of water to zero. Bolton’s formula is not amenable to this approach so a very similar, but more manageable formula with the same accuracy is devised. Last, maximum errors of each formula are determined more systematically than previously by calculating EPT at 1209 points in a grid covering a wide area of a thermodynamic diagram.

2. EPT formulas

As pointed out by Davies-Jones (2008), almost all of the formulas for the EPT $\theta_E$ of an air parcel have the following form:
where \( L_0^* (>0), L_1^* (\geq 0), \) and \( k_2 (\geq 0) \) are constants, \( \theta_E^* \) is the estimate of \( \theta_E \), \( T_L \) is the temperature at the lifting condensation level (LCL), and \( r \) is the mixing ratio. All symbols and a few useful relationships are listed in the appendix. The quantity \( \Theta \) is either the potential temperature \( \theta \) of moist air, which is conserved in unsaturated isentropic motion, or the potential temperature of dry air at the LCL, \( \theta_{DL} \). These formulas apply to all levels because \( T_L, \theta_{DL}, r, \) and \( e \) can be replaced by \( T_K, \theta_D, r(T_K, p), \) and \( e_c(T_K) \), respectively, above the LCL. Associated with (2.1) is the approximate equation for pseudoadiabatic ascent:

\[
\theta_E^* = \Theta \exp \left[ \frac{L_0^* - L_1^* (T_L - C)}{c_{pd} T_L} r (1 + k_2 r) \right], \tag{2.1}
\]

and the adjusted entropy:

\[
c_{pd} d \ln \theta_E^* = c_{pd} d \ln \Theta + \frac{L_0^* - L_1^* (T_L - C)}{T_L} r (1 + k_2 r), \tag{2.2}
\]

The first and least accurate formula for EPT is the one devised by Rossby:

\[
\theta_E^* = \theta_{DL} \exp \left[ \frac{L(T_L) r}{c_{pd} T_L} \right], \tag{2.4}
\]

where \( \theta_E^* \) denotes the Rossby EPT and \( L \) is the actual latent heat. It underestimates the actual EPT by up to 5\(^{\circ} \) in extremely humid air masses because it uses the actual latent heat \( L = L_0 - L_1(T_L - C) \) and ignores the term involving the specific heat of water in the entropy formulation. However, it sufficed for its original purpose, the identification of air masses. Because the omitted entropy term \( c_{pd} d \ln T_K \) is positive, \( \theta_E^* \) is always an underestimate of \( \theta_E \). The Rossby formula has no parameter that can be adjusted to compensate for the missing entropy. A further approximation consists of replacing \( \theta_{DL} \) by \( \theta \) (Iribarne and Godson 1973, p. 122).

When \( k_2 = 0 \), comparison of (2.1) and (2.4) shows that we can regard \( L^* = L_0^* - L_1^* (T_L - C) \) as a surrogate for \( L \). We can obtain more accurate formulas for EPT by adjusting the two latent heat coefficients. Generally, formulas with more degrees of freedom (the number of independent parameters that are varied to obtain a best fit to data) match the data more accurately. The third degree of freedom, \( k_2 \), is needed for a very accurate fit.

Using this approach Betts and Dugan (1973), Simpson (1978), Bolton (1980), and Bryan (2008) have devised formulas that are similar to (2.4), but that match the exact pseudoadiabats far better. Since the equations in Bolton’s paper are referenced many times, we will preface his equations numbers with a “B.” Similarly, Bryan’s (2008) equations are prefaced with “Br.” We do not consider Simpson’s formula [(B33) after Bolton’s modification] further because it is less accurate than (B38) and (B39), has a more cumbersome form, assumes that \( \theta_D \) is conserved during adiabatic ascent to the LCL, and is not easily associated with an imitation first law of thermodynamics. Bolton simplified Betts and Dugan’s (1973) formula to

\[
\theta_E = \theta \exp \left( \frac{L_0^* r}{c_{pd} T_L} \right), \quad L_0^* = 2.690 \times 10^6 \text{ J kg}^{-1}, \quad L_1^* = 0, \tag{2.5}
\]

which is equivalent to (B35). This has one degree of freedom, \( L_0^* \), an optimum constant value of latent heat. Bryan’s (2008) formula (Br13) similarly has one degree of freedom. It has the following form:

\[
\theta_E = \theta_D \mathcal{H}^{-R, c_{pd}} \exp \left( \frac{L^* r}{c_{pd} T_K} \right), \tag{2.6}
\]

with \( L^* = L_0^* = 2.555 \times 10^6 \text{ J kg}^{-1} \). Here, the relative humidity \( \mathcal{H} \) has been introduced to avoid having to compute \( T_L \), and \( \theta_D \) is used instead of \( \theta \).

Bolton’s two new formulas, (B38) and (B39), are the most accurate to date. They have the form (2.1) with \( k_2 \neq 0 \) so they both contain a \( r^2 \) term. The difference between them is the use of \( \theta_{DL} \) in (B39) and \( \theta \) in the less precise (B38). Although (B38) ostensibly has three degrees of freedom, it in fact has only two because \( L^* \) is proportional to \( L \) (i.e., \( L_0^* \) and \( L_1^* \) are not varied independently). Equation (B39) does have three degrees of freedom, and this seems to be the number required for an estimate with maximum error (relative to the results of numerical integration of the differential equation governing pseudoadiabatic ascent) to within 0.03 K. The actual error is around 0.2 K, owing mainly to the specific heat at constant pressure for dry air, assumed to be a constant, varying slightly with temperature and pressure. The corresponding error in pseudo-wet-bulb temperature is less than 0.2 K (Davies-Jones 2008).

The third-order term, \(-k_2 L_1^* (T_L - C) r^2\), in (2.1) is much smaller in magnitude than the term \( k_2 L_0^* r^2 \). When this term is neglected, (2.1) simplifies to
\[ \theta^p_E = \Theta \exp \left\{ \frac{\left( L_0^* - L_1^* (T_L - C) \right) r + K_2 r^2}{c_{pd} T_L} \right\}, \tag{2.7} \]

where \( K_2 (= k_2 L_0^*) \) is a constant. A formula of this form that is as accurate as (B39) is developed later. Note that all the formulas with \( k_2 = 0 \) are included in this form.

3. The formulas’ imitation first law of thermodynamics

In many cloud models, simplified versions of the first law of thermodynamics are used (Wilhelmson 1977). Only some of these are exact differentials for pseudoadiabatic processes and so have exact corresponding analytical formulas for EPT. On the other hand, the approximation to the entropy form of the first law (Holton 1992, p. 51) that corresponds to an analytical approximate formula for EPT can always be found by differentiating the formula as in (2.1) and (2.2).

For use in theoretical studies, Bryan (2008) found an EPT formula that is associated with a simplified version of the first law. Here, we work backward from the entropy formulation to find an imitation first law that the formulas satisfy with the purpose of understanding how the formulas work. As is customary, we neglect the ice phase, assume that the phases of water are in equilibrium (so that there is no supersaturation) and that the volume of any liquid water in the parcel is negligible, and consider the two extreme cases of adiabatic ascent of air that is saturated with respect to a plane surface of water (Bryan 2008). In the first extreme case, the liquid water remains in the parcel and the total water mixing ratio, \( Q = r + r_w \), is constant. This is a reversible process (Saunders 1957) and the liquid water content is said to be adiabatic. The second case is the pseudoadiabatic process where condensate falls out of the parcel immediately after it forms. Normally, the liquid water content in clouds lies between the two extremes (Wilhelmson 1980).

For the reversible process, the differential equation governing entropy has the exact integral:

\[ s_r = c_{pd} \ln \theta_D + c_w \ln T_K - R_y r \ln \mathcal{H} + L(T_K) r / T_K + r_w c_w \ln T_K \tag{3.1} \]

[Emanuel 1994, Eq. (4.5.9)]. After multiplying the differential form of (3.1) by \( T_K \) and using (B23), the ideal gas laws for dry air and vapor, and the Clausius–Clapeyron equation:

\[ \frac{L dT_K}{T_K} = R_y d \ln \mathcal{E}, \tag{3.2} \]

we find that

\[ T_K d s_r = d h - \alpha_d d p = 0, \tag{3.3} \]

where \( h = (c_{pd} + Q c_w) T_K + L r \) is the specific moist (reversible) enthalpy (Emanuel 1994, p. 118). By applying Kirchoff’s equation:

\[ \frac{dL}{dT_K} = c_{pw} - c_w, \tag{3.4} \]

we obtain the first law

\[ T_K d s_r = (c_{pd} + r c_{pw} + r_w c_w) dT_K - \alpha_d d p + L d r = 0. \tag{3.5} \]

In the pseudoadiabatic process, the liquid water is removed immediately from the parcel \( (r_w = 0) \) so total water is no longer conserved. The entropy, EPT and first law are now

\[ s_p = c_{pd} \ln \theta_D + c_w \int_{T_0}^{T_K} Y(\tau) d \tau + L(T_K) Y(T_K) \]

\[ - R_y r \ln \mathcal{H}, \tag{3.6} \]

\[ \theta^p_E = \Theta \mathcal{H}^{-R_y r / c_{pd}} \exp \left\{ Y(T_K) \int_{T_0}^{T_K} \frac{L(T_K) + c_w}{c_{pd} Y(T_K)} \right\} \times \int_{T_0}^{T_K} Y(\tau) d \tau \tag{3.7} \]

and

\[ T_K d s_p = (c_{pd} + r c_{pw}) dT_K - \alpha_d d p + L(T_K) d r = 0, \tag{3.8} \]

where \( Y(T_K) = r / T_K \) and \( T_0 \) is a very cold temperature where the integrand is effectively zero. The integral term in (3.6) is part of the entropy of water vapor. It is not zero because, in the pseudoadiabatic process, water condenses and then is removed instantly, along with its entropy, as the parcel is rising and cooling, not while it is maintained at a constant temperature (Bolton 1980).

The exact formula for \( \theta^p_E \) is (3.7), but it is not useful because it contains an integral. Since the integral is not an exact differential, the integration has to be performed numerically along a pseudoadiabat. This is tantamount to generating pseudoadiabats by stepwise numerical integration so nothing has been gained. Useful formulas
for $s_p$ and $\theta_E$ that depend only on the initial and final states of the parcel have either ignored this term or replaced it implicitly by a compensating term that is an exact differential. Alternatively, we can set $c_w = 0$ and explicitly adjust some other coefficients to minimize the resulting error. This is the approach adopted here.

In fact, the formulas that have the simplified form in (2.7) may all be obtained by setting $c_w = 0$, which eliminates the computationally expensive integral term, and adjusting $L$ (and $K_2$ if nonzero) to minimize error. In other words, $c_w \rightarrow c_w^* = 0$, $L_0 \rightarrow L_0^*$ and $L_1 \rightarrow L_1^*$ where the superscript asterisk denotes that the coefficients have been adjusted. This gives us the estimate

$$\theta_E^* = \theta_D H^{R,rc_{pd}} \exp \left[ \frac{L^*(T_L)r + K_2 r^2}{c_{pd} T_L} \right].$$ (3.9a)

The EPT estimate, $\theta_E^*$, should be conserved below the LCL. We can ensure this by integrating the entropy equation just for the pseudoadiabatic part of the ascent, and evaluating the constant of integration at the LCL. This yields

$$\theta_E^* = \theta_{DL} \exp \left[ \frac{L^*(T_L)r + K_2 r^2}{c_{pd} T_L} \right].$$ (3.9b)

We now find that an imitation first law that corresponds to (3.9). In turns out that in the temperature range $-47^\circ < T_K < C \approx 32^\circ C, 1 < L^*/L < 1.06$ for the formulas with $\Theta = \theta_D$, so to a quite good approximation we can use (3.2) with $L^*$ instead of $L$. From (3.9a) and the modified (3.2), we obtain

$$0 = T_K d s_p^* = c_{pd} T_K - \alpha_d dp + d(L^* r) dh^* + K_2 T_K d(r^2/T_K).$$ (3.10)

Since $r^2$ varies much more rapidly than $T_K$ at high mixing ratios where the last term is important, we can approximate (3.10) by

$$T_K d s_p^* = dh^* - \alpha_d dp = 0,$$ (3.11)

where

$$R^* = c_{pd} T_K + L^*(T_K) + K_2 r^2.$$ (3.12)

is an enthalpy that exists in this simple form because $c_w = 0$. Expanding (3.12) gives us the following “first law”:

$$T_K d s_p^* = (c_{pd} - L_1^* r) dT_K + [L_0^* - L_1^*(T_K - C)] dr + K_2 d(r^2) - \alpha_d dp = 0.$$ (3.13)

If we assume Kirchoff’s Eq. (3.4) for the adjusted parameters, then we can define an adjusted specific heat of water vapor, $c_{pv}^* = -L_1^*$ and (3.13) becomes

$$T_K d s_p^* = (c_{pd} + r c_{pv}^*) dT_K - \alpha_d dp + L^*(T_K) dr + K_2 d(r^2) = 0.$$ (3.14)

Bryan (2008) set $L_1^* = 0$ (and $K_2 = 0$). Then, (3.6), (3.7), and (3.9) reduce to (Br11), (Br13), and (Br10), respectively, and the enthalpy becomes $h^* = c_{pd} T_K + L_0^* r$.

We can see how the formulas work by comparing (3.8) and (3.14). The adjustment of the specific heat of heat to zero and Kirchoff’s equation makes the specific heat of water vapor negative (or zero if $L_1^* = 0$). Thus, the specific heat of moist air is reduced artificially. From a first-law viewpoint, an artificial increase in latent heat, $(L^* - L) dr$, is required to compensate for the decrease in specific heat. The small $r^2$ term is extra compensation needed for very accurate calculations of very warm EPTs. From an entropy viewpoint, the entropy increase associated with the increase in latent heat is needed to compensate the entropy of water vapor for the part that is missing.

Which of the forms (3.9a) or (3.9b) is preferable? Equation (3.9a) has the advantage that it does not require computation of temperature at the LCL. But, unlike (3.9b), it does not conserve $\theta_E$ below the LCL (Bryan 2008). Thus, a disadvantage of (3.9a) is the introduction of additional error below the LCL as shown below. Since $r$ is constant below the LCL, the integrand in (3.7) has an antiderivative and so (3.7) becomes

$$\theta_E = \theta_D H^{R,rc_{pd}} \exp \left\{ \frac{r (L(T_K) - L(T_L))}{c_{pd} T_K T_L} + c_w \ln \frac{T_K}{T_L} \right\}.$$ (3.15)

where $T_1$ is an arbitrary constant of integration. Thus, by conservation of $\theta_E$ and (3.15), $\theta_{DL}$ and $\theta_D$ are related by

$$\theta_{DL} = \theta_D H^{R,rc_{pd}} \exp \left\{ \frac{r (L(T_K) - L(T_L))}{c_{pd} T_K T_L} + c_w \ln \frac{T_K}{T_L} \right\}.$$. (3.16)

Substituting for $\theta_{DL}$ in (3.9b) gives us

$$\theta_E^* = \theta_D H^{R,rc_{pd}} \exp \left\{ \frac{r [L^*(T_K) - L^*(T_L)]}{c_{pd} T_K T_L} + K_2 r \right\} - f(T_K) + f(T_L) \right\}.$$ (3.17)

where
\[
\begin{align*}
    f(\tau) &= \frac{L^*(\tau) - L(\tau)}{\tau} - c_w \ln \tau + \frac{K_r}{\tau}. \\
    \theta_D &= \theta \left[ \frac{\theta}{\theta_D} \right]^{0.28r} \left( 1 + \frac{\theta}{\theta_D} \right)^{5/6}, \\
    \theta_{DL} &= \theta \left[ \frac{\theta}{\theta_{DL}} \right]^{0.28r} \left( 1 + \frac{\theta}{\theta_{DL}} \right)^{5/6}.
\end{align*}
\]

Thus, the extra work involved in computing \( \theta_D \) or \( \theta_{DL} \) is relatively minor, and simply replacing \( \theta \) with \( \theta_D \) in the exact Eq. (3.7) introduces error by tacitly setting the above quantities in square brackets to 1. Since \( \theta < \theta_D \), the adjustment to latent heat has to be larger when \( \Theta = \theta \).

4. Approach for devising new formulas

The approach is to minimize the difference between the true specific pseudoentropy and the one associated with a given formula. From (3.6) and (3.9a) this difference for a saturated parcel is

\[
s_p^* - s_p = Y(T_K, \theta_E) \left\{ L^*(T_K) + K_r r_s - L(T_K) - \frac{c_w}{Y(T_K, \theta_E)} \int_{T_0}^{T_K} Y(\tau, \theta_E) \, d\tau \right\},
\]

for \( \Theta = \theta_p \). [When \( \Theta = \theta \), the term \( c_{pd} \ln(\theta/\theta_D) \) is added to the right side.] Here, \( Y(T_K, \theta_E) = r_s(T_K, \theta_E)/T_K \). To compensate for the omission of the \( c_w \) term in (3.9b), we seek the optimum \( L^* \), defined as the one that minimizes the maximum error, \( |s_p^* - s_p|_{\text{max}} \), in a specified region of a thermodynamic diagram. At cold temperatures and hence small mixing ratios, \( |s_p^* - s_p| \) is small because \( Y \) is small. We need to make this difference small at large \( r_s \). For large mixing ratios, \( |s_p^* - s_p| \) is small only if

\[
L^*(T_K) + K_r r_s(T_K, \theta_E) - L(T_K) \approx \frac{c_w}{Y(T_K, \theta_E)} \int_{T_0}^{T_K} Y(\tau, \theta_E) \, d\tau,
\]

where \( L^* = L \) at the cold temperature \( T_0 \). For estimation of \( L^* + K_r r_s \), the quantity,

\[
\Lambda = L(T_K) + \frac{c_w}{Y(T_K, \theta_E)} \int_{T_0}^{T_K} Y(\tau, \theta_E) \, d\tau = \frac{c_{pd}}{Y(T_K)} \ln \frac{\theta_E}{\theta_D}
\]

[from (3.7)] was evaluated, using results from the numerical integration procedure described below. We may regard \( \Lambda \) as the data that \( L^* + K_r r_s \) should fit at large mixing ratios, and thus satisfy the approximation in (4.2).

The procedure (Bolton 1980) finds the true EPT of a parcel as follows. From (3.6), the differential equation defining a pseudoadiabat is

\[
0 = d \ln \theta_E = d \ln \theta_s + \frac{c_w}{L(T_K)} \chi dT_K,
\]

where \( \ln \theta_s = \ln \theta_D + \chi \) and \( \chi = L_r/c_{pd} T_K \). The saturation vapor pressure is computed from (B10), the dewpoint temperature is calculated from the inverse of (B10), and the temperature at the LCL is given by (B15). The quantities \( \theta_s \) and \( \chi \) along a pseudoadiabat are found by numerically integrating (4.4) from the LCL where \( \theta_s = \theta_{DL} \), using the (second order) modified Euler method here and small temperatures steps. To reduce round-off error slightly, the prognostic variable in (B31) is changed here to \( \ln(\theta_s/\theta_{DL}) \) where \( \theta_{DL} \) is the value of \( \theta_s \) at \( p = p_0 \). At each step, \( \chi \) is recovered from \( \theta_s \) by solving (B32) via Newton's method. The pressure at each step is given by \( p = c_{pd} (1 + c_{pd} T_K \chi) \). The EPT of the pseudoadiabat is the value of \( \theta_s \), obtained by integrating to a very cold temperature \( T_0 \) (in K) where the value of \( \theta_s \) no longer changes.

5. Optimization procedure

First, consider formulas with two degrees of freedom, \( L^*_0 \) and \( L^*_1 \). We seek the values of \( L^*_0 \) and \( L^*_1 \) that minimize the maximum absolute error \( |\delta s|_{\text{max}} \) in the region of a thermodynamic diagram defined by
100 \leq p \leq 1050 \text{ mb} \text{ and } \theta_w \leq 32^\circ \text{C} (\theta_E \leq 400 \text{ K}). \text{ The procedure used to find the optimal values of the parameters } L_0^* \text{ and } L_1^* \text{ in a formula consists of the following steps.}

1) Choose the set of grid points \{-20^\circ, -18^\circ, \ldots, 32^\circ\} \times \{100, 125, \ldots, 1050 \text{ mb}\} \text{ in } (\theta_w, p) \text{ space.}

2) \text{ From the initial condition } T_K = \theta_w, p = p_0, \text{ integrate along pseudoadiabats as described in section 3 to determine the true value of the EPT, } \theta_E, \text{ that corresponds to each WBPT, } \theta_w.

3) \text{ The integration procedure yields pressure on a regular grid in } (\theta_w, T_K) \text{ space. Obtain the values of temperature at the grid points in } (\theta_w, p) \text{ space by interpolation.}

4) Select initial values for } L_0^* \text{ and } L_1^*.

5) Use the formula to compute the estimated EPT, } \theta_E^*, \text{ of a saturated parcel at each point.}

6) \text{ At each point, calculate the difference } \delta \theta_E = \theta_E^* - \theta_E \text{ between the EPT computed from the formula and the true EPT. Save } \delta \theta_E(\theta_w, p).

7) \text{ Find the maximum absolute difference } |\delta \theta_E|_{\text{max}}.

8) \text{ If the formula has two degrees of freedom, set up a mesh system in the 2D } (L_0^*, L_1^*) \text{ space centered on the initial value with uniform spacing } \Delta L_0^* \text{ and } \Delta L_1^* \text{ directions. Repeat steps 4–7 at each grid point and contour the } |\delta \theta_E|_{\text{max}} \text{ field. This is done for (2.7) with } K_2 = 0 \text{ in Figs. 1 and 2 for a } 251 \times 251 \text{ grid in the domain defined by } L_0^* \in [2.5 \times 10^3, 2.75 \times 10^3] \text{ J kg}^{-1} \text{ and } L_1^* \in [0, 2500] \text{ J kg}^{-1} \text{ K}^{-1}. \text{ At this high resolution it is evident that there is only one minimum in the domain when either } \Theta = \theta_{DL} \text{ (Fig. 1) or } \Theta = \theta \text{ (Fig. 2).}

9) \text{ Record the minimum value and locate the grid point where it occurs. At this point center a new } 251 \times 251 \text{ grid with the spacing } \Delta L_0^* \text{ and } \Delta L_1^* \text{ reduced by}
factors of 30 and 10, respectively, and find the minimum value and its location in this grid.

10) The optimum values of the parameters have now been determined. The accuracy $E$ of the formula is the minimum value of $|\delta \theta_{E}|_{\text{max}}$ in $(L_0^*, L_1^*)$ space. The parameters have to be found quite precisely because the error is sensitive to it [as found also by Bryan (2008)].

11) Data are also evaluated at points $[\theta_w = 34^\circ, 36^\circ, 38^\circ, 40^\circ \text{C}] \times \{p = 100, 125, \ldots, 1050 \text{ mb}\}$ in a grid extension for plotting in some of the figures. These data are not used in optimization.

12) Steps 8–10 are similar for a formula with only one parameter $L_0^*$, except the search for the minimum is one dimensional (along the line $L_1^* = 0$). For the formula with three parameters, steps 8–10 are performed for different values of the third parameter, $K_2$, until the minimum in the 3D parameter space is found.

The method is quite general. It can be easily adapted to work with values of the physical constants and formulas for saturation vapor pressure and condensation temperature that differ from those that Bolton used.

6. Optimized formulas

We first consider the class of formulas with $L_1^* = 0$. The optimized version of (B35) is

$$\theta_E = \theta \exp \left( \frac{Ar}{T_L} \right), \quad A = L_0^*/c_{pd} = 2674.5 \text{ K}, \quad (6.1)$$

where the optimum value of $A$ is only slightly different from Bolton’s value of 2675 K. Using $\theta_{DL}$ instead of $\theta$ provides us with a new more accurate formula:

$$\theta_E = \theta_{DL} \exp \left( \frac{L_0^* r}{c_{pd} T_L} \right), \quad (6.2)$$

FIG. 2. As in Fig. 1, but for $\Theta = \theta$. The letters A and the large O locate the points $(L_0^*, L_1^*) = (2689.7 \times 10^3, 0)$ used in (6.1) and $(L_0^*, L_1^*) = (2711 \times 10^3, 1109)$ used in (6.3), respectively.
Table 1. The formulas for $\theta_E$ and their errors. Here $E$ is the maximum error in $\theta_E$ for saturated parcels in the region of a thermodynamic diagram defined by $-20^\circ C \leq \theta_a \leq 32^\circ C$ and $100 \text{ mb} \leq p \leq 1050 \text{ mb}$. The formulas are listed in order of increasing accuracy. The numbers in parentheses in the fourth column are the maximum errors for $\theta_a = 40^\circ C$.

<table>
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<tr>
<th>Formula</th>
<th>Equation</th>
<th>Values of constants</th>
<th>$E$ (K)</th>
<th>Degrees of freedom</th>
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<tr>
<td>(2.4)</td>
<td>$\theta_E = \theta_{DL} \exp \left[ \frac{L_0 - L_1(T_L - C)}{c_{pd}T_L} \right]$</td>
<td>$L_0 = 2.501 \times 10^6 \text{ J kg}^{-1}$, $L_1 = 2370 \text{ J kg}^{-1} \text{ K}^{-1}$</td>
<td>5.0 (11.1)</td>
<td>0</td>
</tr>
<tr>
<td>(Br13)</td>
<td>$\theta_E = \theta_D \exp \left[ \frac{L_0^*}{c_{pd}T_L} \right]$</td>
<td>$L_0^* = 2.555 \times 10^6 \text{ J kg}^{-1}$</td>
<td>0.57 (0.73)</td>
<td>1</td>
</tr>
<tr>
<td>(6.1)</td>
<td>$\theta_E = \theta \exp \left[ \frac{L_0^*}{c_{pd}T_L} \right]$</td>
<td>$L_0^* = 2.6897 \times 10^6 \text{ J kg}^{-1}$</td>
<td>0.49 (1.32)</td>
<td>1</td>
</tr>
<tr>
<td>(6.2)</td>
<td>$\theta_E = \theta_{DL} \exp \left[ \frac{L_0^*}{c_{pd}T_L} \right]$</td>
<td>$L_0^* = 2.5505 \times 10^6 \text{ J kg}^{-1}$</td>
<td>0.38 (0.84)</td>
<td>1</td>
</tr>
<tr>
<td>(6.3)</td>
<td>$\theta_E = \theta \exp \left[ \frac{L_0^* - L_1^*(T_L - C)}{c_{pd}T_L} \right]$</td>
<td>$L_0^* = 2.711 \times 10^6 \text{ J kg}^{-1}$, $L_1^* = 1109 \text{ J kg}^{-1} \text{ K}^{-1}$</td>
<td>0.18 (1.66)</td>
<td>2</td>
</tr>
<tr>
<td>(6.4)</td>
<td>$\theta_E = \theta_{DL} \exp \left[ \frac{L_0^* - L_1^*(T_L - C)}{c_{pd}T_L} \right]$</td>
<td>$L_0^* = 2.569 \times 10^6 \text{ J kg}^{-1}$, $L_1^* = 900 \text{ J kg}^{-1} \text{ K}^{-1}$</td>
<td>0.11 (1.28)</td>
<td>2</td>
</tr>
<tr>
<td>(B38)**</td>
<td>$\theta_E = \theta \exp \left[ \frac{L_0^*}{c_{pd}T_L} (1 + k_2 r) \right]$</td>
<td>$L_0^* = 2.6975 \times 10^6 \text{ J kg}^{-1}$, $L_1^* = 2554.5 \text{ J kg}^{-1} \text{ K}^{-1}$, $k_2 = 0.810$</td>
<td>0.085 (0.94)</td>
<td>2</td>
</tr>
<tr>
<td>(B39)</td>
<td>$\theta_E = \theta_{DL} \exp \left[ \frac{L_0^*}{c_{pd}T_L} (1 + k_2 r) \right]$</td>
<td>$L_0^* = 2.5643 \times 10^6 \text{ J kg}^{-1}$, $L_1^* = 1790 \text{ J kg}^{-1} \text{ K}^{-1}$, $k_2 = 0.448$</td>
<td>0.036 (0.104)</td>
<td>3</td>
</tr>
<tr>
<td>(6.5)</td>
<td>$\theta_E = \theta_{DL} \exp \left[ \frac{L_0^* + K_2 r}{c_{pd}T_L} \right]$</td>
<td>$L_0^* = 2.56313 \times 10^6 \text{ J kg}^{-1}$, $L_1^* = 1754 \text{ J kg}^{-1} \text{ K}^{-1}$, $K_2 = 1.137 \times 10^6 \text{ J kg}^{-1}$</td>
<td>0.015 (0.095)</td>
<td>3</td>
</tr>
</tbody>
</table>

** (B38) has only two degrees of freedom because $L_0^* = 1.0784 L$.

where $L_0^* = 2.5505 \times 10^6 \text{ J kg}^{-1}$ is the optimum value, which is close to the value $2.555 \times 10^6 \text{ J kg}^{-1}$ obtained by Bryan (2008) despite the different optimization criteria.

Optimizing (2.7) with $K_2 = 0$ and two degrees of freedom, $L_0^*$ and $L_1^*$, results in the new formulas:

$$\theta_E = \theta \exp \left[ \frac{L_0^* - L_1^*(T_L - C)}{c_{pd}T_L} \right],$$

$$L_0^* = 2.711 \times 10^6 \text{ J kg}^{-1},$$

$$L_1^* = 1109 \text{ J kg}^{-1} \text{ K}^{-1}$$

and

$$\theta_E = \theta_{DL} \exp \left[ \frac{L_0^* - L_1^*(T_L - C)}{c_{pd}T_L} \right],$$

$$L_0^* = 2.569 \times 10^6 \text{ J kg}^{-1},$$

$$L_1^* = 900 \text{ J kg}^{-1} \text{ K}^{-1}.$$

In addition, a new formula with three degrees of freedom is found:

$$\theta_E = \theta_{DL} \exp \left[ \frac{[L_0^* - L_1^*(T_L - C) + K_2 r]}{c_{pd}T_L} \right],$$

where $L_0^* = 2.56313 \times 10^6 \text{ J kg}^{-1}$, $L_1^* = 1754 \text{ J kg}^{-1} \text{ K}^{-1}$, and $K_2 = 1.137 \times 10^6 \text{ J kg}^{-1}$.

7. Accuracy of the formulas

The previous and new formulas are ranked in terms of the accuracy $E$ in Table 1 and their maximum errors as functions of $\theta_a$ are shown in Figs. 3–5. The accuracies of the formulas in the region $100 \leq p \leq 1050 \text{ mb}$ and $\theta_a \leq 32^\circ C$ are determined first of all by the number of degrees of freedom and then, with one exception, by the choice for $\Theta$ as anticipated in section 3. The Rossby equation (2.4), which has no degrees of freedom, shows the maximum error (up to 5 K in the above region) that...
arises from neglecting the $c_{w}$ term and utilizing the actual latent heat. The Rossby formula has a maximum error with respect to pressure (with $\theta_{w}$ held fixed) that varies almost exponentially with $u_{w}$ (Fig. 3).

Of the formulas with a constant adjusted latent heat (one degree of freedom), (6.2) has the least maximum error, 0.38 K (Fig. 3) because it uses $\theta_{DL}$. The modified Betts–Dugan equation (6.1), which contains $\theta$, is accurate to 0.49 K. Even though it has the same form as (6.2) for a saturated parcel, Bryan’s formula (Br13) with a maximum error of 0.57 K is slightly less accurate than (6.1). This is probably due to it being optimized by minimizing the mean square error (instead of the maximum error) for a limited number of cases.

Because it uses $\theta$, (6.3) with a maximum error of 0.18 K is the least accurate of the formulas with two degrees of freedom (Fig. 4). Even though it also uses $\theta$ and (6.4) uses $\theta_{DL}$, (B38) is slightly the more accurate with a maximum error of 0.09 K compared to 0.11 K. The better accuracy of (B38) is probably because of its use of $k_{2}$ [see (2.1)] as the second adjustable parameter and because of the fixed parameter $L_{E}$ being determined by $L_{E} \propto L$ instead of being set to zero.

In the category with three degrees of freedom, (6.5) is accurate to 0.15 K and (B39) to 0.035 K (Fig. 5). However, (B39) is more accurate for $\theta_{w} < 24^\circ$C so there is little to choose between the two formulas. Surprisingly, (B38) is used most frequently (Emanuel 1994; Bryan 2008) even though it is less accurate than (B39).

The maximum errors are not restrained by optimization beyond $\theta_{w} = 32^\circ$C and grow rapidly (Figs. 3–5). For $\theta_{w} \leq 40^\circ$C, the widely used Eq. (B38) has a maximum error of 0.94 K (Table 1). In contrast, (6.5) and (B39) contain the error to 0.1 K (Fig. 5).

The quantity $\Lambda$ defined by (4.3) is the “adjusted latent heat according to the data.” Figure 6 is a scattergram of
\( \Lambda \) versus \( T \) for all the points in the grid for which \( rs \approx 0.002 \). Lower values of \( rs \) were excluded because \( \Lambda \rightarrow 0/0 \) as \( rs \rightarrow 0 \). The data do not lie precisely on a straight line, indicating that, although \( \Lambda \) is mainly a linear function of \( T_K \), it is slightly dependent on \( r \). The straight lines are the actual latent heat of vaporization according to (B2), the constant adjusted latent heat used in (6.2), and the adjusted latent heats used in (6.4), (6.5), and (B39), respectively. The data for \( \theta_a > 32^\circ C \) is not used in the determination of the adjusted latent heats (and \( K_2 \)).

For formulas with \( L^a = k_2 = K_2 = 0 \), it is apparent from (2.7) that the adjusted latent heat for a formula that uses \( \theta \) must be larger than the corresponding formula that uses \( \theta_{DL} \), by an amount that is approximately equal to \( (c_{pd} T/\rho_r) \ln(\theta_{DL}/\theta) \). Over the grid this quantity ranges from \( 1.2 \times 10^5 \) to \( 1.6 \times 10^5 \) J kg\(^{-1}\) with a mean around
1.4 \times 10^5 \text{ J kg}^{-1}. From Table 1 we see that this agrees with the difference (1.39 \times 10^5 \text{ J kg}^{-1}) between \( L_0^* \) for (6.1) and (6.2).

Finally, we stress that all the accuracies given above are relative to the results from numerical integration of (B25). Owing to \( cpd \) not being strictly constant, there is an uncertainty of up to 0.2 K, which has not been taken into account (Bolton 1980).

8. Summary

The accuracies of several formulas for equivalent potential temperature have been assessed by computing their maximum error throughout the region of a thermodynamic diagram defined by 100 \( \leq p \leq 1050 \text{ mb} \) and \( \theta_w \leq 32^\circ \text{C} \) (\( \theta_e \leq 400 \text{ K} \)). The more accurate formulas contain \( \theta_{DL} \) rather than \( \theta \), and \( T_L \), not \( T \). Some new formulas have been devised. These have two or three degrees of freedom and minimize the above maximum error. With one degree of freedom, a constant latent heat, it is possible to fit the data to 0.4 K. This error is reduced to 0.1 K [in (6.4)] by utilizing an adjusted latent heat that depends linearly with temperature. The residual error is not a function of temperature. It must therefore be a function of \( r \) since \( T_L \) and \( r \) alone (or \( T_K \) and \( r_s \) if the parcel is saturated) determine \( \theta_E \). To achieve greater accuracy than 0.1 K, a formula must contain this extra dependence on \( r \). Bolton’s (B38) and (B39) do so by containing an \( r^2 \) term. Equation (B39) is the more accurate because it uses \( \theta_{DL} \) and has an extra degree of freedom.

Finding the iteration first law of thermodynamics that the formulas satisfy approximately reveals how the formulas work. Adjusting the adjusted specific heat of water to zero to eliminate the integral term in the entropy formulation and Kirchoff’s equation makes the specific heat of water vapor negative (or zero if the formula uses a constant latent heat). Thus, the specific heat of moist air is reduced artificially. An artificial increase in latent heat, \((L^* - L) \, dr\), is required to compensate for the decrease in specific heat.

The highly accurate Eqs. (B38) and (B39) are not easily reconciled with an iteration first law. Therefore, a formula, (6.5), is devised herein that has a slightly simpler formula than (B39), has comparable accuracy, and can be related to an iteration first law if a small \( r^2 \) term is added to the iteration enthalpy. This small boost in enthalpy is additional compensation that is needed at high mixing ratios.

Acknowledgments. This work was supported in part by NSF Grant ATM-0733539. I thank Dr. George Bryan and two anonymous reviewers for their thought-provoking comments.

APPENDIX

List of Symbols

\[ A = 2675 \text{ K}, \text{ the constant in (B35)}. \]
\[ C = 273.15 \text{ K (absolute zero = 0 K = -273.15°C)}. \]
\[ E: \text{ Maximum absolute error in} \theta_E \text{ for} \theta_w = 32^\circ \text{C}, 100 \leq p \leq 1050 \text{ mb}. \]
\[ \%: \text{ Relative humidity} = e/e_s. \]
\[ K_2: \text{ A constant in (2.7) and (6.5)} = 1.137 \times 10^6 \text{ J kg}^{-1} \text{ in (6.5)}. \]
\[ L(T_K): \text{ Latent heat of vaporization,} = L_0 - L_1(T_K - C). \]
\[ L_0 = 2.501 \times 10^6 \text{ J kg}^{-1}. \]
\[ L_1 = 2370 \text{ J kg}^{-1} \text{ K}^{-1} (c_w - c_p, \text{ Kirchoff's equation}). \]
\[ L^*: \text{ Adjusted latent heat,} = L_0^* - L_1^*(T_K - C). \]
\[ Q: \text{ Total water mixing ratio (g g}^{-1}). \]
\[ R_{d}: \text{ Gas constant for dry air} = 287.04 \text{ J kg}^{-1} \text{ K}^{-1}. \]
\[ R_{m}: \text{ Gas constant for moist air} = R_d(1 + r/s) + (1 + r) = R_d(1 + 0.608r). \]
\[ R_c: \text{ Gas constant for water vapor} = 461.50 \text{ J kg}^{-1} \text{ K}^{-1} \text{ (} = R_d(2e)/e). \]
\[ T, T_K: \text{ Temperature (°C and K, respectively)}. \]
\[ T_L: \text{ Temperature at the LCL (K)}. \]
\[ T_0: \text{ A very cold temperature (K)}. \]
\[ Y = r/T_K (\text{ K}^{-1}). \]
\[ cpd: \text{ Specific heat at constant pressure of dry air} = 1005.7 \text{ J kg}^{-1} \text{ K}^{-1} (= 3.504 R_d). \]
\[ c_{pm}: \text{ Specific heat at constant pressure of moist air} = (c_{pd} + rc_{pv})(1 + r) = c_{pd}(1 + 0.887r). \]
\[ c_{pw}: \text{ Specific heat at constant pressure of water vapor} = 1875 \text{ J kg}^{-1} \text{ K}^{-1} (\text{ } = 4R_d). \]
\[ c_w^*: \text{ Adjusted value of} c_{pw} = c_w + L_1^*. \]
\[ c_w, c_w^*: \text{ Specific heat of water,} = 4190 \text{ J kg}^{-1} \text{ K}^{-1} \text{ and } \text{ adjusted value (}=0). \]
\[ e: \text{ Vapor pressure (mb)}, e = e_s(T_D), D = \text{ dewpoint temperature (K)}. \]
\[ e_s(T_K): \text{ Saturation vapor pressure (mb)} = 6.112 \times \exp[17.67(T_K - C)/(T_K - C + 243.5)]. \]
\[ de/dT_K = L_e/R_dT_K^2 \text{ (Clausius–Clapeyron equation)}. \]
\[ h, h^*: \text{ Specific enthalpy, adjusted specific enthalpy (J kg}^{-1}). \]
\[ k_2: \text{ A nondimensional constant equal to 0.81 in (B38)} \text{ and 0.448 in (B39)}. \]
\[ p: \text{ Pressure (mb)}. \]
\[ p_0 = 1000 \text{ mb}. \]
\[ r: \text{ Mixing ratio (g g}^{-1}), \text{ } = e/e_s(p - e). \]
\[ r_s: \text{ Saturation mixing ratio (g g}^{-1}), \text{ } = e/e_s(p - e_s). \]
\[ r_w: \text{ Liquid water mixing ratio (g g}^{-1}). \]
\[ sp, s^*: \text{ Actual and adjusted specific pseudoadiabatic entropy}. \]
\[ s: \text{ Total (reversible) specific entropy of moist air} = s_d + r_w s_w. \]
\[ \Delta L_0^*, \Delta L_1^*: \text{ Grid spacings in} (L_0^*, L_1^*) \text{ space}. \]
\[ \Lambda: \text{ Defined in (4.3)}. \]
\[ \theta: \text{ Either} \theta \text{ or} \theta_{DL}. \]
\( \alpha_d \): Specific volume of dry air (m
\textsuperscript{3}).
\( \chi = L_r / c_p T_K \).
\( \varepsilon = R_d / R_v = 0.6220 \).
\( \kappa_d = R_d / c_p d = 0.2854 (\approx 2/7) \).
\( \kappa_m = R_m / c_p m = \kappa_d (1 - 0.28 r) \).

\( \theta \): Potential temperature of moist air (K) = \( T_K (p_0 / p)^{k_m} \).

\( \theta_D \): Potential temperature of dry air (K) = \( T_K (p_0 / (p - \varepsilon))^{k_d} \).

\( \theta_{DL} \): Value of \( \theta_D \) at the LCL (K).
\( \theta_E, \theta_E^b \): EPT and Rossby EPT (K).
\( \theta_E^b \): Estimate of \( \theta_E \) by a formula.
\( \theta_w \): Wet-bulb potential temperature (WBPT) (°C).
\( \theta_s = \theta_D \exp(\chi) \).

\( |\delta \theta_E|_{max} \): Maximum error in \( \theta_E \) as a function of \( L_0^k \) and \( L_1^k \).

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