The Impact of Noisy Physics on the Stability and Accuracy of Physics–Dynamics Coupling

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ABSTRACT

The coupling of the dynamical core of a numerical weather prediction model to the physical parameterizations is an important component of model design. This coupling between the physics and the dynamics is explored here from the perspective of stochastic differential equations (SDEs). It will be shown that the basic properties of the impact of noisy physics on the stability and accuracy of common numerical methods may be obtained through the application of the basic principles of SDEs. A conceptual model setting is used that allows the study of the impact of noise whose character may be tuned to be either very smooth (red) or white (noisy). The change in the stability and accuracy of common numerical methods as the character of the noise changes is then studied. Distinct differences are found between the ability of multistage (Runge–Kutta) schemes as compared with multistep (Adams–Bashforth/leapfrog) schemes to handle noise of various characters. These differences will be shown to be attributable to the basic philosophy used to design the scheme. Additional experiments using the decentering of the noisy physics will also be shown to lead to strong sensitivity to the quality of the noise. As an example, the authors find the novel result that noise of a diffusive character may lead to instability when the scheme is decentered toward greater implicitness. These results are confirmed in a nonlinear shear layer simulation using a subgrid-scale mixing parameterization. This subgrid-scale mixing parameterization is modified stochastically and shown to reproduce the basic principles found here, including the notion that decentering toward implicitness may lead to instability.

1. Introduction

A numerical weather prediction (NWP) model can broadly be decomposed into two main components. On the one hand we have the dynamical core of the model that solves the well-known Navier–Stokes (NS) equations (Batchelor 1977, 147–151) in a rotating frame of reference and on the sphere. On the other hand, for NWP models to accurately simulate the general circulation of the earth’s atmosphere they must also simulate the collection of physical processes distinct from those represented by the dynamical core. These processes are referred to as the physics of the model and are associated with the effects of unrepresented physical processes, such as solar and outgoing longwave radiation, cloud microphysical processes, convective clouds, etc. Another group of processes also referred to as physics and represented by this portion of the model are the unresolved motions (e.g., gravity wave breaking, turbulence, etc.). These unresolved motions would be simulated by the NS equations, and therefore by the dynamical core of the model, if the model had sufficient resolution to resolve them. Unfortunately, there are always motions smaller than the smallest resolvable scale of our spatial discretization and some accounting for these motions is made by representing their sum total (or average) effect on a grid cell.

The governing equations for the average effects across a grid cell are not known to the same standard as the NS equations. Because the equations governing this average gridcell response to unresolved motions are largely unknown, these routines typically contain “switches” to activate different hypothesized physical processes at different times based on the various properties of the state of the model. These switches may lead to rapid changes with time in the forcing produced by this portion of the model. This potential for rapidly varying forcing functions (tendencies) in the model is what we will be referring to as “noisy physics.” In addition to these implicitly noisy physics parameterizations there has recently been substantial research into the development of parameterizations that are explicitly stochastic (e.g., Leith 1990; Majda and Khouider 2002; Plant and Craig...
2008; Teixeira and Reynolds 2008; Berner et al. 2009; Eckermann 2011). These types of parameterizations attempt to account for the uncertainty in unresolved, subgrid-scale phenomena through the explicit use of stochastic (i.e., explicitly including random numbers in the calculation of the tendency) forcing functions.

Irrespective of whether the physics is implicitly or explicitly stochastic, a significant amount research has been expended on the study of the proper methods to link the physics to the dynamics within the model algorithm. These different methods, for the most part, can be broken into groups determined by their choice of when to evaluate the physics across the present time step. This apparently arbitrary choice of when to evaluate the physics across the present time step will be the subject of this manuscript. Recent experiments with NWP models (e.g., Cullen 2001; Cullen and Salmond 2003; Diamantakis et al. 2007) have shown significant benefit from predictor–corrector-type schemes for the physics. This evaluation of the physics more than once per time step allows for an effective linear interpolation of the evaluation point across the time step. Usually, the benefit from predictor–corrector-type schemes for the physics is attributed to the apparently beneficial properties of implicitness (evaluating the physics at the most forward point in time of the present time step). A major point of our work here is to provide an alternative interpretation for the beneficial attributes of predictor–corrector-type schemes for the physics.

A wide body of theoretical studies, most similar to the tack taken in the work to be presented here, has been performed with a simple idealized model setting that illustrates the basic properties of the stability and accuracy of potential physics–dynamics coupling methods (e.g., Caya et al. 1998; Staniforth et al. 2002a,b; Egger 2003; Dubal et al. 2004, 2005, 2006; Termonia and Hamdi 2007). This body of work has generally focused on answering questions about numerical methods applied to forcing functions (physics) that are both state independent and also at frequencies that, while they may indeed be fast, are nevertheless resolved by the temporal discretization. We will extend this work by answering questions about the properties of numerical methods applied to state-dependent forcing functions that oscillate at frequencies unresolved by the temporal discretization. The proper way to answer questions of this nature is from the perspective of a stochastic differential equation (SDE).

An SDE is simply a differential equation being disturbed by random noise. Significant discussion of the properties of SDEs will be presented in section 2. The theory for SDEs will be used here to provide a formal framework to understand the impact of noise on the stability and accuracy of a particular numerical method. The textbook on this subject by Kloeden and Platen (1999) provides the basis for everything to be discussed below and the interested reader is encouraged to consult chapter 10 of their work. While a wide body of work has been constructed describing a variety of stochastic versions of traditional numerical methods (e.g., Ewald and Temam 2003; Ewald et al. 2004; Ewald and Penland 2009; Ewald 2012), we choose here not to employ these methods. We do this because NWP models do not make use of these schemes and also because these stochastic schemes typically require some knowledge about the properties of the stochastic term. These properties are simply unknown for the implicitly stochastic parameterizations mentioned above. In contrast, our goal here is to answer questions about the properties of the schemes presently being used in NWP using the theory of SDEs. Our approach will be to make a connection between traditional numerical analysis and the theory of SDEs by constructing a simple model setting that allows the study of the impact of noise whose character may be tuned to be either very red (smooth) or white (noisy). We then study the change in the stability and accuracy of common numerical methods as the character of the noise goes from red to white.

The plan of this paper is as follows. In section 2 we briefly review the relevant theory of stochastic differential equations. In section 3 we illustrate why different numerical methods behave differently in the presence of stochastic forcing terms and then go on in section 4 to illustrate our main results with a nonlinear model of evolving shear instabilities. Section 5 closes the article with a summary and a discussion of the main conclusions.

2. A heuristic connection between nonlinear dynamical systems and SDEs

In this section we will provide a brief, heuristic description of the relationship between a simple nonlinear dynamical system and the theory of SDEs. The interested reader is encouraged to see the more complete treatment of these and other ideas in Papanicolaou and Kohler (1974), Kloeden and Platen (1999), and Jazwinski (2007).

Imagine a physical system \( u \) governed by the following partial differential equation:

\[
\frac{\partial u}{\partial t} = M(u, x, t),
\]

where \( x \) and \( t \) represent space and time, respectively, and \( M \) represents the various terms (typically in the form of spatial derivatives of the state \( u \)) of the governing equations. We assume the existence of two distinct physical
processes: one process is characterized by motions on a long spatial scale and slow temporal scale and the other process is characterized by motions on a short spatial scale and fast temporal scale. This assumption can be made explicit by defining an additional slow scale $T = \varepsilon t$ and long scale $X = \varepsilon x$ where $\varepsilon \ll 1$, and decomposing the state $u$ as

$$u = u_s(X, T) + \varepsilon u_f(x, t; X, T),$$

where $u_s$ represents the slow/long motions and $u_f$ represents the fast–short motions. We may perform a Taylor expansion of $M$ around the slow motions in (2.1) to obtain an equation for the slow motions:

$$\frac{\partial u_s}{\partial T} = M(u_s; X, T) + \frac{1}{\varepsilon} \frac{dM}{du} u_s + \varepsilon u_f,$$

where we have assumed that $M(u_s; X, T)$ is of $O(\varepsilon^2)$ because it is largely composed of terms proportional to spatial derivatives. Equation (2.3) shows that the lowest-order effect of the fast motions on the slow motions is to produce a small disturbance to the slow motions according to the slope of the model $M$ with respect to the state. While we will not develop an equation for the fast motions we will assume that $u_f$ has the following characteristics:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} u_f dT = w_T,$$

where $w_T$ is Gaussian white noise with mean zero and variance equal to $dT$ and the subscript is a reminder that this function $w_T$ is to be thought of as a function of the long time scale $T$.

Equation (2.4) simply states that as the difference between the spatial–temporal characteristics of the slow–long motions and the fast–short motions increases (i.e., $\varepsilon \to 0$), the fast–short motions appear as white noise on the long time scale $T$. Given this assumption we may write (2.3) in the limit as $\varepsilon \to 0$ as

$$du_s = M(u_s) dT + g(u_s) d\beta_T,$$

where, by convention, we have replaced $w_T$ with $d\beta_T$ because the increment of Brownian motion [i.e., $\Delta \beta_T = \beta_{T+\Delta T} - \beta_T$ where $(\Delta \beta_T)^2 = \Delta t$] is synonymous with white noise (Jazwinski 2007, 70–74) and $g(u_s) = \varepsilon^2 dM/du$. Note that (2.5) has been written in differential form to avoid writing the “noise” term in a form where it would be required to take the temporal derivative of the noise, which, of course, would lead to an undefined (infinite) variance. Equation (2.5) states that the slow–long motions $u_s$ evolve on the long time scale $T$ but are also modulated by the projection of the fast–short motions onto the slow–long motions through $\beta_T$. This requires interpreting the slow–long motions $u_s$ as a function of both the long time scale $T$ and what is effectively a second time scale $\beta_T$, that is $u_s = u_s(X, T, \beta_T)$.

We interpret (2.5) with respect to our present goal of understanding physics–dynamics coupling as follows: $M(u_s) dT$ represents the “dynamics” and is composed of terms such as what would be found in the NS equations as well as all of the collection of other processes that are resolved and that may also act upon the slow–long motions. We will assume that the meaning of slow motions here implies that they are entirely resolved by any spatial–temporal discretization that we might employ. The term $g(u_s) d\beta_T$ represents the “physics” and its goal is to replicate the dynamical processes that are missing from this dynamical core and are implied by the fast motions (e.g., unresolved and unrepresented fast motions).

The function $g(u_s)$ is interesting because it is potentially state dependent. The reason this is interesting is because the impulse provided by this noise term induces the state to jump from $u_s \to u_s + du_s$ in a time $dT$. Of course, over the much shorter time $d$ the action of the term $g(u_s) d\beta_T$ would not be impulsive (or discontinuous) in nature; it would be entirely resolved. In any event, when $g(u_s)$ is independent of the state the “size” of the jump is determined largely by the noise $d\beta_T$. However, when $g(u_s)$ is state dependent the size of the impulse quite obviously depends on the value of the state $u_s$ at some time during the interval $[T_0, T_0 + dT]$. A schematic illustration of how the state varies across the jump is provided in Fig. 1.

As discussed by van Kampen (1981), the specification of $when$ to evaluate the state $u_s$ within the interval may lead to significantly different state-dependent impulses $g(u_s) d\beta_T$ and therefore may lead to significantly different temporal evolutions of our simulation of the state $u_s$. This has important implications for the physics–dynamics coupling problem addressed in this manuscript because one of the central questions the modeler must answer in the development of their algorithm is at what point in time during each time step to couple the physics to the dynamics.

In the theory of SDEs this question of $when$ to evaluate the function $g(u_s)$ has been analyzed in great detail. Although one may technically evaluate the function $g(u_s)$ at any time within $[T_0, T_0 + dT]$, two distinct evaluation times have emerged as most important. On the one hand, we might evaluate the function $g(u_s)$ before (at time $T_0$) the impulse is applied. An SDE defined such that it evaluates the function $g(u_s)$ before the impulse is applied was first analyzed in great detail by Itô (1951) and will be
referred to here as an Itô SDE. An Itô SDE has the peculiar property that it does not in fact obey the traditional rules of calculus. For example, the Itô interpretation of the SDE (2.5) for which \( M(u_t) = u_t/2 \) and \( g(u_t) = u_t \) has for its solution \( u_t = Ae^{bT} \). Note that for an SDE to have this same solution and satisfy the traditional rules of calculus would require \( M(u_t) = 0 \) and \( g(u_t) = u_t \).

On the other hand, there does exist a time at which one can evaluate the function \( g(u_t) \) and receive an SDE that can be understood with the traditional rules of calculus. This happens only when one evaluates the function \( g(u_t) \) at precisely the center of the interval \( T_0 + dT/2 \). This type of SDE was first studied by Stratonovich (1966) and will be referred to here as a Stratonovich SDE. Interpreting the SDE (2.5) as Stratonovich [which means evaluating \( g(u_t) \) at \( T_0 + dT/2 \)] with \( M(u_t) = 0 \) and \( g(u_t) = u_t \) returns the solution \( u_t = Ae^{bT} \). Hence, two different SDEs, fundamentally differing in when the function \( g(u_t) \) (the physics) is evaluated across the interval, have identical solutions. This fact was deduced by Stratonovich (1966) and lead to his noting that by adding the term

\[
I_c = \frac{1}{2} g \frac{dg}{du} dT,
\]

(2.6)

which we will refer to as the Itô correction, to the right-hand side of (2.5) an Itô SDE will then obtain the same answer as the Stratonovich SDE, even though the function \( g(u_t) \) is still evaluated at the beginning of the interval. We will show in the next section that the Itô correction can be a very useful tool to answer questions about the impact of decentering on the stability and accuracy of different numerical methods in the presence of noise. This is because (2.6) may be extended to an arbitrary decentering parameter \( \delta \in [0, 1] \) of

\[
d_c = \left( \frac{1}{2} - \delta \right) g \frac{dg}{du} dT.
\]

(2.7)

Therefore, by adding (2.7) to the right-hand side of an SDE where the function \( g(u_t) \) has been evaluated at \( T_0 + dT \) one may obtain the same answer as that of the Stratonovich SDE (Kloeden and Platen 1999, 99–102).

Because a given SDE with a state-dependent \( g(u_t) \) can only be either Itô or Stratonovich it is important to know which it is so one knows whether one can or even should apply the Itô correction. From the work of Papanicolaou and Kohler (1974) we know that the SDE in (2.5), in which we start by assuming the traditional rules of calculus apply and then work toward the white noise limit, is to be interpreted in the sense of Stratonovich. As a general rule (see van Kampen 1981; Jazwinski 2007), if the physical process being studied is fundamentally discrete, and the limit of small time is taken to obtain an SDE, that SDE will be Itô. On the other hand, if the process is fundamentally continuous, and therefore satisfies the traditional rules of calculus, and then the white-noise limit is taken, the resulting SDE will be Stratonovich. Consequently, we hypothesize that the appropriate starting point for the study of physics–dynamics coupling in NWP is that of Stratonovich because of our

![FIG. 1. How the state varies across the impulse. Each solid dot depicts the state as it evolves in time on a grid of spacing \( \Delta T \). As the impulse is applied across the interval \( T_0 \) to \( T_0 + \Delta T \) the state jumps across that interval and the value of the state within this interval takes distinctly different values. Note that this is unlike the continuous case in that in the limit as \( \Delta T \) goes to zero there is still a jump and therefore there is still a difference in the state across the jump.](image-url)
strong foundation within the continuum hypothesis of the NS equations (Batchelor 1977, 4–5). However, we admit that some physical processes in meteorology may in fact be fundamentally discrete (e.g., raindrops, clouds, etc.) and therefore this needs further study. Given this assumption we proceed in the next section to illustrate why some numerical techniques from the literature of numerical methods produce solutions consistent with the Stratonovich calculus and others do not.

3. Physics–dynamics coupling in a simplified framework

This section is broken into two parts. In the first part we explain in detail why a numerical method converges to one calculus or the other. In this way the reader may see how the issues of section 2 may play a role in model development. In the second part of this section we back up the analytical results of the first section with explicit examples illustrating the ways different numerical methods may fail to give the expected answer.

a. Numerical analysis

In traditional numerical analysis the Taylor expansion has been used to great effect in the study and development of the techniques to numerically integrate differential equations (Duran 1999). In the theory of SDEs an extension of the Taylor expansion, often referred to as the stochastic Taylor expansion, is used to similarly understand the properties of SDEs (Jazwinski 2007, chapter 4; Kloeden and Platen 1999, chapter 5). We choose here, however, to present our analysis using the language of the traditional Taylor expansion in order to clearly draw parallels to previous work in numerical analysis and NWP modeling, although we will at times bring the stochastic components in as needed. By beginning with the traditional Taylor expansion we are implicitly making the assumption that the traditional rules of calculus apply and therefore that the appropriate calculus is that of Stratonovich.

To this end and for the remainder of this section we will imagine we are solving a differential equation of the following form:

$$\frac{\partial u}{\partial t} = M(u) = D(u) + P(u),$$

(3.1)

where $D$ represents the dynamical core and $P$ represents the noise. Note that for ease of presentation we will hereafter drop the distinction between the long time $T$ and the short time $t$; the reader should simply remember that we are always solving our model equations on the resolved temporal and spatial scales. Lastly, we emphasize that the schemes presented below will be implemented as in traditional numerical analysis, like what might be done in an NWP model, and not as their stochastic counterparts. We do this because the methods of stochastic numerical analysis often require explicit knowledge of the function $g(u)$ and this function is unknown for the typical physics parameterizations used in NWP.

1) FORWARD EULER

The forward Euler scheme is interesting because it is the canonical example of a numerical method that converges to the Itô calculus. We construct a temporal discretization of $N$ points with spacing $\Delta t$ to solve (3.1). A Taylor expansion to second order would be

$$u_{t_i + \Delta t} = u_{t_i} + \left. \frac{du}{dt} \right|_{t_i} \Delta t + \frac{d^2 u}{dt^2} \frac{\Delta t^2}{2},$$

(3.2)

which, after use of (3.1), gives

$$u_{t_i + \Delta t} = u_{t_i} + M(u_{t_i}) \Delta t \tag{Forward Euler}$$

$$+ \left( \frac{\partial D}{\partial u} \bigg|_{t_i} + \frac{\partial P}{\partial u} \bigg|_{t_i} \right) [D(u_{t_i}) + P(u_{t_i})] \frac{\Delta t^2}{2}. \quad (3.3)$$

The first two terms on the right-hand side of (3.3) constitute the traditional forward Euler scheme. Note that both the dynamics and the physics are evaluated at the beginning of each time interval $t_i$, which as discussed in section 2 leads to the Itô calculus. This fact can be understood if we first assume the physics can be modeled as

$$P(u_{t_i}) \Delta t \rightarrow g(u_{t_i}) \Delta \beta_{t_i},$$

(3.4)

where $\Delta \beta_{t_i} = \beta_{t_i + \Delta t} - \beta_{t_i} = w_{t_i}$ is the increment of Brownian motion consisting of Gaussian white noise with mean zero and variance equal to $\Delta t$. We focus here on one of the terms in (3.3) that is ostensibly of second order but is actually one order larger [the remaining terms in (3.3) may be shown to be smaller]:

$$\frac{\partial P}{\partial u} \bigg|_{t_i} \frac{\Delta t^2}{2} \rightarrow \frac{\partial g}{\partial u} \frac{(\Delta \beta_{t_i})^2}{2}. \quad (3.5)$$

Note that because the variance of the noise is $O(\Delta t)$ [i.e., $(\Delta \beta_{t_i})^2 = \Delta t$], the term identified in (3.5) is actually first order rather than second order in $\Delta t$ when the assumption in (3.4) is satisfied. Physically, one can understand this by recalling that this term arose from the second derivative of the state function with respect to time and that a temporally noisy field may have a second
derivative that is as large as or larger than its first derivative. In addition, a comparison of (3.5) with the Itô correction of (2.6) reveals that (3.5) is simply the Itô correction. The fact that this term is as large as the retained terms in the forward Euler scheme but nevertheless still neglected by this scheme explains why this scheme does not converge to the solutions of traditional calculus when the physics is noisy. Indeed, we show next that when we appropriately attempt to evaluate the physics at the center of the window we automatically include the Itô correction, which leads to a numerical scheme that converges to the calculus of Stratonovich.

2) SECOND-ORDER RUNGE–KUTTA

The second-order Runge–Kutta\(^1\) scheme can be written as
\[
\begin{align*}
    u_{t_0 + \Delta t}^e &= u_{t_0} + M(u_{t_0}) \Delta t, \\
    u_{t_0 + \Delta t} &= u_{t_0} + \frac{1}{2} [M(u_{t_0}^e) + M(u_{t_0})] \Delta t, 
\end{align*}
\] (3.6a, 3.6b)

which can be thought of as a predictor–corrector scheme that attempts to linearly interpolate the tendencies to the center of each time window. Note that we may expand the dynamics and physics tendencies to first order as
\[
D(u_{t_0}^e + \Delta t) = D(u_{t_0}) + \frac{dD}{dt}|_{t_0} (u_{t_0}^e + \Delta t - u_{t_0}),
\]
\[
P(u_{t_0}^e + \Delta t) = P(u_{t_0}) + \frac{dP}{dt}|_{t_0} (u_{t_0}^e + \Delta t - u_{t_0}).
\] (3.7, 3.8)

Using (3.7) and (3.8) in (3.6b) obtains (3.3), which shows that the second-order Runge–Kutta scheme can be thought of as the forward Euler scheme with the Itô correction already included. We emphasize that the inclusion of the Itô correction was not due to the fact that the scheme was second order, but rather because the scheme achieves second-order accuracy by explicitly creating all of the terms in the Taylor expansion (3.3). We explicitly show that this is the case in the next example by examining another second-order scheme that attempts to achieve second-order accuracy by canceling (rather than creating) terms in the Taylor expansion, but does not in fact converge to the Stratonovich calculus.

3) SECOND-ORDER ADAMS–BASHFORTH

The second-order Adams–Bashforth scheme is written as
\[
u_{t_0 + \Delta t} = u_{t_0} + \frac{1}{2} \left[ 3 M(u_{t_0}^e) - M(u_{t_0 - \Delta t}) \right] \Delta t, \tag{3.9}\]

where one should note that the terms in brackets are an extrapolation to the center of each time window. One can understand this extrapolation by constructing two Taylor series approximations to second order,
\[
M_1(u_{t_0 + \Delta t/2}) = M(u_{t_0}) + \frac{dM}{dt}|_{t_0} \Delta t + \frac{d^2 M}{dt^2}|_{t_0} \frac{\Delta t^2}{2},
\]
\[
M_2(u_{t_0 + \Delta t}) = M(u_{t_0 - \Delta t}) + \frac{dM}{dt}|_{t_0} \frac{3 \Delta t}{2} + \frac{d^2 M}{dt^2}|_{t_0 - \Delta t} \frac{9 \Delta t^2}{8},
\] (3.10, 3.11)

and then appropriately weighting and subtracting obtains
\[
M(u_{t_0 + \Delta t/2}) = \frac{3}{2} M_1 - \frac{1}{2} M_2 = \frac{3}{2} M(u_{t_0}) - \frac{1}{2} M(u_{t_0 - \Delta t})
\]

Extrapolation

\[
+ \left( \frac{dM}{dt}|_{t_0} - \frac{dM}{dt}|_{t_0 - \Delta t} \right) \frac{3 \Delta t}{4} + O(\Delta t^2).
\] (3.12)

Note that the next-order term after the two extrapolation terms in (3.12) appears first order in \(\Delta t\). However, when the model physics are slowly varying in time the difference between the two tendency terms in parentheses is of order \(\Delta t\), meaning that the actual error in (3.12) is of second order. In contrast, when the model physics are not slowly varying in time and can be modeled by (3.4), then the difference between the two tendency terms in parentheses is
\[
\left( \frac{dM}{dt}|_{t_0} - \frac{dM}{dt}|_{t_0 - \Delta t} \right) \frac{\Delta t^2}{2} - \frac{\partial g}{\partial u} \left( \Delta \beta_{t_0} \right)^2 \frac{\Delta t}{2} + \ldots \ldots.
\] (3.13)

Note that the terms on the right-hand side of (3.13) are clearly the Itô correction of (3.5) evaluated at two different time levels. Because the noise in one time interval \((\Delta \beta_{t_0})\) is independent of the noise in the previous interval \((\Delta \beta_{t_0 - \Delta t})\) there can be no expectation that these

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\(^1\) Throughout we will refer to the collection of single-step, multistage schemes that explicitly create the terms of the corresponding Taylor expansion as “Runge–Kutta” methods.
terms will cancel. Hence, (3.12) appears to imply that the scheme in (3.9) is evaluating the physics at the center of the window and therefore should be Stratonovich. However, because these terms in (3.13) do not cancel when the physics is noisy leads to solution behavior more like the calculus of Itô. Consequently, a scheme that is constructed by assuming the unconditional cancelation of these terms in (3.13) to obtain higher-order accuracy will not be Stratonovich. Another example of this is presented next.

4) LEAPFROG

The leapfrog scheme will be the only three-time-level method we examine. Nevertheless, it is an important scheme because of its extensive use in NWP modeling. The scheme can be understood by performing two Taylor expansions:

\[ u_{t_0 + \Delta t} = u_{t_0} + M(u_{t_0}) \Delta t + \frac{dM}{dt} \left|_{t_0} \right. \Delta t^2, \]

(3.14a)

\[ u_{t_0 - \Delta t} = u_{t_0} - M(u_{t_0}) \Delta t + \frac{dM}{dt} \left|_{t_0} \right. \Delta t^2, \]

(3.14b)

and subsequently subtracting to obtain

\[ u_{t_0 + \Delta t} = u_{t_0 - \Delta t} + 2M(u_{t_0}) \Delta t + \left( \frac{dM}{dt} \left|_{t_0} \right. - \frac{dM}{dt} \left|_{t_0} \right. \right) \Delta t^2. \]

(3.15)

The notation on the model time-derivative terms denotes the time the derivative is valid (scripted) and the arrow represents the side at which the limit was taken to obtain the derivative; that is, a right arrow \((\rightarrow)\) denotes that the limit was taken from \(t_0 + \Delta t\) to \(t_0\) and the left arrow denotes that the limit was taken from \(t_0\) to \(t_0 - \Delta t\). Note that when the model physics are smooth (and hence continuous) this difference in parentheses in (3.15) must vanish (i.e., the two limits on either side must converge). However, when the model physics is noisy and satisfies (3.4) this difference will be identical to (3.13) and the size of this term in (3.15) will again be \(O(\Delta t)\), which is of course the same size as the terms retained by the leapfrog scheme. By design the leapfrog scheme achieves second-order accuracy by assuming the cancelation of these terms and therefore it clearly cannot converge to the calculus of Stratonovich when the physics is sufficiently noisy. In the numerical experiments to be presented below the leapfrog scheme has been implemented using Asselin time filtering (Durran 1999), which reduces its accuracy to first order but, as will be shown, does not change the fact that it does not converge to the calculus of Stratonovich.

b. Example simulations

The focus here will be on a simplified framework in the spirit of Staniforth et al. (2002a, b) and Egger (2003) but incorporating a general stochastic representation of the forcing terms that we will interpret as broadly representing physics parameterizations. A key difference between this previous work and what we will present will be our focus on state-dependent forcing terms and the key differences that arise when the physics is noisy. The numerical methods examined will be the four methods discussed above.

The basic model problem will be advection–diffusion of some field, \(u = u(x, t)\), on the circle, \(x \in [0, 2\pi]\), and integrated for two units of time with a state-dependent forcing term, namely,

\[ \frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + g(u) n(t), \]

(3.16)

where \(c = 1\) and \(\mu = 0.1\) are the advection and diffusion parameters, respectively, and \(g(u) n(t)\) is intended to broadly represent the physics routines. The function \(n(t)\) is described in appendix A and produces random noise with a given spectral character that ranges from red noise to white noise. The initial condition for all simulations below will be

\[ u(x, 0) = \cos(k_0 x), \]

(3.17)

where \(k_0 = 1\). Because we will focus on only a single, well-resolved spatial wavenumber here we will not be examining the case where noise arises because of a lack of spatial resolution; all of the key issues below will arise entirely from a lack of temporal resolution.

1) STATE-INDEPENDENT NOISE

To begin we wish to present results for state-independent noise in (3.16). The goal here is to make a connection to previous work with state-independent forcing functions (e.g., Staniforth et al. 2002a, b; Egger 2003) and therefore to show the behavior when the Itô correction vanishes. The analytical solution in this case is presented in appendix B.

Simulations for four different schemes are presented in Fig. 2 for \(g(u)\) equal to (B.4) with \(\rho = 0.2\) and six different types of noise (from very red to white). Each curve in these figures is the root-mean-square (RMS) error averaged over 100 trials with different random noise in the noise term. The key feature of Fig. 2 is that all schemes converge to the true solution as \(\Delta t \to 0\), although the rate of convergence generally differs for differing
qualities of noise. For large $\alpha$ (very red spectrum) each scheme converges at a rate proportional to the rate predicted by traditional numerical analysis. For $\alpha = 0$ (white spectrum) the forward Euler and second-order Runge–Kutta schemes converge toward the true solution at $\Delta t$. Discussion of this convergence property for the Euler scheme can be found in Kloeden and Platen (1999, section 10.2) and for the second-order Runge–Kutta scheme in Buckwar and Winkler (2006) and Buckwar et al. (2010). In contrast, the second-order Adams–Bashforth and leapfrog schemes both converge at the much slower rate of $\Delta t^{1/2}$ (Buckwar and Winkler 2006).

When the character of the noise is neither completely red nor white, but somewhere in between, all numerical methods converge, for sufficiently small $\Delta t$, at a rate faster than that determined by stochastic numerical analysis. In
general, this rate of convergence for sufficiently small $\Delta t$ is
that predicted by deterministic numerical analysis though we note that this was not the case for the second-order Adams–Bashforth scheme for all values of $a$. The "critical
time step" ($\Delta t_c$) derived in appendix C denotes the largest
time step required for a given temporal discretization to
resolve noise of a specific spectral character. Given the
noise source in appendix A, we find that when the time
step in our simulation is larger than $\Delta t_c$ the rate of con-
vergence of all methods is that of the stochastic theory.
Hence, when the temporal discretization is coarse relative
to the noise the numerical method behaves as if the model
problem is that of an SDE even when the noise is not
white. However, when the temporal discretization is fine
relative to the noise these numerical methods converge at
a rate faster than the stochastic theory.

2) STATE-DEPENDENT NOISE

Simulations for our four different schemes are pre-
sented in Fig. 3 for $g(u) = \rho \partial u / \partial x$ with $\rho = 0.2$. The
analytic solution to which all methods should converge
is presented in appendix B. Again, for large $a$ (very red
spectrum) all methods converge at the rate predicted
by traditional numerical analysis. However, in contrast
to the state-independent noise simulations, for $a = 0$
(white spectrum) only the second-order Runge–Kutta
scheme converges toward the true solution. As discussed
in section 3a, the other three schemes converge to the
calculus of Itô and therefore those three schemes produce
a solution that differs from the true solution by the Itô
correction, which in this case is

$$I_c = \frac{1}{2} \rho \partial^2 u / \partial x^2 \, dt.$$ (3.18)

The Itô correction in this case is a diffusion and it implies
that the numerical methods that converge to the cal-
culus of Itô grow the solution too much relative to the
Stratonovich solution. This prediction was verified by
rerunning the experiments with $\alpha = 0$ but this time with
the addition of the Itô correction applied to the numerical solver (not shown) and it was found that all methods then converged to the true solution at the rate $\Delta t^{1/2}$. While the fact that these Itô schemes are associated with too much growth is consistent with the notion that an “explicit” scheme (evaluated at the beginning of a time step) may lead to an instability, we will by contrast show in the next section that this is in fact a result of our particular choice for the function $g(u)$. In fact, there exists a wide class of functions $g(u)$ in which the Itô correction predicts damping and therefore an implicit scheme in these cases must lead to an instability. Finally, similar to the state-independent noise simulations, when the spectrum is between that of being very red or white, the character of the convergence of the numerical method is again determined by the critical time step. This can be seen in Fig. 3 where again to the right of the critical time step all numerical methods behave as predicted by the theory of SDEs and to the left these same methods behave according to that predicted from traditional numerical analysis.

3) DECENTERING SCHEMES

The last numerical experiment of this section involves the examination of a common variation on the second-order Runge–Kutta scheme in which “decentering” is employed. In this case we need only modify (3.6b) of the second-order Runge–Kutta scheme to

$$u_{t_0+\Delta t} = u_{t_0} + [\delta M(u_{t_0+\Delta t}) + (1-\delta)M(u_{t_0})] \Delta t, \quad (3.19)$$

where $\delta \in [0,1]$ is the decentering parameter such that $\delta = 0$ approximates the tendencies at the beginning of the window, $\delta = 1$ approximates the tendencies at the end of the window, and $\delta = 1/2$ approximates the tendencies at the center of the window. Typically the decentering parameter is $\delta > 1/2$ because this leads, in traditional numerical analysis, to damping. By contrast, we will present the counterintuitive result that when the physics can be approximated as state-dependent noise we find that there exists some state dependencies that lead to growth for $\delta > 1/2$.

We provide simulations with $\rho = 0.2$ and $\alpha = 0$ (white spectrum) for two kinds of state-dependent noise functions. One of the functions is $g(u) = \rho \partial u/\partial x$ and is used in section 3b above with further discussion in appendix B. The other function is $g(u) = \rho u$ and is discussed in appendix B. Experiments (not shown) of the same sort as in sections 3b(1) and 3b(2) above reveal that as expected $\delta = 1/2$ is required for the solutions from (3.19) to converge to the correct answer because that is the only parameter setting for which the numerics leads to the calculus of Stratonovich. Because we have already discussed the convergence properties of the second-order Runge–Kutta scheme our goal here is not to focus on whether or not the solutions converge to the true solution but rather to ask this question: When (3.19) is decentered (i.e., $\delta \neq 1/2$) and the solutions do not converge to the correct answer, are the solutions unstable or damped relative to the true solution? To answer this question we define a new norm to measure the error in the amplitude of the wave as simulated from (3.19), namely,

$$A_{\text{err}} = (|F|^2 - |F^t|^2), \quad (3.20)$$

where the norm defined by the vertical bars is the complex magnitude, $F^t$ is the final time solution in Fourier space for wavenumber $k_0$, $F^t$ is the final time true solution in Fourier space from appendix B, and the angle brackets denote an average over 100 trials consisting of different random noise in the noise term.

The result of the numerical simulations at a variety of time steps employed in (3.19) is reported in Fig. 4. To begin we note that a stability analysis of the second-order Runge–Kutta scheme with $\delta = 1/2$ for the advection equation shows that it is absolutely unstable with a growth term that is proportional to the time step raised to the fourth power (Durran 1999, 53–56). This instability leads to the significant growth with increasing time step seen on the right-hand side of Fig. 4. This effect happens regardless of the properties of the decentering or the stochastic term. Our interest here, however, lies in the behavior for decreasing time step. For small time step and $g(u) = \rho \partial u/\partial x$, decentering $\delta < 1/2$ leads to too much growth while decentering toward $\delta > 1/2$ leads to damping with respect to the true solution. This result is consistent with traditional numerical analysis and the prediction from the theory of SDEs as the decentering correction (2.7) does predict this result. However, when $g(u) = \rho u$ decentering $\delta < 1/2$ leads to too much damping while decentering toward $\delta > 1/2$ leads to too much growth with respect to the true solution. For this function $g(u)$ the decentering correction (2.7) is

$$d_c = \left(\frac{1}{2} - \delta\right) \rho^2 u dt, \quad (3.21)$$

which predicts that for $\delta = 0$ the Itô solution will be damped with respect to the true solution and for the “implicit” solution with $\delta = 1$ (3.21) predicts too much growth. Additionally, we have performed experiments (not shown) with higher-order derivatives for the function $g(u)$. This behavior in Fig. 4a continues for $g(u)$ defined from any odd-order derivatives (advective and dispersive processes) and for $g(u)$ defined from any even-order derivatives (diffusive processes) the behavior is
4. An application to stochastic subgrid-scale mixing in a shear flow

In this section we will construct a stochastic subgrid-scale mixing parameterization for the purpose of illustrating the result we found in the previous section that when applying a scheme based on decentering to a stochastic diffusive process an implicit scheme may lead to instability while an explicit scheme may actually result in damping. A second goal of this section is to present an example where the stochastic parameterization is not explicitly created in the form $g(u)\Delta\theta_\delta$, but nevertheless we are still able to predict its behavior through the proper application of the decentering correction.

We will focus our attention to a nonrotating, stably stratified, nonhydrostatic Boussinesq fluid bounded above and below by rigid, horizontal boundaries, but periodic in the horizontal. The governing equations along an $x$–$z$ cross section may be combined to two equations in two unknowns:

\[
\frac{\partial \zeta}{\partial t} = - \left( \frac{\partial u}{\partial x} + w \frac{\partial \zeta}{\partial z} \right) + F, \quad (4.1)
\]

\[
\frac{\partial \theta}{\partial t} = - \left( \frac{\partial u}{\partial x} + w \frac{\partial \theta}{\partial z} \right) + H, \quad (4.2)
\]

along with the diagnostic relations

\[
u = \frac{\partial \psi}{\partial z}, \quad (4.3a)
\]

\[
w = - \frac{\partial \psi}{\partial x}, \quad (4.3b)
\]

\[
\zeta = \nabla^2 \psi, \quad (4.4)
\]

where $\nabla^2$ is the Laplacian operator in the $x$–$z$ plane and $u$, $w$, and $\zeta$, are the zonal wind, vertical wind, and vorticity, respectively. Note that with the definition of the streamfunction in (4.3a,b) the positive vorticity is consistent with clockwise rotation in the $x$–$z$ plane. The buoyancy frequency of the reference state $\theta_\delta$ is $N_0^2 = (g/\theta_\delta) \frac{du_0}{dz} = 10^{-4} \text{s}^{-1}$, which defines the background potential temperature field. The reference state for the zonal wind consists of a vertically sheared wind profile, namely

\[
U_0 = \frac{V}{2} \left[ 1 + \tanh \left( \frac{z - \zeta_0}{L} \right) \right], \quad (4.5)
\]

with $V = 10 \text{m s}^{-1}$, $\mu = 8$, $L = 1 \text{km}$, and $\zeta_0 = 0.5 \text{km}$. This reference shear flow has a Richardson number less than $\frac{1}{4}$ and is therefore unstable, which gives rise to amplifying wave trains that eventually wrap up and break nonlinearly.

Equations (4.1)–(4.4) will be solved subject to the boundary conditions that the vertical velocity $w$ vanishes along the upper and lower rigid boundaries located at $z = 0, 1 \text{km}$. In the $x$ direction, the vorticity and the
potential temperature $[\zeta, \theta]$ will be assumed to be periodic with a channel length of $4\pi$ km. The vorticity source $F$ and heat source $H$ are the physics consisting of a subgrid-scale mixing parameterization that uses a traditional first-order closure that depends only on the strength of stratification and shear (please see appendix D).

The dynamical core in (4.1) and (4.2) will be solved using a pseudospectral technique with the third-order Adams–Bashforth scheme in time and Fourier in both the $x$ and $z$ directions. The boundary conditions at $z = 0$, 1 km are enforced through a mirror image of the state occupying an imaginary region above the model domain that both effectively forces the vertical velocity to vanish on the boundaries and ensures the vertical direction is exactly periodic such that a Fourier expansion is sensible. We use the nonlinear transform method of Orszag (1970) to deal with the nonlinear instability arising from aliasing due to computing the nonlinear terms in (4.1) and (4.2).

Our use here of the third-order Adams–Bashforth scheme for the dynamics but the centered version of the second-order Runge–Kutta scheme for the physics was done consciously to show that mixing two different schemes between that for the dynamics and that for the physics does not change the essential behaviors found in section 3, which used the same scheme for both the dynamics and the physics. We have, however, also performed this experiment with a multistage method (RK2) rather than a multistep method for the dynamics and did not find a difference in the solution behaviors to be described below. This notion that we may integrate the dynamics by a method different from the stochastic terms is well known in the theory of SDEs and arises because the integral over the dynamics is a Riemann integral. This is because for Riemann integrals (but not stochastic integrals) as the time step of any partition to evaluate a Riemann sum approaches zero, this sum converges independent of where within the time step one evaluates the integrand. [Please see chapter 3 of Kloeden and Platen (1999) for the stochastic counterpoint to this argument.]

In the nonstochastic simulations the subgrid-scale mixing will be modeled through the application of the subgrid-scale mixing parameterization of Durran and Klemp (1983) applied using a decentering parameter of $\delta = 1/2$. Experiments with different decentering parameters in these nonstochastic simulations did not lead to different spectral slopes in their vorticity spectrums. The standard model grid has 128 points in the $x$ direction and 33 points (not counting the mirror region) in the $z$ direction. A high-resolution simulation will also be presented with 256 points in the $x$ direction and 65 points (not counting the mirror region) in the $z$ direction.

Each simulation is begun from an initial condition composed of the sum of the background state and a random field of sinusoidal waves whose spectrum in vorticity varies as $k^{-1}$. We do this because we have found empirically from long time integrations that the equilibrium slope of the energy $|E|$ varies as $k^{-3}$ in this model configuration and therefore, because the enstrophy $|H|$ is related to the energy through $|H| = k^2|E|$, this predicts that the slope in vorticity varies as $k^{-1}$. We emphasize that simulations begun from small-amplitude white noise would reach the same spectral character if integrated for significantly longer than the simulations presented here as the developing instabilities on this shear flow would carry the solution to finite amplitude with the appropriate spectral character. In any event, this random initial state is sampled 1024 times to produce an ensemble of states that are then integrated for 30 min. In Fig. 5 is shown the ensemble mean vorticity spectrum as a function of wavenumber at the final time averaged over the 1024 simulations. One can see that the high-resolution spectrum is more consistent with the equilibrium slope of $k^{-1}$ than the standard resolution simulation. In fact, the standard resolution simulation appears to be missing energy at the smallest wavelengths of the numerical simulation. This is a typical way by which a low-resolution simulation can misrepresent the small-scale nonlinear interactions that are not included in the model simulation (Shutts 2006; Berner et al. 2009).

We then ran additional standard resolution simulations using the stochastically modified scheme of appendix D with three different decentering parameters: $\delta = 0$, $1/2$, 1. Each version of the model with different decentering parameters uses the same 1024 initial conditions to produce an ensemble of simulations at the final time whose spectrum is plotted in Fig. 5. Figure 5 shows that the implicit scheme with $\delta = 1$ produces a slope with more energy at the smallest wavelengths as well as an incorrect slope as compared to the centered scheme $\delta = 1/2$, which appears to have a slope that is most consistent with the theoretical result and the high-resolution simulation. This incorrect slope from the implicit scheme can be seen by extending its curve across the figure such that it becomes obvious that it crosses the curve for the high-resolution simulation, which clearly implies a difference in slope. This high-wavenumber instability of the implicit scheme leads to an increased graininess of its simulations, which in turn leads to what appears to be a nonlinear computational instability (Phillips 1959; Lauritzen 2011) and the eventual blowing up of the simulation in approximately 33% of the 1024 simulations. By contrast, none of the simulations with $\delta = 1/2$ lead to the eventual blowup of the simulations. In addition, note that the explicit scheme $\delta = 0$ is damped (weakly) for the smallest
scales and therefore again implies a different slope than that produced by the centered scheme $d = 1/2$.

All of these results may be predicted by the decentering correction (2.7) as applied to a physics parameterization that can be modeled as a stochastic diffusive process, that is,

$$P(u_i)\Delta t \to \rho \frac{\partial^2 u}{\partial x^2} \Delta \beta_i.$$  \hspace{1cm} (4.6)

A physics parameterization that can be modeled as in (4.6) implies a decentering correction of

$$d_e = \left(\frac{1}{2} - \delta\right) \rho \frac{\partial^4 u}{\partial x^4} dt.$$  \hspace{1cm} (4.7)

Equation (4.7) states that the decentering correction will have a $k^4$ wavenumber dependence such that explicit schemes with $\delta = 0$ will be excessively damped with the greatest damping occurring at the smallest scales and implicit schemes with $\delta = 1$ will be excessively amplified with the greatest amplification occurring at the smallest scales.

5. Summary and conclusions

The impact of noisy physics as represented by stochastic forcing terms was studied here in the context of the numerical solution of stochastic partial differential equations (SDEs). In particular, the Itô correction and its generalization to the decentering correction of (2.7) were argued to provide a simple and concise description of the impact of noise on the physics–dynamics coupling problem.

It was shown that the ability of a particular numerical method to handle general qualities of noise stems from the manner by which it was created. Methods created by attempting to develop each term in its corresponding Taylor expansion generally appear to be able to handle noise better than those methods that are derived through the assumed cancellation of particular terms. In all cases, however, it was shown that when the temporal discretization is fine enough to resolve the noise these schemes generally behave as in deterministic numerical analysis. It is only when the temporal discretization does not resolve the noise that the scheme behaves as predicted by the stochastic theory. This point is important because it means that the noise present in an NWP model does not need to be of the type described in appendix A for the above results to hold. In fact, this implies that it is only necessary that general variability of whatever form in an NWP model is simply unresolved by the temporal discretization for the above results to hold.

The second-order Runge–Kutta scheme was shown to be the simplest scheme that was able to handle all qualities of noise in a systematic way. The major benefit provided by the second-order Runge–Kutta scheme is seen in the presence of state-dependent noise. In this case, this scheme was the only scheme presented that converged to the true solution for all qualities of noise. Additional experiments (not shown) reveal that the result
that the second-order Runge–Kutta scheme converges for all qualities of noise also translates to the third- and fourth-order Runge–Kutta schemes (Rümelin 1982).

We have shown that schemes based on the cancelation of terms to obtain high-order accuracy may lead to a general sensitivity to noise. We presented results for a second-order Adams–Bashforth scheme as well as the leapfrog scheme using Asselin time filtering to damp its computational mode. These simulations showed that in the presence of state-independent noise these schemes failed in fact to converge at all. Additional simulations using the stochastic advection–diffusion equation of section 3 (not shown) with a third-order Adams–Bashforth scheme revealed a similar lack of convergence to the Stratonovich result.

The common practice of centering the physics across the time step was shown to lead to sensitivity to noise for certain types of state-dependent noises. When the noise has the character of stochastic advective and dispersive processes the standard result that explicit schemes ($\delta < 1/2$) are unstable and implicit schemes ($\delta > 1/2$) are damped emerges. However, when the noise has the character of a stochastic diffusive process one finds the opposite result, in which the explicit schemes are damped and the implicit schemes are unstable. In all cases, however, the centering correction (2.7) properly predicts the behavior of centering in this model setting.

The ideas presented here were applied to a common form of physics parameterization that aims at introducing small-scale energy that goes missing because of the spatial discretization eliminating small-scale nonlinear interactions. This form of physics parameterization was implemented in a nonlinear model of shear flow instabilities in which the damping owing to subgrid-scale mixing was being performed with the parameterization of Durran and Klemp (1983). It was shown here that the equilibrium slope of the vorticity spectrum was most easily attainable at low resolution by using a scheme that converged to the calculus of Stratonovich and that an implicit scheme was actually unstable. It was argued that this occurred because the physics parameterization behaved as a stochastic diffusive process and this type of process has a centering correction that leads to a fourth-order diffusive term that, for implicit centering, leads to instability.

Presently, we are examining which and to what extent common physics parameterizations behave according to stochastic advective, diffusive, or dispersive processes. Additionally, we are examining to what extent common stochastic physics methodologies may benefit from their implementation within a predictor–corrector-type scheme.

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APPENDIX A

Constructing Random Noise

Given a temporal discretization of $N$ points (assume $N$ is odd), with spacing $\Delta t$, we use a sinusoidal basis of $N$ functions to describe noise of either a “smooth” (red) or a “noisy” (white) character on that grid:

$$n(t) = C(\omega_0) \frac{b_0}{\sqrt{2}} + \sum_{i=1}^{N_f} C(\omega_i) [a_i \sin(\omega_j t) + b_i \cos(\omega_j t)],$$

(A.1)

where the number of sinusoidal functions varies as $N_f = (N - 1)/2$ and $\omega_i = 2\pi i/(N\Delta t)$ with $i = 0, 1, 2, \ldots, N_f$. The coefficients $a_i$ and $b_i$ are drawn from normal distributions and give the noise its random character: $a_i, b_i \sim N(0, 1)$. The function $C$ carries the information describing the character of the noise as either red or white. We define the function $C$ as

$$C(\omega) = \frac{1}{\sqrt{\Delta t N_f}} \exp(-\alpha \omega^2),$$

(A.2)

where $\alpha$ is a tunable parameter for which $\alpha = 0$ produces white noise in (A.1) and $\alpha > 0$ produces red noise in (A.1) with the noise becoming redder as $\alpha$ increases. The square root of $\Delta t$ in the denominator of (A.2) is there to ensure that the variance of Brownian motion defined from this noise with $\alpha = 0$, that is,

$$\beta_i = \int_0^t n(t') \, dt',$$

(A.3)

is a linearly increasing function of time. Throughout the numerical experiments described in this manuscript the integral (A.3) is solved analytically term by term over (A.1).

APPENDIX B

Solutions to a Class of SDEs

The numerical experiments of section 4 rely on knowing the true solution the numerical techniques should be converging toward. Our ability to define the following solutions hinges on the linearity of the function $g(u)$ with
respect to the state and the use of Fourier transforms. Similarly, in the white noise limit, we will be interpreting (3.16) as a Stratonovich SDE.

a. State-dependent noise

By assuming a single Fourier mode in space as

\[ u = F(k, t) \exp(ikx), \]  
(B.1)

we may write (3.16) as

\[ \frac{\partial F}{\partial t} + ikcF = -\mu k^2 F + \sigma n(t)F, \]  
(B.2)

where the linearity of \( g(u) \) has been used explicitly to replace that function with its Fourier transform, which would be simply \( F \) times a possibly complex constant \( \sigma \).

The solution to (B.2) for the single Fourier mode in (B.1) is simply

\[ F = A \exp \left[ -(ikc + \mu k^2)t + \sigma \int_0^t n(t') \, dt' \right], \]  
(B.3)

where \( A \) is a complex constant. Note that the integral in (B.3) goes to Brownian motion in the limit of \( \alpha \) approaching zero. Physically, this implies that in the limit as \( \alpha \) approaches zero the wave defined by (B.1) undergoes Brownian motion in either in its amplitude or its phase. Whether it is one or the other is determined by the state-dependent forcing function \( g(u) \).

When the function \( g(u) = \rho u \), where \( \rho \) is a constant, then \( \sigma = \rho \) and the noise induces the amplitude of the wave to undergo Brownian motion. In contrast, when the function \( g(u) = \rho \partial u / \partial x \), then \( \sigma = ipk \), which implies the wave now undergoes Brownian motion in its phase only; there exists no amplitude variation in this case. In fact, one can show that all linear functions \( g(u) \) that are proportional to even spatial derivatives (diffusive processes) induce the wave to undergo Brownian motion in its amplitude. In contrast, all linear functions \( g(u) \) that are proportional to odd spatial derivatives (advective and dispersive processes) induce the wave to undergo Brownian motion in its phase.

b. State-independent noise

For the ostensibly simpler case of state-independent noise the solution to (4.15) is actually significantly more complicated. We will assume in this case that the state-independent forcing function is

\[ g = \rho \sin(k_0 x), \]  
(B.4)

which, after assuming a particular solution of the form,

\[ u \]

obtains the following equation for the amplitudes \( a \) and \( b \):

\[ \begin{bmatrix} a \\ b \end{bmatrix} = e^{At} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^{-At} \int_0^t e^{Ar} \begin{bmatrix} n(t') \\ 0 \end{bmatrix} \, dt', \]  
(B.6)

where

\[ e^{At} = \begin{bmatrix} \cos(k_0 ct) & \sin(k_0 ct) \\ -\sin(k_0 ct) & \cos(k_0 ct) \end{bmatrix} \exp(-\mu k_0^2 t). \]  
(B.7)

Like (A.3), the integral in (B.6) is evaluated analytically in order to ensure convergence at the known rate for large \( \alpha \) (resolvable noise) for each numerical method.

APPENDIX C

Resolving the Noise: The Critical \( \Delta t \)

We estimate the temporal grid spacing necessary to resolve the noise by finding the frequency at which the spectrum drops off the fastest (Fig. C1). The basic idea is that if the temporal discretization can only resolve the frequencies to the left of the vertical line in Fig. C1 then the spectrum appears relatively white to the numerical
method because the smallest resolved scale has a relatively large amount of energy. However, if the temporal discretization does resolve the frequencies to the right of the vertical line in Fig. C1, then the spectrum will appear relatively red because the numerical method will see that the spectrum has relatively little energy at the smallest scales.

We will accomplish this task by finding the frequency for which the slope of the spectrum is maximized, that is,

$$\frac{d^2 C(\omega_c)}{d\omega^2} = 0. \quad (C.1)$$

When this procedure is applied to the spectrum in (A.2) we find that this frequency is

$$\omega_c = \frac{1}{\sqrt{2\alpha}}. \quad (C.2)$$

We may translate this into a period of time through the definition of the frequency:

$$\tau = 2\pi\sqrt{2\alpha}. \quad (C.3)$$

As a general rule, one can only resolve on a particular grid a shortest period of approximately $2\Delta t$ to $4\Delta t$. To be conservative we define the critical time step $\Delta t_c$, for which the spectrum begins to be resolved as

$$\Delta t_c = \frac{\tau}{4}. \quad (C.4)$$

### APPENDIX D

**Stochastic Subgrid-Scale Mixing**

A simple stochastic subgrid-scale mixing parameterization is constructed through the appropriate modification of the deterministic parameterization of Durran and Klemp (1983). The subgrid-scale mixing parameterization of Durran and Klemp is a traditional first-order closure that depends only on the strength of stratification and shear. The effects of subgrid-scale motions are introduced through zonal–vertical momentum and heat sources, respectively:

$$D_u = \frac{\partial}{\partial x} (K_M A) + \frac{\partial}{\partial z} (K_M B), \quad (D.1)$$

$$D_w = \frac{\partial}{\partial x} (K_M B) - \frac{\partial}{\partial z} (K_M A), \quad (D.2)$$

$$H = \frac{\partial}{\partial x} \left( K_H \frac{\partial \theta}{\partial x} \right) + \frac{\partial}{\partial z} \left( K_H \frac{\partial \theta}{\partial z} \right), \quad (D.3)$$

where

$$A = \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}, \quad (D.4a)$$

$$B = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \quad (D.4b)$$

The vorticity source for (5.1) is constructed from

$$F = \frac{\partial D_u}{\partial x} - \frac{\partial D_w}{\partial z}. \quad (D.5)$$

In the original formulation of Durran and Klemp (1983) the gridscale coefficient is calculated at every grid point from

$$K_M = k^2 \Delta x \Delta z \sqrt{A^2 + B^2} \left[ \max \left( 1 - \frac{K_H}{K_M} \text{Ri}, 0 \right) \right]^{1/2}, \quad (D.6)$$

where the following relationship was assumed: $K_H/K_M = 3$. In (D.6) $\Delta x$ and $\Delta z$ are the grid interval in the $x$ and $z$ directions, respectively. The parameter $k = 0.21$ and the Richardson number was calculated from

$$\text{Ri} = \frac{N^2}{A^2 + B^2}. \quad (D.7)$$

This parameterization was modified stochastically in the spirit of Leith (1990) such that we simply multiplied (D.6) by the following stochastic function:

$$S = 1 + b \frac{\phi}{\sqrt{\Delta t}}, \quad (D.8)$$

where $b = 3$ is simply a tunable parameter controlling the amplitude of the noise and $\phi \sim N(0, 1)$. The noise in the function $S$ is drawn independently at every point in space and time where $K_M$ is to be evaluated. As discussed in Leith (1990) and Schumann (1995), the function (D.8) allows for random yet sporadic situations where the coefficients $K_M$ and $K_H$ become negative such that the subgrid-scale turbulence may induce the realistic emergence of random coherent waves (i.e., upscale energy cascade).

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