The three-dimensional dynamical core of an atmospheric general circulation model employing Yin–Yang grid is developed and examined. Benchmark test cases based on the shallow-water model configuration are first performed to examine the validity of two-dimensional calculations. The experiments show that the model simulates reasonable flow fields with second-order accuracy. The model validation is then extended to three-dimensional features where the capability of the dynamical core on the Yin–Yang grid has not been tested before: the global mountain gravity wave, long-term integration, and life cycle experiments. The simulated flow fields are in good agreement with the results of original experiments in all three experiments. The sensitivity of the model flow field to the overset region is also tested. The experiments reveal that the presence of the overset region does not significantly affect the dynamics on both long and short time scales, if the number of overset grids is fixed to three and the high-order interpolation method is applied for data interpolation between the Yin–Yang grids.
two-dimensional shallow-water model. Rancic et al. (1996) and McGregor (1996) also developed and validated the cubed sphere grid using conformal mapping. Although these grids can overcome the latitude–longitude grid’s problems, the composite grid has also a cause of disadvantage, namely the overset region where different grids are overlapped. This overset region is known to generate numerical noise or error increase due to the violation of continuity at the connected grids’ bound (Burton and Eaton 2002; Sugimura et al. 2006; Williamson 2007).

The Yin–Yang grid (Kageyama and Sato 2004) is an alternative composite grid that can overcome the latitude–longitude grid’s problems, and can obtain some advantages of other composite grids. In addition, the Yin–Yang grid can avoid complicated coordinate conversion between the connected grids, because the grid is composed of two identical latitude–longitude grids connected perpendicular each other. The advantages of using the Yin–Yang grid that we expect are summarized as follows (see also Kageyama and Sato 2004):

- The grid distribution is quasi-uniform over the spherical geometry, thus an explicit large time interval can be chosen, and filtering at the Pole point is unnecessary.
- The grid is composed of only two identical latitude–longitude grids, therefore existing and newly developed methods in the latitude–longitude grid can be straightforwardly applied.
- Simple perpendicular grid connection allows us to avoid complicated coordinate conversion between the connected grids, and discretization in each grid becomes identical.
- The AGCM using Yin–Yang grid can directly switch between global or regional configuration since the grid is an extension of the latitude–longitude grid.

To take these advantages in atmospheric simulation, Peng et al. (2006) and Li et al. (2008) validated their two-dimensional shallow-water models on the Yin–Yang grid using the constrained interpolated profile (CIP) method. Unfortunately, their studies are limited to two-dimensional cases, and they did not use a general method for connecting the two latitude–longitude grids. In the composite grids, three-dimensional dynamical core of AGCM has been developed and tested on both the polar-capped grid (Dudhia and Bresch 2002) and the cubed sphere grid (Adcroft et al. 2004). However, the dynamical core on the Yin–Yang grid has not been developed or examined before. In this paper, we will focus on the feasibility of the Yin–Yang grid on simulating three-dimensional features in three experiments: in the cases of atmospheric flows with topography, long-term integration, and short-term nonlinear property. In the all experiments, sensitivity of the overset region, which is the cause of Yin–Yang grid’s disadvantage, is investigated and the features of the grid depending on configuration of the overset region are analyzed.

In the next section, we will give an overview of the Yin–Yang grid and details of the method for connecting two latitude–longitude grids. Then in section 3, verification experiments on the dynamical core are shown by two-dimensional test cases (i.e., Williamson benchmark test cases; Williamson et al. 1992). Although two-dimensional cases have been already validated by Li et al. (2008), we will again show the results in order to verify the model with simplified and generalized configuration. Formulations and numerical methods for the dynamical core and verifications of the three-dimensional test cases are presented in section 4 using three experiments, based on Qian et al. (1998), Held and Suarez (1994), and Polvani et al. (2004). Summary and conclusion are presented in section 5.

2. The Yin–Yang grid

The Yin–Yang grid (Kageyama and Sato 2004) is a composite grid consisting of two identical latitude–longitude grids connected perpendicular each other. Two grids are identified as the Yin grid and Yang grid, and are essentially the same as original latitude–longitude grid. The Yin grid ($n$ grid) has a Pole axis perpendicular to the equatorial plane; the Yang grid ($e$ grid) has a Pole axis along the equatorial plane (Fig. 1). The region where the Yin grid covers is defined in polar coordinates as $\pi/4 \leq \lambda \leq 7\pi/4$ and $-\pi/4 \leq \varphi \leq \pi/4$, where $\lambda$ and $\varphi$ are longitude and latitude, respectively. The remaining region is covered by the Yang grid.

There are some possible overset configurations for connecting the Yin and Yang grids. As shown in Fig. 1c, we will basically consider the case where three grids points are overlapped in the minimum overset region. However, to investigate the influence of width of the overset region, we will consider an additional two cases for the overset configurations, namely two and four grids points are overlapped in the minimum overset region. Hereinafter, these overset configurations are denoted as over3, over2, and over4, respectively.

a. Coordinate definition

We will describe the formulation of coordinates and their conversions for connecting the Yin and Yang grids.

The coordinates of grid points in both longitude $\lambda$ and latitude $\varphi$ are defined as

\[
\lambda^i_l = \lambda_{\min} + i\Delta\lambda, \quad (i = 0, \ldots, N_\lambda - 1),
\]
\[
\varphi^j_e = \varphi_{\min} + j\Delta\varphi, \quad (j = 0, \ldots, N_\varphi - 1),
\]

(1)
\[ \Delta \lambda = (\lambda_{\text{max}} - \lambda_{\text{min}})/(N_{\lambda} - 1), \]
\[ \Delta \phi = (\phi_{\text{max}} - \phi_{\text{min}})/(N_{\phi} - 1), \] (2)

where \( \lambda \) represents \( n \) (Yin grid) or \( e \) (Yang grid). The each domain bound is defined as \( \lambda_{\text{min}} = \pi/4, \lambda_{\text{max}} = 7\pi/4, \phi_{\text{min}} = -\pi/4, \) and \( \phi_{\text{max}} = \pi/4. \) Both \( N_{\lambda} \) and \( N_{\phi} \) are the total numbers of grid points in the longitude and latitude, respectively.

Then, we have a relationship in the spherical coordinate \((r, \lambda, \phi)\) between Yin and Yang grids as
\[ r^y = r^e, \quad \cos \phi^y \cos \lambda^y = -\cos \phi^e \cos \lambda^e, \]
\[ \cos \phi^y \sin \lambda^y = \sin \phi^e, \quad \sin \phi^y = \cos \phi^e \sin \lambda^e, \] (3)

where \( r \) is the radial coordinate. Here \((r^y, \lambda^y, \phi^y)\) and \((r^e, \lambda^e, \phi^e)\) are the coordinates of Yin and Yang grids, respectively. Data interpolation between Yin and Yang grids are done using the relation of Eq. (3).

b. Data interpolation

The Yin grid and Yang grid exchange data at each grid’s bound during the computation. The data exchange is done at the halo layer that is aligned to each domain’s bound (Fig. 2). The coordinate of grid point located in the Yang grid domain can be expressed by that of the Yin grid domain as
\[ \phi^e = \sin^{-1}(\cos \phi^y \sin \lambda^y), \quad \lambda^e = \sin^{-1}(\sin \phi^y / \cos \phi^e). \] (4)

From this relationship, the coordinate of the grid point located in the Yang grid’s halo layer, \((\lambda_h^y, \phi_h^y)\), can be expressed by the coordinate of the grid point within the Yin grid, \((\lambda_{in}^y, \phi_{in}^y)\), as
\[ \phi_h^y = \sin^{-1}(\cos \phi_{in}^y \sin \lambda_{in}^y), \quad \lambda_h^y = \sin^{-1}(\sin \phi_{in}^y / \cos \phi_h^y). \] (5)

The grid point data at \((\lambda_h^y, \phi_h^y)\), is not updated by governing equations, but is interpolated by grid point data at \((\lambda_{in}^y, \phi_{in}^y)\).

From Yin grid to Yang grid, the value of grid point at \((\lambda_h^y, \phi_h^y)\) is interpolated by applying one-dimensional (1D) Lagrange interpolation in both longitudinal and latitudinal directions (Li et al. 2006). Let \(x_0, \ldots, x_n\) are distinct points in the longitude, then the coefficients of Lagrange interpolation are given as
\[ I_i(x) = \prod_{j=0,j \neq i}^n \frac{x - x_j}{x_i - x_j}, \quad (i = 0, 1, \ldots, n). \] (6)

The order of Lagrange interpolation that uses the above coefficients is \(n\)th order. In addition, let \(y_0, \ldots, y_n\) are distinct points in the latitude. The value at point A (see

Fig. 2. Schematic diagram of data interpolation at the grids’ bound. Third-order Lagrange interpolation is used in the over3 overset configuration.
Fig. 2) is obtained from following equation using 1D Lagrange interpolation:

\[
q^n_A(x_p, y_p) = l_0(x_0)q^n(x_0, y_0) + l_1(x_1)q^n(x_1, y_0) + l_2(x_2)q^n(x_2, y_0) + l_3(x_3)q^n(x_3, y_0),
\]

where \(q^n\) and \((x_i, y_i)\) represent a certain dependent value and the coordinate of grid point within the Yin grid, respectively. The subscript \(A\) denotes that the value is located at point \(A\). Here \(x_p\) is the interpolated point’s coordinate in the longitude. Values at points \(B\), \(C\), and \(D\) are obtained in the same manner. Applying 1D interpolation to remaining points \(B\), \(C\), and \(D\), we have values of \(q^n_B\), \(q^n_C\), and \(q^n_D\). Next, by applying the 1D interpolation in the latitude, we have value at the point \(E\) as

\[
q^5_E(x_p, y_p) = l_0(y_0)q^5_A(x_p, y_0) + l_1(y_1)q^5_B(x_p, y_1) + l_2(y_2)q^5_C(x_p, y_2) + l_3(y_3)q^5_D(x_p, y_3),
\]

where \(y_p\) is the interpolated point’s coordinate in the latitude. The value of \(q^5_E\) is given to the value at the Yang grid’s halo layer grid point, that is, \((\lambda^n, \phi^n)\).

If the interpolated variable is a vector variable, the transformation matrix is adopted to convert the vector direction between the Yin and Yang grids. The horizontal vector direction is rotated by following matrix \(P\) and its inversion \(P^{-1}\). The vector rotation is performed as follows, from the Yang grid to the Yin grid:

\[
\begin{bmatrix}
\nu^n \\
\nu^t
\end{bmatrix} = P
\begin{bmatrix}
\nu^e \\
\nu^e
\end{bmatrix},
\]

\[
P = \begin{bmatrix}
-\sin \lambda^e \sin \nu^n & -\cos \lambda^e / \cos \phi^n \\
\cos \lambda^e / \cos \phi^n & -\sin \lambda^e \sin \nu^n
\end{bmatrix},
\]

and from the Yin grid to the Yang grid:

\[
\begin{bmatrix}
\nu^e \\
\nu^e
\end{bmatrix} = P^{-1}
\begin{bmatrix}
\nu^n \\
\nu^n
\end{bmatrix},
\]

\[
P^{-1} = \begin{bmatrix}
-\sin \lambda^n \sin \nu^e & -\cos \lambda^n / \cos \phi^e \\
\cos \lambda^n / \cos \phi^e & -\sin \lambda^n \sin \nu^e
\end{bmatrix},
\]

where \((\nu^n, \nu^t)\) and \((\nu^e, \nu^e)\) are horizontal vectors in the Yin and Yang grids, respectively. The two grids are in symmetric relation; therefore boundary data interpolation from the Yang grid to the Yin grid is the same as the presented above.

Interpolation processes for over2 and over4 are the same as that for over3 described above. If the targeted interpolation point is found, surrounding 16 grid points used for the interpolation are automatically determined.

In over2 and over4, the targeted grid point with the surrounding grid points only move near or far from the Yin and Yang grids’ interface by one grid, compared to the over3.

3. Two-dimensional test cases

a. The governing equations and numerical methods

A shallow-water model configuration is first implemented to examine the validity of our model with the described interpolation method using the Williamson benchmark test cases (Williamson et al. 1992). The governing equations are the conservation equations of mass and momentum:

\[
\frac{\partial h^*}{\partial t} + \nabla \cdot (h^* \mathbf{v}) = 0,
\]

\[
\frac{\partial h^* \mathbf{v}}{\partial t} + \nabla \cdot (h^* \mathbf{v} \mathbf{v}) = -f \mathbf{k} \times h^* \mathbf{v} - gh^* \nabla h,
\]

where \(h^*\) is the fluid depth, \(h = h^* + h_z\) is the free surface height, and \(h_z\) is the mountain height at fluid bottom. Here \(\mathbf{v} = (u, v)\) is the horizontal velocity vector, where \(u\) and \(v\) are longitudinal and latitudinal velocities, respectively. Here \(\mathbf{k}\) is the vertical unit vector. Constants used in the equations are the gravity acceleration \(g = 9.80616 \text{ m s}^{-2}\), and angular velocity \(\Omega = 7.292 \times 10^{-5} \text{ s}^{-1}\). Here, \(f = 2\Omega \sin \phi\) is the Coriolis parameter.

The discretization of Eqs. (11) and (12) is done with the second-order central finite difference method on the Arakawa C grid (Arakawa and Lamb 1977) except for the lateral advection terms. The third-order upwind scheme of Wicker and Skamarock (2002, hereafter WS03) is adapted to the advection terms since the scheme shows the best balance between the accuracy and performance in our preliminary investigation. Time advance is performed using the fourth-order Runge–Kutta method to exclude dependence of convergence rate on time discretization error. The overset configuration is over3, and the data interpolation between the Yin and Yang grids is done with the third-order Lagrange interpolation method. The Shapiro filter (Shapiro 1971) is applied when unphysical numerical noise is detected as is commonly done in general circulation models (e.g., Fox-Rabinovitz et al. 1997; Klinger et al. 2006; Schroeder et al. 2006). The horizontal resolutions used for these benchmark test cases are 225, 113, 56, and 28 km at the equator.

b. The Williamson benchmark test results

We will give a brief description of the Williamson benchmark test results. Four test cases, namely cases 1, 2, 5, and 6 are conducted [see Williamson et al. (1992) for each case].
In the all test cases, the model results show a qualitatively good agreement with the results of original experiments. The cosine bell is advected and the shape is conserved in the test case 1 (Fig. 3). The stable balance between advection, the pressure gradient, and the Coriolis force is maintained in the testcase 2 (Fig. 4). Qualitatively same fluid height distributions with the original results are obtained in the test case 5 (Fig. 5). Symmetric distributions across the equator are advected without varying the shapes in the test case 6 (Fig. 6).

Since test cases 1 and 2 have theoretical solutions, normalized errors at the last defined day can be estimated by comparing the fluid height with that of theoretical solution. The method of the error estimation follows Li et al. (2008). Estimated errors show that the convergence rates are second order regardless of the rotation angle $\alpha$ (Tables 1 and 2).

Conservation errors are estimated from mass $h^*$ and total energy $E = (h^*|v| + gh^*^2)/2$, which are required to be invariant values. The conservation errors decrease as the resolution increases, and the values are kept to be sufficiently small even at the last defined day (Table 3).

Based on these two-dimensional test cases, the model appears to be capable of simulating two-dimensional circulation on the sphere.

### 4. Three-dimensional test cases

The validity of the dynamical core based on the Yin–Yang grid is investigated using three-dimensional test cases: the global mountain gravity wave experiment (Qian et al. 1998), the Held–Suarez experiment (Held and Suarez 1994), and the life cycle experiment (Polvani et al. 2004). Before describing the experimental results, the governing equations and numerical methods of our model will be shown.

#### a. The governing equations

Starting equations of our model are compressible Euler's equations consisting of mass, momentum, and energy conservation equations, given as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (13)$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla p - \rho \mathbf{g} \kappa - 2\rho \mathbf{\Omega} \times \mathbf{v} + \mathbf{F}_v, \quad (14)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = -(\gamma - 1)p \nabla \cdot \mathbf{v} + (\gamma - 1)q_{\text{heat}}, \quad (15)$$

where $\nabla$ is the divergence operator, $\rho$ is the density, $\mathbf{v}$ is the velocity, $\mathbf{F}_v$ are the sources and sinks of momentum, $p$ is the pressure, $\gamma = C_p/C_v$ is the specific heat ratio ($C_p$ and $C_v$ are specific heats at constant pressure and volume, respectively), $q_{\text{heat}}$ is the heating rate, $\kappa$ is the vertical unit vector, and $\mathbf{\Omega}$ is the angular velocity of the earth. The governing equations transformed into the terrain-following coordinate in which shallow atmosphere approximation (White et al. 2005) is applied are given as
\[
\frac{\partial R}{\partial t} + \mathbf{V}_H \cdot \mathbf{V}_H + \frac{\partial}{\partial z^*} \left( \frac{W}{G^{1/2}} \right) = 0,
\]

\[
\frac{\partial \mathbf{V}_H}{\partial t} + \mathbf{V}_H \mathbf{P} + \frac{\partial}{\partial z^*} \left( \mathbf{G}^2 \mathbf{P} \right) = - \mathbf{A}_H - \mathbf{C}_H + \mathbf{F}_{\mathbf{V}_H},
\]

\[
\frac{\partial W}{\partial t} + \frac{\partial}{\partial z^*} \left( \frac{P}{G^{1/2}} \right) + R_g = - A_z + F_W,
\]

\[
\frac{\partial P}{\partial t} + \mathbf{V}_H \cdot (p \mathbf{V}_H) + \frac{\partial}{\partial z^*} \left( p \mathbf{V}_H \cdot \mathbf{G}^2 + \frac{pw}{G^{1/2}} \right) = -(\gamma - 1) p \left[ \mathbf{V}_H \cdot \mathbf{V}_H + \frac{\partial}{\partial z^*} \left( \mathbf{V}_H \cdot \mathbf{G}^2 + \frac{w}{G^{1/2}} \right) \right] + (\gamma - 1) q_{\text{heat}},
\]

where \( \mathbf{V}_H \) is the horizontal divergence operator defined as

\[
\mathbf{V}_H \cdot \mathbf{V}_H = \frac{1}{G^{1/2} a \cos \phi} \left( \frac{\partial G^{1/2} U}{\partial \lambda} + \frac{\partial G^{1/2} \cos \phi V}{\partial \phi} \right),
\]

Likewise, \( \mathbf{V}_H \cdot (p \mathbf{V}_H) \) is obtained as Eq. (20). Here \( z^* \) is the terrain-following vertical coordinate. Here \( R = \rho' \) and \( P = p' \) denote variances of \( \rho \) and \( p \) from given background values. The background values are set to satisfy the hydrostatic condition and those horizontal distributions are set to be homogeneous. The horizontal momenta presented here are expressed as \( \mathbf{V}_H = (U, V) = (p u, p v) \), and horizontal velocities are expressed as \( \mathbf{v}_H = (u, v) \). Here \( W = \rho w \) and \( w \) are vertical momentum and velocity, which directions are parallel to the radial direction; \( F_{\mathbf{V}_H} \) and \( F_W \) are sources and sinks of momentum; and \( \mathbf{A}_H \) and \( A_z \) are advection terms of momentum conservations, given as

\[
\mathbf{A}_H = \mathbf{V}_H \cdot (\mathbf{V}_H \mathbf{v}_H) + \frac{\partial}{\partial z^*} \left[ (\mathbf{V}_H \mathbf{v}_H) \cdot \mathbf{G}^2 + \frac{\mathbf{V}_H \mathbf{w}}{G^{1/2}} \right],
\]

\[
A_z = \mathbf{V}_H \cdot (\mathbf{W} \mathbf{v}_H) + \frac{\partial}{\partial z^*} \left[ (\mathbf{W} \mathbf{v}_H) \cdot \mathbf{G}^2 + \frac{\mathbf{W} \mathbf{w}}{G^{1/2}} \right].
\]
The metric vector $G^z$ and metric term $G^{1/2}$ are defined as

\[
G^z = \left( \frac{G^{13}}{a \cos \phi}, \frac{G^{23}}{a} \right), \quad G^{1/2} = 1 - \frac{z}{H},
\]

\[
G^{13} = \frac{1}{G^{1/2}} \left( \frac{z^*}{H} - 1 \right) \frac{\partial z}{\partial \lambda}, \quad G^{23} = \frac{1}{G^{1/2}} \left( \frac{z^*}{H} - 1 \right) \frac{\partial z}{\partial \phi},
\]

where $z^* = H(z - z_s)/(H - z_s)$ is the height in the terrain-following coordinate. Here $z$, $z_s$, and $H$ are physical height, ground surface height, and physical height of the model’s top, respectively.

Here $C_H = (C_{H,U}, C_{H,V})$ are terms including the Coriolis and centrifugal forces, given as

\[
C_{H,U} = 2 \Omega V + \frac{V u \tan \phi}{a}, \quad C_{H,V} = -2 \Omega U - \frac{U u \tan \phi}{a},
\]

where $\Omega = \sin \phi \Omega$ with $\Omega = 7.292 \times 10^{-5}$ s$^{-1}$ and $a = 6.37122 \times 10^6$ m being the angular velocity and the earth radius, respectively.

### b. The numerical methods

In our model, the horizontal explicit and vertical implicit (HEVI) method (Klemp and Wilhelmson 1978; Satoh 2002; Tomita and Satoh 2004) is employed to solve the presented equations. This method uses a Runge–Kutta time-splitting scheme by dividing the integrations for fast and slow mode terms (Wicker and Skamarock 2002; Skamarock and Klemp 2008). The fast mode terms are assumed to relate to both acoustic and gravity waves. The computational sequences for integrating the fast mode terms use the following equations:

\[
\frac{R^{*+\Delta \tau} - R^{*}}{\Delta \tau} + V_H \cdot \nabla H = \frac{W^{*+\Delta \tau} - W^{*}}{G^{1/2}} = \frac{R^{*+\Delta \tau}}{G^{1/2}} A^* + F^*_W,
\]

\[
\frac{V^{*+\Delta \tau} - V^{*}}{\Delta \tau} = V_H P^* + \frac{\partial}{\partial z^*} \left( G^z P^* \right) - A^*_H + C^*_H + F^*_V,
\]

\[
\frac{W^{*+\Delta \tau} - W^{*}}{\Delta \tau} = - \frac{\partial}{\partial z^*} \left( \frac{P^* + \Delta \tau}{G^{1/2}} \right) + R^{*+\Delta \tau} a^* + F^*_W,
\]
Table 1. Normalized errors and convergence rates for test case 1.

<table>
<thead>
<tr>
<th>Resolution (km)</th>
<th>$l_1$</th>
<th>$l_1$ order</th>
<th>$l_2$</th>
<th>$l_2$ order</th>
<th>$l_w$</th>
<th>$l_w$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>225</td>
<td>$2.31 \times 10^{-1}$</td>
<td>—</td>
<td>$1.54 \times 10^{-1}$</td>
<td>—</td>
<td>$1.14 \times 10^{-1}$</td>
<td>—</td>
</tr>
<tr>
<td>113</td>
<td>$5.02 \times 10^{-2}$</td>
<td>2.20</td>
<td>$3.42 \times 10^{-2}$</td>
<td>2.17</td>
<td>$2.86 \times 10^{-2}$</td>
<td>1.99</td>
</tr>
<tr>
<td>56</td>
<td>$9.08 \times 10^{-3}$</td>
<td>2.47</td>
<td>$7.36 \times 10^{-3}$</td>
<td>2.22</td>
<td>$7.83 \times 10^{-3}$</td>
<td>1.87</td>
</tr>
<tr>
<td>28</td>
<td>$1.67 \times 10^{-3}$</td>
<td>2.45</td>
<td>$1.77 \times 10^{-3}$</td>
<td>2.05</td>
<td>$2.38 \times 10^{-3}$</td>
<td>1.72</td>
</tr>
</tbody>
</table>

α = π/4

<table>
<thead>
<tr>
<th>Resolution (km)</th>
<th>$l_1$</th>
<th>$l_1$ order</th>
<th>$l_2$</th>
<th>$l_2$ order</th>
<th>$l_w$</th>
<th>$l_w$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>225</td>
<td>$2.51 \times 10^{-1}$</td>
<td>—</td>
<td>$1.54 \times 10^{-1}$</td>
<td>—</td>
<td>$1.42 \times 10^{-1}$</td>
<td>—</td>
</tr>
<tr>
<td>113</td>
<td>$5.33 \times 10^{-2}$</td>
<td>2.23</td>
<td>$3.25 \times 10^{-2}$</td>
<td>2.24</td>
<td>$2.29 \times 10^{-2}$</td>
<td>2.63</td>
</tr>
<tr>
<td>56</td>
<td>$9.98 \times 10^{-3}$</td>
<td>2.42</td>
<td>$7.12 \times 10^{-3}$</td>
<td>2.19</td>
<td>$6.46 \times 10^{-3}$</td>
<td>1.82</td>
</tr>
<tr>
<td>28</td>
<td>$1.87 \times 10^{-3}$</td>
<td>2.41</td>
<td>$1.75 \times 10^{-3}$</td>
<td>2.02</td>
<td>$1.96 \times 10^{-3}$</td>
<td>1.72</td>
</tr>
</tbody>
</table>

α = π/2

<table>
<thead>
<tr>
<th>Resolution (km)</th>
<th>$l_1$</th>
<th>$l_1$ order</th>
<th>$l_2$</th>
<th>$l_2$ order</th>
<th>$l_w$</th>
<th>$l_w$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>225</td>
<td>$2.54 \times 10^{-1}$</td>
<td>—</td>
<td>$1.61 \times 10^{-1}$</td>
<td>—</td>
<td>$1.20 \times 10^{-1}$</td>
<td>—</td>
</tr>
<tr>
<td>113</td>
<td>$5.50 \times 10^{-2}$</td>
<td>2.21</td>
<td>$3.59 \times 10^{-2}$</td>
<td>2.17</td>
<td>$2.91 \times 10^{-2}$</td>
<td>2.05</td>
</tr>
<tr>
<td>56</td>
<td>$1.01 \times 10^{-2}$</td>
<td>2.45</td>
<td>$7.71 \times 10^{-3}$</td>
<td>2.22</td>
<td>$7.93 \times 10^{-3}$</td>
<td>1.88</td>
</tr>
<tr>
<td>28</td>
<td>$1.86 \times 10^{-3}$</td>
<td>2.44</td>
<td>$1.86 \times 10^{-3}$</td>
<td>2.05</td>
<td>$2.39 \times 10^{-3}$</td>
<td>1.73</td>
</tr>
</tbody>
</table>

In short step integration for the fast mode, a short time interval $\Delta \tau$ is used independently of the large time interval $\tau$. The time derivative from large step variables is defined as $\phi_{\tau} = \phi - \phi'$. The $\phi_{\tau+\Delta \tau}$ is the value to be integrated, $\phi'_{\tau}$ is the value at the time $\tau$ for short steps, and $\phi'$ is the value at time $\tau$. Based on Eqs. (25)–(28), time integrations are performed for variances of the prognostic variables in the short step.

Beginning of the short step, $\mathbf{V}_{h_{\tau+\Delta \tau}}$ is first predicted using Eq. (26), then 1D Helmholtz equation formed from Eqs. (25), (27), and (28) will be solved. Formulation of the 1D Helmholtz equation is as follows. By converting the energy equation using the relation of $w_{\tau+\Delta \tau} = W_{\tau+\Delta \tau} / \rho'$, and moving the terms unrelated to $w_{\tau+\Delta \tau}$ except for the time derivative to the right-hand side, Eq. (28) becomes

$$
\frac{p_{\tau+\Delta \tau} - p_{\tau}}{\Delta \tau} + (\gamma - 1)p_{\tau} \left[ \mathbf{V}_{h_{\tau}} \cdot \mathbf{V}_{h_{\tau}} + \frac{\partial}{\partial \tau} \left( \mathbf{V}_{h_{\tau}} \cdot \mathbf{G} + \frac{w_{\tau+\Delta \tau}}{G_{\tau}^{1/2}} \right) + \mathbf{V}_{h_{\tau}} \cdot (p' \mathbf{V}_{h_{\tau}}) \right] + \frac{\partial}{\partial \tau} \left( p' \frac{\partial w_{\tau+\Delta \tau}}{\partial \tau} \right) \mathbf{G} + \left( \gamma - 1 \right) q_{\text{heat}}^{\tau}.
$$

(28)

$$
S_{p} = - (\gamma - 1)p_{\tau} \left[ \mathbf{V}_{h_{\tau}} \cdot \mathbf{V}_{h_{\tau}} + \frac{\partial}{\partial \tau} \left( \mathbf{V}_{h_{\tau}} \cdot \mathbf{G} + \frac{w_{\tau+\Delta \tau}}{G_{\tau}^{1/2}} \right) \right] - \mathbf{V}_{h_{\tau}} \cdot (p' \mathbf{V}_{h_{\tau}}) - \frac{\partial}{\partial \tau} \left( p' \frac{\partial w_{\tau+\Delta \tau}}{\partial \tau} \right) \mathbf{G} + \frac{p' w'_{\tau+\Delta \tau}}{G_{\tau}^{1/2}}.
$$

(30)

Because we assume that the background fields for density and pressure satisfy the hydrostatic condition, Eq. (29) can then be transformed into

$$
\frac{p_{\tau+\Delta \tau} - p_{\tau}}{\Delta \tau} + \gamma p_{\tau} \frac{\partial}{\partial \tau} \left( \frac{W_{\tau+\Delta \tau}}{\rho' G_{\tau}^{1/2}} \right) + \frac{W_{\tau+\Delta \tau} \tilde{g}}{G_{\tau}^{1/2}} = S_{p'},
$$

(31)

where

$$
\tilde{g} = g - \frac{1}{\rho'} \frac{\partial}{\partial \tau} \left( \frac{P'}{G_{\tau}^{1/2}} + R'g \right),
$$

(32)

is the pressure gradient and buoyancy force term evaluated in the large step. We now have Eq. (31) and following two equations to be combined for 1D Helmholtz equation:

The WS03 scheme is adapted to the lateral advection method. The three equations, Eqs. (31), (33), and (34) are identical to those used in Tomita and Satoh (2004), so the following steps for solving the 1D Helmholtz equation and variable correction steps by are identical to those used in Tomita and Satoh (2004), then we have both and one short step computation is finished. Computations of both large and short steps are repeated several times in one time step, depending on the applied time integration method (Skamarock and Klemp 2008).

The WS03 scheme is adapted to the lateral advection terms in the dynamical core, and the other terms are discretized with the second-order central finite difference method. The third-order Runge–Kutta time-splitting scheme (Wicker and Skamarock 2002) is used in all the experiments because this method also has the best balance between low computational load and high accuracy. Nonuniformly spaced 30 vertical layers are used and 30-km height is considered. Divergence damping (Skamarock and Klemp 1992) is not used in the all following experiments.

c. The global mountain gravity wave experiment

The test case of the global mountain gravity wave proposed by Qian et al. (1998) examines the validity of vertical calculation in the global model. The model configurations are as follows.

A westerly constant wind of $U = 40$ m s$^{-1}$ is set for the initial fields, considering stable balance over a rotating solid sphere. Our model has a circular mountain with a height ($h_0$) of 1000 or 4000 m and half-width of 1250 km (Former and latter cases are denoted as mild nonlinear and highly nonlinear cases, hereafter). Three mountain center locations, (180°, 0°), (45°E, 0°), and (45°W, 0°) are chosen to test the influence of the overset region. [The

TABLE 2. Normalized errors and convergence rates for test case 2.

<table>
<thead>
<tr>
<th>Resolution (km)</th>
<th>$l_1$</th>
<th>$l_1$ order</th>
<th>$l_2$</th>
<th>$l_2$ order</th>
<th>$l_3$</th>
<th>$l_3$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>225</td>
<td>1.21 × 10^{-4}</td>
<td>—</td>
<td>1.38 × 10^{-4}</td>
<td>—</td>
<td>2.70 × 10^{-4}</td>
<td>—</td>
</tr>
<tr>
<td>113</td>
<td>3.03 × 10^{-5}</td>
<td>2.00</td>
<td>3.45 × 10^{-5}</td>
<td>2.00</td>
<td>6.49 × 10^{-5}</td>
<td>2.00</td>
</tr>
<tr>
<td>56</td>
<td>7.61 × 10^{-6}</td>
<td>1.99</td>
<td>8.62 × 10^{-6}</td>
<td>2.00</td>
<td>1.59 × 10^{-5}</td>
<td>2.00</td>
</tr>
<tr>
<td>28</td>
<td>1.91 × 10^{-6}</td>
<td>2.00</td>
<td>2.15 × 10^{-6}</td>
<td>2.00</td>
<td>3.90 × 10^{-6}</td>
<td>2.00</td>
</tr>
</tbody>
</table>

$\alpha = 0$

$\alpha = \pi/4$

$\alpha = \pi/2$

$\frac{R^{\phi+\Delta t} - R^\phi}{\Delta t} + \frac{\partial}{\partial \phi^*} \left( \frac{W^{\phi+\Delta t}}{G^{1/2}} \right) = S_R$, (33)

$\frac{W^{\phi+\Delta t} - W^\phi}{\Delta t} + \frac{\partial}{\partial \phi^*} \left( \frac{P^{\phi+\Delta t}}{G^{1/2}} \right) + R^{\phi+\Delta t} = S_W$, (34)

where

$S_R = \left[ \mathbf{v}_H \cdot \mathbf{v}_H^{+\phi+\Delta t} + \frac{\partial}{\partial \phi^*} \left( \mathbf{v}_H^{+\phi+\Delta t} \cdot \mathbf{G} + \frac{W^t}{G^{1/2}} \right) \right]$, (35)

$S_W = \left[ \frac{\partial}{\partial \phi^*} \left( \frac{P^t}{G^{1/2}} \right) + R^t \right] - A_2^t + F^t$. (36)

By substituting Eqs. (31) and (33) into Eq. (34), we can form the 1D Helmholtz equation to be solved for $W^{\phi+\Delta t}$. The three equations, Eqs. (31), (33), and (34) are identical to those used in Tomita and Satoh (2004), so the following steps for solving the 1D Helmholtz equation and variable correction steps by $W^{\phi+\Delta t}$ are the same. Performing the correction steps by $W^{\phi+\Delta t}$, then we have both $R^{\phi+\Delta t}$ and $P^{\phi+\Delta t}$, and one short step computation is finished. Computations of both large and short steps are repeated several times in one time step, depending on the applied time integration method (Skamarock and Klemp 2008).

The WS03 scheme is adapted to the lateral advection terms in the dynamical core, and the other terms are discretized with the second-order central finite difference method. The third-order Runge–Kutta time-splitting

TABLE 3. Normalized errors at last defined day for test cases 5 and 6.

<table>
<thead>
<tr>
<th>Resolution (km)</th>
<th>Case 5</th>
<th>Case 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>225</td>
<td>6.47 × 10^{-6}</td>
<td>1.17 × 10^{-5}</td>
</tr>
<tr>
<td>113</td>
<td>5.20 × 10^{-7}</td>
<td>1.06 × 10^{-6}</td>
</tr>
<tr>
<td>56</td>
<td>2.85 × 10^{-8}</td>
<td>4.51 × 10^{-8}</td>
</tr>
<tr>
<td>28</td>
<td>1.28 × 10^{-8}</td>
<td>2.38 × 10^{-8}</td>
</tr>
</tbody>
</table>
The configuration of over3 (see Fig. 1c) is basically used for 5 days and the results are compared at 5 days to the horizontal resolutions. Time integration is performed for 5 days and the results are compared at 5 days. The configuration of over3 (see Fig. 1c) is basically used in this experiment.

Nondimensional mountain height scales ($Nh/U$, where $N = 0.0187$ s$^{-1}$ is Brunt–Väisälä frequency), which indicate nonlinearity of the gravity wave for 1000- and 4000-m mountains, are 0.47 and 1.87, respectively. In the latter highly nonlinear case, the value of $Nh/U$ is much larger than those of other mountain gravity wave experiments (e.g., Epifanio and Qian 2008; Wedi and Smolarkiewicz 2009). Therefore in the highly nonlinear cases, we adopted a turbulence model (Smagorinsky 1963) to model unresolved fluctuations induced from high nonlinearity of the gravity wave (Tomita and Satoh 2004).

Gravity waves of the mild nonlinear case are simulated reasonably well (Fig. 7). The gravity wave propagates from the mountain location toward the upstream and upward region. The distribution becomes stable just after 2 days. Even though the mountain locates at the overset region, gravity wave behavior is the same as when the mountain locates far from the overset region. The sensitivity of the gravity wave behavior to the horizontal resolution is also found in Fig. 7. The behavior of the gravity wave in high resolution is simulated almost same as that in the low resolution.

Gravity waves of highly nonlinear cases are also simulated reasonably well (Fig. 8). The shape of the gravity waves becomes steeper than that of mild nonlinear case. It is also found that the difference of the mountain location does not have influence on the gravity wave behavior. In addition to these experiments, we performed highly nonlinear cases with varying width of the overset region from over3 to over2 and over4. Nevertheless different overset width is set, we obtained similar results compared to the results of Fig. 8 (Fig. 9). This fact indicates that width of the overset region does not affect the behavior of the gravity wave.

Both the mild and highly nonlinear cases are reasonably well simulated regardless of the mountain’s center location, horizontal resolution, and width of the overset region. In addition, the simulated behaviors are similar to each other. These results show that the dynamical core using the Yin–Yang grid is feasible to simulate gravity waves with topography.

d. The Held–Suarez experiment

The test case of Held and Suarez (1994) examines the model validity of long-term integration, which is an important ability for climate studies. Our model is first integrated for 200 days for spinup. Then it is integrated for another 1000 days to examine the model. The experiment requires forcing and dissipation terms for the governing equations; we use identical forcing terms given in Held and Suarez (1994). In this experiment, 450-, 225-, and 113-km horizontal resolutions are used. The configuration for the overset region is basically over3.

Good agreements with the results of original experiment are obtained in the vertical distributions of 1000-day-averaged zonal mean longitudinal velocity and temperature (Fig. 10). Since the forcing is given symmetrically across the equator, symmetrical vertical distributions should be obtained. It is found that symmetrical distributions are observed as the horizontal resolution increases.

To investigate sensitivity and characteristics of the overset region on the long-term integration experiment, the configuration of the overset region is modified to over2 and over4. To extract the difference in fluctuating fields, eddy variances of the temperature are compared as done in Adcroft et al. (2004) (Fig. 11). Both zonal mean distributions become asymmetrical as the overset region becomes wider. In the longitudinal velocity distributions, a strong wind speed region shows asymmetrical distributions in the case of over4. Moreover, in this case, noiselike structure, which might be derived from the overset region appears in the eddy variance distribution. This noiselike structure is considered to occur because number of grid points, which have independent values at the overset region increases, and that results in creating discontinuity. The longitudinal velocities of over2 and over3 are similar, but fluctuation of the temperature field of over2 is slightly weak compared to the original results.

Mass conservation in this experiment is analyzed in terms of the configuration of the overset region. We varied the width of the overset region and the order of the Lagrange interpolation. The initial total mass over the sphere surface is used as the reference value, and the errors are computed as normalized errors (i.e., $l_2$ norm). The mass conservation errors do not grow as long as high-order interpolation is used, while the errors grow and become significant when first-order interpolation is used (Fig. 12). This is because in the overset grid, the order of the interpolation should be higher than the order of discretization to ensure the continuity over the overset region (Burton and Eaton 2002). On the other hand, the difference in width of the overset region does not have influence on the error growth.

From this long-term experiments, we conclude that wide overset region might affect long-term atmospheric circulation. However, the wide overset region does not have influence on the mass conservation error growth. It is also found that the order of the interpolation accuracy...
Fig. 7. Vertical velocity distributions at 5 day in the case of mild nonlinear case: (a) 225- and (b) 113-km resolutions: \( \lambda_c, \varphi_c \) = (180°, 0°), (middle) 45°E, and (bottom) 45°W. The contour interval is 0.005 m s\(^{-1}\).
Fig. 8. As in Fig. 7, but for the highly nonlinear case. The contour interval is 0.01 m s$^{-1}$. 

$$(\lambda_c, \varphi_c) = (180^\circ, 0^\circ)$$

$$(\lambda_c, \varphi_c) = (45^\circ E, 0^\circ)$$

$$(\lambda_c, \varphi_c) = (45^\circ W, 0^\circ)$$
should be higher than that of the discretization; otherwise, mass conservation becomes worse.

This experiment examines the long-term and large-scale dynamics; however, it is insufficient to examine the more short-term and small-scale dynamics. Therefore to examine the validity of the dynamical core in the short-term highly nonlinear experiment, the experiment of Polvani et al. (2004) is conducted.

e. The life cycle experiment

The life cycle experiment, proposed by Polvani et al. (2004), is intended for examining a three-dimensional hydrostatic dynamical core, but we will use it to examine our model nonetheless. Unlike the Held–Suarez experiment, this test case involves a baroclinically unstable flow field (i.e., nonlinear instability and initial value problems). It is therefore believed to be a useful validation test for short-term atmospheric flow. Jablonowski and Williamson (2006) is similar useful experiment, but we follow Polvani et al. (2004) for its simplicity. This experiment principally examines two features: the qualitative tendency of the model flow field and the grid convergence (i.e., whether the simulated flow field converges with increasing horizontal resolution).

The basic initial longitudinal velocity and temperature fields are set to

\[
\begin{align*}
u(p, \varphi) &= \begin{cases} u_0 \sin^2(\pi \sin^2 \varphi)F(z) & \varphi \geq 0 \\ 0 & \varphi < 0 \end{cases}, \\
F(z) &= \frac{1}{2} \left[ 1 - \tanh^3 \left( \frac{z - z_0}{A} \right) \right] \sin \left( \frac{\pi z}{z_1} \right), \quad (37) \\
T(\varphi, z) &= \int_\varphi^{\pi} \left[ -HR^{-1}(af + 2u \tan \varphi') \frac{\partial u}{\partial z} \right] d\varphi', \quad (38)
\end{align*}
\]

where \( z = -H \log(p/p_0) \) and \( f = 2 \Omega \sin \varphi \). The constant variables shown in Table 1 of Polvani et al. (2004) are
also used. The temperature field is then perturbed with a simple shape temperature anomaly $T'$ defined by

$$T'(\lambda, \varphi) = \bar{T} \operatorname{sech}^2\left(\frac{\lambda - \lambda_o}{\alpha}\right) \operatorname{sech}^2\left(\frac{\varphi - \varphi_o}{\beta}\right),$$  

(39)

where $(\lambda_o, \varphi_o) = (0^\circ, 0^\circ)$ is the anomaly center location, and the amplitude of the anomaly is $\bar{T} = 1.0 \text{ K}$. Note that because the experiment of Polvani et al. (2004) is originally proposed for hydrostatic models, the initial condition also needs to satisfy hydrostatic balance. We

Fig. 10. The 1000-day-averaged zonal mean longitudinal (top) velocity and (bottom) temperature: (a) 450-, (b) 225-, and (c) 113-km horizontal resolutions.

Fig. 11. The zonal mean longitudinal (top) velocity and (bottom) eddy variance of temperature: (a) over2, (b) over3, and (c) over4 with 113-km horizontal resolution used.
use 5 horizontal spatial resolutions, 563, 282, 141, 70, and 35 km at the equator, which are the same as those used in the original test cases. Time integration is performed for 12 days. Diffusion terms need to be introduced to the horizontal momentum and the energy conservation equations to obtain the identical experimental setup of Polvani et al. (2004). The diffusion terms are split into the short step computation as fast mode terms. We perform this case without using filtering, damping, or hyper-diffusion (Polvani et al. 2004; Iga et al. 2007).

We find our model flow field qualitatively in good agreement with the original results of Polvani et al. (2004), Iga et al. (2007), and Barros and Carcia (2007). Grid convergences in the horizontal distributions are observed (Fig. 13). The temperature field shows the initial perturbation growing as it propagates to the east until it finally reaches (180°, 0°) on 12 days.

To examine the grid convergence more quantitatively, we estimate the convergence error, regarding the highest-resolution case, 35 km, as the true solution. Figure 14 shows the time developments of $l_2$ norms at $\sigma = 0.975$, where $\sigma = p/p_s$ and $p_s$ is the surface pressure. Here, the $l_2$ norm follows Polvani et al. (2004), defined by

$$l_2 = \frac{|\zeta - \zeta_t|_2}{|\zeta_t|_2},$$

where $\zeta_t$ is the vorticity computed from the 35-km resolution. Here $\zeta$ is computed from coarser-resolution cases. As seen in Fig. 14, the values of $l_2$ norms converge. Values of $l_2$ at 12 days are 0.73, 0.36, 0.13, and 0.03 for 563, 282, 141, and 70 km, respectively. These errors
indicate that convergence rates increase with an increase of the horizontal resolution, and the convergence rate approaches second order. This is attributed to the fact that the vertical calculation is much affected by the error of coarse horizontal resolution since we do not employ any numerical stabilization such as hyper-diffusion or divergence damping (Skamarock and Klemp 1992).

We further examine the effect of the overset region by locating the largest perturbation within the overset region. Two cases (experiments 45°W and 45°E), where the initial temperature anomalies are located at 135°E, 0° and 135°W, 0°, show basically the same results as the original experiment (Fig. 15). The same experiments are performed with other coarser resolutions and the time evolutions of the $l_2$ norms of vorticity fields at $\sigma = 0.975$ are computed. The evolutions of $l_2$ norms are found to be similar to those in original experiments, except for around 10 days (Fig. 16). The error growth difference is due to the difference of the anomaly initial location, but the difference is considered to be insignificant. If the overset region affects the flow field, the results are likely not to converge with the resolution increase.

We therefore conclude that the overset region does not affect the flow field in this experiment. The life cycle experiment shows that the dynamical core employing the Yin–Yang grid is feasible to simulate short- and small-scale dynamics.

5. Summary and conclusions

The dynamical core of an atmospheric general circulation model employing the Yin–Yang grid is developed and examined. Two- and three-dimensional validation
experiments are performed. The results are summarized below.

Validation experiments are first performed using the two-dimensional Williamson benchmark test cases (Williamson et al. 1992) where the third-order upwind scheme of Wicker and Skamarock (2002) is used for computing the advection. The results show good agreement with the original cases and the model accuracy is found to be second-order accuracy.

In the global mountain gravity wave experiment (Qian et al. 1998), behaviors of gravity waves of both mild and highly nonlinear cases are simulated reasonably well. By locating the mountain over the overset region, it is confirmed that the overset region does not affect the behavior of the gravity wave. In addition, differences in horizontal resolution and width of the overset region do not have influence on the behavior of the gravity wave.

The feasibility for long-term integration is examined by the Held and Suarez (1994) experiment. Our model succeeds in reproducing flow fields in agreement with the results of original experiment. We further analyze the mass conservation by varying the width of the overset region and the order of the data interpolation method, which is used for connecting Yin and Yang grids. It is found that the wide overset region causes errors at the overset region and makes the results worse because of the discontinuity at the grid’s bound. It is also found that a high-order interpolation method is required to ensure the mass conservation.

The life cycle experiment (Polvani et al. 2004) is performed to examine the properties of a short-term, nonlinear, small-anomaly-initiated flow evolution. We find that the results agree with the results of original experiment, with the horizontal distributions converging as resolution increases. We also perform the same experiments but with the center of the largest perturbation located over the overset region. The results show that the flow field is not significantly affected by the presence of the overset grid. Moreover, grid convergence is equally obtained.

Based on two- and three-dimensional test cases, we conclude that the dynamical core employing a Yin–Yang grid is feasible to simulate atmospheric flows and circulation. However, treatment of the overset region should be done in the appropriate manner. The moderate narrow overset region and high-order data interpolation method for connecting Yin and Yang grids are necessary to obtain accurate results.

Acknowledgments. The authors thank Dr. Shin Kida of ESC, JAMSTEC for his helpful advice and inspiring discussions. The authors are also grateful to Mr. Hiromitsu Fuchigami of NEC Informatec Systems for his technical support.

REFERENCES


