Potential Vorticity Dynamics of Forecast Errors: A Quantitative Case Study

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ABSTRACT

Synoptic-scale error growth near the tropopause is investigated from a process-based perspective. Following previous work, a potential vorticity (PV) error tendency equation is derived and partitioned into individual contributions to yield insight into the processes governing error growth near the tropopause. Importantly, we focus here on the further amplification of preexisting errors and not on the origin of errors. The individual contributions to error growth are quantified in a case study of a 6-day forecast. In this case, localized mesoscale error maxima have formed by forecast day 2. These maxima organize into a wavelike pattern and reach the Rossby wave scale around forecast day 6. Error growth occurs most prominently within the Atlantic and Pacific Rossby wave patterns. In our PV framework, the error growth is dominated by the contribution of upper-level, near-tropopause PV anomalies (near-tropopause dynamics). Significant contributions from upper-tropospheric divergent flow (prominently associated with latent heat release below) and lower-tropospheric anomalies (tropospheric-deep (i.e., baroclinic) interaction) are associated with a misrepresentation of the surface cyclone development in the forecast. These contributions are, in general, of smaller importance to error growth than near-tropopause dynamics. This result indicates that the mesoscale errors generated near the tropopause do not primarily project on differences in the subsequent baroclinic growth, but instead directly project on the tropopause evolution and amplify because of differences in the nonlinear Rossby wave dynamics.

1. Introduction

Numerical weather prediction has improved remarkably over the last decades (e.g., Bauer et al. 2015). Occasionally, however, very poor medium-range forecasts do still occur (Rodwell et al. 2013). Forecast errors arise due to errors in the initial conditions and due to model deficiencies (e.g., Palmer and Hagedorn 2006). After 1–2 forecast days, localized errors may form that start to affect the synoptic-scale flow (e.g., Davies and Didone 2013; Martínez-Alvarado et al. 2016). Subsequently, these errors further amplify and propagate downstream (e.g., Langland et al. 2002; Anwender et al. 2008; Pantillon et al. 2013). The focus of this study is on the amplification of existing errors, as illustrated in Fig. 1. At forecast day 2, a localized PV error is, in our case, generated in the Atlantic ridge (Fig. 1a). Within the next day, significant error amplification occurs, and the localized error evolves into a wavelike pattern at forecast day 3 (Fig. 1b). This study aims to understand such error growth by quantifying the processes governing the error amplification, rather than identifying the initial source of the error growth. Gaining deeper insight into the dynamics of error growth, and thus into the atmospheric conditions that exhibit high or low intrinsic predictability (e.g., Melhauser and Zhang 2012), can be expected to improve the future interpretation of the forecast uncertainty observed in ensemble forecast systems.
Synoptic-scale error growth is often related to baroclinic instability (e.g., Tribbia and Baumhefner 2004; Hakim 2005; Hohenegger and Schä 2007; Zhang et al. 2007; Boisserie et al. 2014; Selz and Craig 2015; Sun and Zhang 2016). Baroclinic instability leads to exponential growth of wave amplitudes (e.g., Eady 1949). Therefore, a small difference between the baroclinic growth in the forecast and the analysis can be expected to yield exponential error growth. A number of other studies have highlighted the importance of processes that are unrelated to baroclinic instability: Snyder (1999) and Plu and Arbogast (2005) emphasize that errors in the propagation characteristics and nonlinear error growth mechanisms can be as important for the error growth as barotropic and baroclinic instabilities. In addition, more recent studies indicate that synoptic-scale error growth can be induced by misrepresented deformation fields or potential vorticity (PV) anomalies near the tropopause (Davies and Didone 2013), a systematic underestimation of the background PV gradient in the forecast (Harvey et al. 2016), and bifurcation-like behavior (Riemer and Jones 2014). A quantitative analysis of the relative importance of individual processes, however, is still missing.

This study adopts the PV perspective of forecast errors to gain insight into the processes governing the error growth. This perspective provides a useful framework to analyze error growth from a dynamical, process-oriented perspective (Davies and Didone 2013). The largest values of the PV error occur near the midlatitude tropopause and are related to erroneous displacements of the tropopause (e.g., Dirren et al. 2003; Gray et al. 2014). It is these near-tropopause errors that we will focus on in this study. Davies and Didone (2013) have derived a tendency equation for the PV error and analyzed it qualitatively. Our study considers a similar tendency equation in combination with a partitioning of the PV tendencies, as applied in an investigation of Rossby wave packet dynamics (Teubler and Riemer 2016). This partitioning allows us to quantify the relative importance of four individual processes to PV error growth near the tropopause: 1) near-tropopause dynamics, 2) tropospheric-deep interaction, 3) upper-tropospheric divergent flow, and 4) nonconservative processes.

The contribution of near-tropopause dynamics quantifies the influence of upper-level, near-tropopause PV anomalies on the tropopause evolution, which is mostly related to Rossby wave dynamics, while the contribution of tropospheric-deep interaction quantifies the influence of lower-tropospheric anomalies on the tropopause evolution, which signifies baroclinic interaction in the PV framework (Eady 1949; Hoskins et al. 1985; Davis and Emanuel 1991; Heifetz et al. 2004). Baroclinic instability can be described by the mutual amplification of upper- and lower-level PV anomalies. Its potential influence near the tropopause can therefore be assessed by the influence of tropospheric-deep interaction. In addition to these contributions from balanced dynamics, we quantify the influence of upper-tropospheric divergent flow. Upper-tropospheric divergence can be associated with balanced dynamics and diabatic processes [see considerations with the omega equation; e.g., Holton and Hakim (2013)]. For Rossby wave dynamics, a few cases indicate that strong upper-tropospheric divergence is mostly associated with latent heat release below (e.g., Davis et al. 1993; Riemer et al. 2014; Quinting and Jones 2016). Finally, the direct impact of nonconservative processes is quantified.

This study is organized as follows. In section 2, we describe the data and tendency equation used to quantify PV error growth and the relative importance of individual processes. Section 3 presents our results for a case study of medium-range forecast error growth. In section 4, we discuss some conceptual aspects of error growth based on the error patterns observed in our case study. Finally, section 5 contains our summary and conclusions of the results.
2. Data and diagnostics to quantify the individual contributions to PV error growth

a. Data

We use analysis and deterministic forecast data from the Atmospheric Model high-resolution 10-day forecast (HRES) of the European Centre for Medium-Range Weather Forecasts (ECMWF). Error growth is analyzed for the forecast starting at 0000 UTC 12 November 2013. This forecast covers an extreme precipitation event over Sardinia on 18–19 November 2013, which was closely connected with the preceding Rossby wave packet, wave breaking, and the formation of a cutoff low. The error growth within this Rossby wave packet is of particular interest.

Six-hourly data are used on pressure levels and a $1^\circ \times 1^\circ$ grid. The piecewise PV inversion used to partition the PV tendencies is performed on pressure levels, as in Davis and Emanuel (1991) and Davis (1992). For the further analysis of PV error growth, all variables are interpolated to isentropic levels. Similar to the Year of Tropical Convection (YOTC) data (Moncrieff et al. 2012), accumulated temperature and wind tendencies from the physical parameterization schemes were saved for our case study. Note that the computations of the physical parameterization schemes are performed in the vertical only (ECMWF 2013); in particular, they account only for vertical diffusion and not for horizontal mixing. Instantaneous tendencies are approximated by centered differences of the 3-h accumulated tendency data.

Tendency data for the analysis are estimated with accumulated data from short-range forecasts.

b. PV error and tendency equation

The hydrostatic form of Ertel PV (Ertel 1942) is used in isentropic coordinates:

$$ PV = -g \frac{\partial \theta}{\partial p} (\zeta_\theta + f), \quad (1) $$

where $g$ is the gravitational acceleration, $\theta$ is the potential temperature, $p$ is the pressure, $\zeta_\theta$ is the vertical component of the isentropic relative vorticity, and $f$ is the Coriolis parameter.

Forecast errors (index *) are defined as the difference between the forecast (index fc) and the analysis (index an); for example, $\text{PV}^* := \text{PV}_{fc} - \text{PV}_{an}$. Following the approach of Davies and Didone (2013), we derive a PV error tendency equation. Our tendency equation is based on the local tendency equation for PV in isentropic coordinates, which is given by (e.g., Davies and Didone 2013)

$$ \frac{\partial \text{PV}}{\partial t} = -\mathbf{v} \cdot \nabla \cdot \text{PV} + \text{NonCons} + \text{RES} , \quad (2) $$

where

$$ \text{NonCons} = -\theta \frac{\partial \text{PV}}{\partial \theta} + \text{PV} \frac{\partial \theta}{\partial \theta} - \frac{g}{\partial \theta} \left( \mathbf{k} \times \frac{\partial \mathbf{v}}{\partial \theta} \right) \cdot \nabla \theta $$

$$ - \frac{g}{\partial \theta} \mathbf{k} \cdot (\nabla \times \mathbf{v}) $$

describes the nonconservative PV tendency due to diabatic heating and nonconservative momentum change as measured by the parameterization schemes. The heating rate $\theta$ comprises heating due to the cloud, convection, radiation, and turbulence parameterization, while the horizontal wind tendency components of $\mathbf{v}$ ($\mathbf{u}$ and $\mathbf{v}$) comprise tendencies from the convection and turbulence parameterization. The residual term RES comprises the influence of those processes that cannot be quantified with the available data (i.e., numerical diffusion, analysis increments due to data assimilation, and numerical inaccuracies due to the discretization and interpolation of data).

As the error is defined by the difference between the forecast and analysis, the local tendency for the PV error is given by (see Davies and Didone 2013)

$$ \frac{\partial \text{PV}^*}{\partial t} = \frac{\partial \text{PV}_{fc}}{\partial t} - \frac{\partial \text{PV}}{\partial t} $$

$$ = -\mathbf{v}^* \cdot \nabla \cdot \text{PV}^* - \mathbf{v} \cdot \nabla \cdot \text{PV}^* $$

$$ + \text{NonCons}^* + \text{RES}, \quad (3) $$

where variables without an index refer to analysis variables. Note that the PV error is not materially conserved under conservative conditions and that the tendency equation for the PV error is thereby more complicated than Eq. (2).

Similar to the error enstrophy equation by Boer (1984), we consider the local tendency of error potential enstrophy (the squared PV error, $\text{PV}^*^2/2$), which is a positive-definite error metric and, thus, ensures that a positive error tendency is associated with error amplification. The according tendency equation is given by
where $v_s$ describes the motion of the integration area. As in Boer (1984), the first term on the right-hand side of Eq. (5) can be interpreted as a (nonlinear) production term. Since the second term in Eq. (4) merely redistributes existing errors, it can be evaluated in terms of an error flux across the boundary of the integration domain [second term in Eq. (5)]. Not explicitly included in the barotropic framework of Boer (1984), the third term in Eq. (5) constitutes an error source due to the divergence of the quasi-horizontal (adiabatic) flow. The remaining terms describe the influence of nonconservative processes (term 4), the boundary contribution due to changes in the integration area (term 5), and the residual (term 6).

We evaluate the PV error tendency equation on an isentropic level intersecting the midlatitude tropopause, namely, on 320 K. PV anomalies on this isentropic level are representative for midlatitude Rossby waves during the time of year under consideration (Liniger and Davies 2004; Martius et al. 2010). A comparison with the error growth on nearby isentropes (315 and 325 K) showed qualitatively very similar results.

c. Further separation of the advective term

To gain deeper insight into the dynamics of the PV error growth, we further partition the nonlinear production term (term 1 on the right-hand side of Eq. (5)), following Teubler and Riemer (2016). This partitioning builds on the PV perspective of midlatitude dynamics. In essence, the evolution of PV anomalies near the tropopause can be described by advective tendencies associated with 1) the upper-level (near tropopause) PV anomalies themselves, 2) $\theta$ anomalies just above the boundary layer and “interior” PV anomalies in the lower troposphere, and 3) upper-tropospheric divergent flow. In addition, upper-level PV anomalies are modified by 4) nonconservative processes.

The influence of upper-level PV anomalies on upper-level dynamics can be related to Rossby wave dynamics. As mentioned in the introduction, baroclinic instability is in the PV framework described by the mutual amplification of upper- and lower-level anomalies (Eady 1949; Hoskins et al. 1985; Davis and Emanuel 1991; Heifetz et al. 2004). Since our focus is on error growth near the tropopause, the contribution of tropospheric-deep interaction is quantified by the influence of lower-level errors on the upper-level errors. This part of the interaction, however, provides a necessary condition that amplification due to baroclinic instability does occur. Upper-tropospheric divergent flow can significantly influence Rossby wave dynamics—in particular, ridge building (e.g., Grams et al. 2011; Teubler and Riemer 2016). It can be associated with dry balanced dynamics and diabatic processes [see considerations with the omega equation as discussed in, e.g., chapter 6.4 in Holton and Hakim (2013)]. A few cases (e.g., Davis et al. 1993; Riemer and Jones 2010; Riemer et al. 2014) indicate that pronounced upper-level divergence is mostly related to latent heat release below. This indirect diabatic impact can be of larger importance for Rossby wave dynamics than direct diabatic PV modification (e.g., Riemer and Jones 2010; Teubler and Riemer 2016).

Figure 2 illustrates that the errors in our case concentrate in two regions: errors exhibit their strongest signal in PV close to the tropopause and, equivalently,
as $\theta$ errors at the top of the boundary layer (875 hPa). There is a clear vertical separation between these upper- and lower-level PV and $\theta$ errors. Therefore, it is reasonable to separate the error features at a midlevel, where the error features are, in general, of small amplitude. Our results are not sensitive to the exact choice of the separation level (tested for 700, 600, and 500 hPa). The following results are shown for a separation level of 600 hPa.

The upper-level wind associated with the lower-tropospheric (PV and $\theta$) anomalies will be referred to as “tropospheric-deep” ($v_{TPd}$), the upper-level wind associated with upper-level anomalies as “near-tropopause” ($v_{nTP}$), and the upper-level wind associated with upper-level divergence as “divergent” ($v_{div}$). Thus, the full wind field can be partitioned as

$$v = v_{nTP} + v_{TPd} + v_{div} + v_{unc}. \quad (6)$$

The uncertainty term $v_{unc}$ describes the uncertainties associated with the partitioning of the wind field. It is calculated as the difference between the wind field given by ECMWF and the sum of the near-tropopause, tropospheric-deep, and divergent wind fields. This uncertainty turns out to be small and does not affect the physical interpretation of our results. Technical details on our flow partitioning are provided in appendix A.

Quantitatively, we separate the relative importance of the individual processes by spatially integrating their contribution to the error tendency [Eq. (5)]:

$$\text{OBS} = nTP + TPd + DIV + \text{NONCONS} + BND + UNC + RES,$$

where

$$\text{OBS} := \frac{d}{dt} \int_A \frac{\text{PV}^2}{2} dA \approx \frac{\text{PV}^2_{t+6h} - \text{PV}^2_t}{6h},$$

$$nTP := -\int_A \text{PV}^* v_{nTP} \cdot \nabla_\theta \text{PV} dA,$$

$$TPd := -\int_A \text{PV}^* v_{TPd} \cdot \nabla_\theta \text{PV} dA,$$

$$DIV := -\int_A \text{PV}^* v_{div} \cdot \nabla_\theta \text{PV} dA + \int_A \frac{\text{PV}^2}{2} \nabla_\theta \cdot (v_{div} + v_{unc}) dA,$$

$$\text{NONCONS} := \int_A \text{PV}^* \text{NonCons} dA,$$

$$BND := -\int_s \frac{\text{PV}^2}{2} (v^* + v - v_3) \cdot \mathbf{n} dS,$$

$$UNC := -\int_A \text{PV}^* v_{unc} \cdot \nabla_\theta \text{PV} dA,$$

$$RES := \text{OBS} - (nTP + TPd + DIV + \text{NONCONS} + BND + UNC). \quad (7)$$

The residual term is determined by the difference between the observed error evolution (OBS; approximated by 6-h finite differences) and the sum of the
quantified contributions to error growth (averaged over the two time steps used for the finite differences).

We emphasize that Eq. (7) describes the influence of individual processes to error growth at an instantaneous time step. This approach is different from Lagrangian approaches that investigate the accumulated effect of, for example, diabatic processes on the PV distribution (e.g., Gray 2006; Chagnon et al. 2013) or that track errors back to their initial source (e.g., Martínez-Alvarado et al. 2016). In particular, we do not explicitly analyze the indirect effect of diabatic processes by having created PV anomalies at earlier times [see summary and discussion by Davis et al. (1993)]. As the processes governing the error growth may differ at different stages of the error growth (e.g., Zhang et al. 2007; Hohenegger and Schär 2007; Selz and Craig 2015), analysis of the instantaneous tendencies provides helpful insight into the error-growth dynamics.

3. Error growth in an illustrative case

In this section, the error growth in our case is first illustrated by the time evolution of the PV error. Then, we show the spatial patterns of the individual contributions to net error amplification. Finally, the relative importance of these contributions to error growth in specific regions is quantified by time series of spatially integrated tendencies.

a. Error evolution

Figure 3 illustrates the PV error evolution of our case study on 320 K, together with the location of the dynamical tropopause in the forecast and the analysis (here, defined as the 2-PVU surface; 1 PVU = 10⁻⁶Km²kg⁻¹s⁻¹), which serves as an indication of the synoptic evolution in the forecast and in the analysis. The tropopause is, in general, undulated by Rossby wave patterns. Two regions, the Atlantic and Pacific regions, turn out to be associated with a significant error evolution and will be discussed in more detail.

The synoptic evolution in the Atlantic starts with a ridge building around 30°W (Fig. 3a), which is in association with cyclogenesis off the coast of Newfoundland. The cyclone’s intensity and ridge building were underestimated in the forecast, leading to a localized positive error in surface pressure and PV around 40°–20°W at forecast day 2. In the following 2 days, the downstream trough breaks anticyclonically, and a cutoff is formed at day 4 (10°W–20°E; Fig. 3b). This wave breaking occurred too late in the forecast and was associated with significant error amplification between days 2 and 4. At day 6 (Fig. 3c), the wave patterns in the forecast and in the analysis exhibit large differences, including local phase shifts of approximately half a wavelength and significant amplitude errors.

The synoptic evolution in the Pacific starts with anticyclonic wave-breaking and cutoff formation around 140°W before forecast day 2 (not shown). The error pattern at day 2 is dominated by an error in the meridional location of the tropopause (Fig. 3a). In the following 2 days, ridge building associated with cyclogenesis governs the evolution (150°W–170°E; Fig. 3b). The cyclogenesis and ridge building were much stronger in the forecast than in the analysis, leading to significant error amplification in surface pressure and PV. Further error amplification occurs between days 4 and 6, leading to large differences between the wave patterns in the forecast and the analysis at day 6 (Fig. 3c).

Consistent with, for example, Dirren et al. (2003) and Davies and Didone (2013), large absolute values of the PV error are related to a displacement of the dynamical tropopause. Both in the Atlantic and the Pacific, a first localized error pattern changes into a wavelike error pattern between days 2 and 4 (Figs. 3a,b). Error extrema in both regions are associated with an erroneous representation of the Rossby wave pattern. Compared to the PV anomalies (defined by the deviation of the PV from a background PV; see appendix A for details), the PV error exhibits, in general, a smaller horizontal scale, particularly in the meridional direction (Figs. 3b,d).

b. Spatial patterns of the individual contributions to error growth

In Fig. 4, the individual contributions to the net error amplification in Eq. (7) are illustrated for forecast day 4 (note the different color bars in this figure). The advective tendencies are further discussed with focus on the Pacific region (Fig. 5). All individual tendencies exhibit their largest amplitude and a relatively high degree of spatial coherence within the Rossby wave pattern in the Atlantic and in the Pacific. The Rossby wave patterns thus stand out as important regions of the error evolution, corroborating our notion from the previous subsection.

The near-tropopause tendency (Figs. 4a, 5b) exhibits the largest absolute values, compared to the other individual tendencies. The spatial pattern of the total tendency (not shown) is almost entirely determined by the near-tropopause tendency. The dominant features of the spatial pattern are dipoles of positive and negative values. A comparison with the PV error (Fig. 5a) indicates that the near-tropopause error wind and near-tropopause error tendency are clearly related to the PV error: positive PV errors (error feature A in Fig. 5a) are associated with a cyclonic error circulation, whereas negative PV errors (error feature B in Fig. 5a) are associated with an anticyclonic error circulation. The associated error
tendency (Fig. 5b), which is defined by the advection of the analysis PV by the near-tropopause error wind [Eq. (7)], leads to error amplification on the western side and error decay on the eastern side of the individual PV error features (see error tendency within error features A and B in Fig. 5b). The tendency pattern is thus dominated by a displacement of the PV error along the dynamical tropopause. From Figs. 4a and 5b, it is not directly clear whether the net effect of the near-tropopause tendency is positive or negative.

The tropospheric-deep tendency (Figs. 4b, 5d) leads to error amplification in the Pacific ridge (140°W–180°) and possibly also in the Atlantic ridge (near 10°W). In the Pacific region, temperature error features at the top of the boundary layer and PV error features near the tropopause have a phase shift of about π/4 to π/2 (see Figs. 5a,c). The tropospheric-deep error wind associated with the positive potential temperature error (error feature D in Fig. 5c) exhibits a cyclonic pattern, leading to the tropospheric-deep error amplification observed in the Pacific ridge (Fig. 5d). The absolute values of the tropospheric-deep tendency are, however, almost one order of magnitude smaller than those of the near-tropopause tendency.
The divergent tendency (Figs. 4c, 5f) leads to error amplification over coherent regions, particularly in ridges and cutoffs. A comparison with the error in midtropospheric specific humidity and mean sea level pressure in the Pacific region (Fig. 5e) indicates that the divergent tendency is directly linked to the misrepresentation of the surface cyclone around 190°W and the associated latent heat release. This cyclone (error feature E in Fig. 5e), its accompanying warm conveyor belt (error feature F in Fig. 5e), and the associated upper-tropospheric divergent flow and divergent ridge amplification were too strong in the forecast. The divergent tendency (Fig. 5f) thus increases the difference between the ridge amplification in the forecast and the analysis, leading to error amplification in the Pacific region as seen by the coherent region of positive error tendency in the ridge (Fig. 5f). A similar investigation shows that strong divergent tendencies in other regions can also be related to latent heat release (not shown).

The largest values of the nonconservative tendency (approximated by the wind and temperature tendencies from the parameterization schemes; Fig. 4d) are found in ridges and cutoffs. A further partitioning of this
tendency (not shown) reveals that the radiative tendency leads to the error amplification in ridges and cutoffs, whereas the turbulence tendency leads to the error decay in regions of a strong PV gradient (e.g., in the Pacific ridge). These two processes have mostly opposite effects. Compared to the advective tendencies, the nonconservative tendency exhibits the smallest absolute values.
c. Spatially integrated metric of the error growth

Insight into the relative importance of the individual processes can be gained by considering spatially integrated tendencies. Integration over large, sensibly defined regions diminishes the compensation between positive and negative tendencies prominent in Figs. 4 and 5. The areas used for integration are chosen such that the distinct error patterns in the Atlantic and Pacific are included during the whole forecast time. These areas are indicated by the blue boxes in Fig. 3. Time series of the spatially integrated tendencies are depicted in Fig. 6.

Comparing the observed error evolution with the quantified contributions to the error growth [RES in Eq. (7)], reveals a systematic negative residual of considerable amplitude, in particular after day 2. From the processes contributing to the residual (see section 2b), numerical diffusion can be expected to have a systematic negative effect on the error evolution. Nonconservation of PV by the dynamic core, which is related to numerical diffusion, has been shown to yield PV tendencies of similar magnitude as diabatic processes (Chagnon and Gray 2015; Saffin et al. 2016). From a physical point of view, numerical diffusion is associated with a smoothing of small-scale PV features. PV forms rich filamentary structures near the tropopause (e.g., Appenzeller et al. 1996) related to the downscale cascade of potential enstrophy (PV^2). Errors that are associated with such filaments (e.g., displacement errors) are thus eliminated when the scale of the filament reaches the resolution of our data. Effectively, this elimination constitutes irreversible horizontal mixing (e.g., Nie et al. 2016), a nonconservative process that cannot be quantified in this study.

A comparison of the individual tendencies clearly demonstrates that the error growth, integrated over the Northern Hemisphere from 30° to 80°N, is dominated by the near-tropopause tendency, which yields the largest contribution to the error growth at all time steps (Fig. 6a). The most conservative estimate of the near-tropopause tendency would be to add all uncertainty of our diagnostic [residual and uncertainty of PPVI, i.e., terms RES and UNC in Eq. (7)] to the near-tropopause tendency. However, even together with this error sink, the near-tropopause tendency still makes by far the largest contribution to error growth. The divergent tendency makes the second largest contribution to the error growth, but is at all times much smaller than the near-tropopause tendency. Furthermore, we find only a minor contribution from the tropospheric-deep tendency to the error growth. This tendency contributes, on average, only 14% to the observed error growth.

The nonconservative tendency makes a small, but continuously negative, contribution to the error evolution. A further partitioning of this tendency reveals that it is mostly governed by the radiation and turbulence tendency (Fig. 7). As mentioned in section 3b, these tendencies have rather opposite effects on the error evolution. While the radiation tendency makes a significant positive contribution to error growth, the turbulence tendency makes a contribution of similar magnitude, but of opposite sign.

As demonstrated above, the error growth in this case most prominently occurs within the Atlantic and Pacific Rossby wave patterns (Figs. 3, 4). The error features in the Pacific and Atlantic are clearly separated, and their amplification may exhibit different characteristics. In the following, these two regions are therefore investigated in isolation.

Within the Atlantic region, error growth during wave breaking and the highly nonlinear stage of the wave pattern (after forecast day 4) is clearly dominated by the near-tropopause tendency (Fig. 6b). Preceding the wave breaking, a significant contribution is also given by the divergent tendency, which is of similar magnitude as the conservative estimate of the near-tropopause tendency. During this time, the relative importance of the divergent tendency is larger than when integrated hemispherically (30°–80°N; Fig. 6a). This large divergent tendency can be attributed to the surface cyclone located around 40°W at forecast day 2 and its accompanying warm conveyor belt that were underestimated in the forecast (surface pressure error shown in Fig. 3a). The forecast of the associated ridge building was of particular importance to the subsequent wave breaking, cutoff formation, and high-impact weather over Sardinia. Thereby, the error growth associated with this high-impact weather was strongly related to near-tropopause dynamics and upper-tropospheric divergence.

Within the Pacific region (Fig. 6c), the largest contribution to error growth is again given by the near-tropopause tendency. The tropospheric-deep tendency contributes more prominently to error growth than when integrated hemispherically (30°–80°N; Fig. 6a). Averaged over forecast days 4 and 5, it accounts for about 1/3 of the observed error growth. Interestingly, this significant contribution from the tropospheric-deep tendency is accompanied by a contribution of similar magnitude from the divergent tendency. These significant contributions from the tropospheric-deep and divergent tendencies can be attributed to the cyclone development on the upstream side of the Pacific ridge that was overestimated in the forecast (surface pressure error around 170°E in Figs. 3b, 5e,f).
Fig. 6. Individual contributions (colored lines) to spatially integrated error growth on the 320-K surface as indicated by Eq. (7) as a function of forecast time. The integration area is (a) the Northern Hemisphere (30°–80°N) and (b), (c) the Rossby wave pattern in the Atlantic and Pacific, respectively. The integration areas in the Atlantic and Pacific are indicated by the blue boxes in Fig. 3. The shading next to the near-tropopause and tropospheric-deep tendencies indicates the uncertainty of the result due to the uncertainty of PPVI [UNC in Eq. (7)].
Some insight into error-growth dynamics

The previous section has shown that PV error growth in our case is not primarily governed by baroclinic instability (tropospheric-deep interaction), as assumed by some previous studies. Instead, we find a large contribution of near-tropopause dynamics to the error growth. This section provides a plausible explanation for the limited importance of tropospheric-deep interaction to error growth. Furthermore, we discuss differences between the nonlinear Rossby wave dynamics in the analysis and in the forecast, which can explain a large part of the observed error evolution in our case. Finally, we compare the error evolution with the underlying waviness evolution in the forecast and the analysis.

a. Limited importance of tropospheric-deep interaction to error growth

To better understand the limited importance of tropospheric-deep interaction to the error growth in our case, we consider the Eady (1949) model in the reformulation of Davies and Bishop (1994) as a well known and, arguably, the simplest model that describes baroclinic instability. Details about the model setup are provided in appendix B. The basic idea is to prescribe idealized error patterns in this model and to apply our error tendency derived in section 2 to quantify the tropospheric-deep contribution to error growth in the Eady model. Because of the linearity of the model, it turns out that error growth in this model exhibits the same characteristics as generic baroclinic growth.

The Eady model is nondivergent and conservative, and we will consider error growth globally (i.e., integrated over the whole domain). In the integrated PV error tendency equation [Eq. (5)], all terms on the right-hand side therefore vanish, except for the first term. The dynamics of the Eady model are completely governed by the distribution of \( \theta \)-edge waves at the upper and lower boundaries, effectively representing the PV distribution, from which the wind field can be derived directly [Eqs. (B6) and (B7)]. Evaluating the error tendency equation in the Eady model requires specification of the upper-level analysis PV and of the PV error at the upper and lower levels. A zonal background superposed by a wavenumber-6 wave (representing a synoptic-scale Rossby wave pattern) constitutes our idealized upper-level analysis state. Because of the linearity of the Eady model, arbitrary error patterns can be prescribed by the superposition of individual Fourier modes. It is therefore sufficient to consider error growth in the presence of one arbitrary wave mode in the analysis.

The dependence of the tropospheric-deep error tendency on the wavenumber and phase shift of the upper- and lower-level errors is investigated by varying these two parameters. These investigations show that tropospheric-deep error growth is only possible for a rather specific configuration: namely, for identical wavenumbers of the upper and lower error waves and for a favorable phase shift between 0 and \( \pi \) (Fig. 8a). Furthermore, the error growth shows a prominent scale dependence. For the optimal growth configuration (identical wavenumbers and phase shift of \( \pi/2 \)), the error tendency decreases exponentially with increasing wave number (Fig. 8b). Tropospheric-deep error growth in...
the Eady model thus shows the same behavior as baroclinic growth in general.

In light of these results from an idealized model, we now examine in more detail the error patterns observed in our case study. In the Pacific, the region in which tropospheric-deep error growth is most prominent, the error pattern is indeed as suggested by Fig. 8a: error features at upper and lower levels have a similar wavelength and a phase shift of $\pi/4 - \pi/2$ (indicated by labels in Fig. 5c). It is evident that the associated tropospheric-deep error tendency is mostly positive in this region (Fig. 5d). Apart from this region, a pronounced coupling between the upper- and lower-level errors is not noticeable (Fig. 4b). From a qualitative perspective, a favorable configuration for tropospheric-deep interaction is evident in the Atlantic ridge (similar wavelength of upper- and lower-level error features and phase shift of approximately $\pi/2$), and positive tendencies prevail in the western part of the ridge. In the integrated sense of Fig. 6b, however, the tropospheric-deep error tendency is much weaker than in the Pacific due to cancellation with negative values in the eastern part of the ridge and near the cutoff.

Evidently, the favorable configuration for tropospheric-deep error growth is achieved only over a limited region. In our case, the organization of the error pattern into such a configuration requires 3–4 forecast days, which is significantly longer than assumed in previous studies (e.g., Zhang et al. 2007; Hohenegger and Schär 2007). In addition, we note that the spatial scale of the error features is smaller than that of the anomalies governing baroclinic development of the midlatitudes (Figs. 3b,d). Because of the dependence of vertical penetration on the horizontal scale (Fig. 8b), the tropospheric-deep interaction of PV and $\theta$ errors can be expected to be less prominent than the tropospheric-deep interaction of respective anomalies in a baroclinically unstable Rossby wave. Our considerations with the Eady model thus provide theoretical support for our observation that tropospheric-deep error growth (i.e., error growth due to differences in the release of baroclinic instability) is not a dominant error-growth mechanism in our case study.

b. Importance of nonlinear near-tropopause dynamics

Further insight into the important role of near-tropopause dynamics to error growth can be gained by considering the differences in the underlying linear and nonlinear dynamics. Recall from Eq. (3) that the error tendency is defined as the difference between the PV tendency in the forecast and in the analysis. Introducing a stationary background state and deviations thereof (see appendix A for details), we further partition the advective part of the respective tendency as

$$\frac{\partial PV}{\partial t}\bigg|_{\text{adv}} = -v \cdot \nabla_\theta PV - v' \cdot \nabla_\theta PV' - v' \cdot \nabla_\theta PV'',$$  \hspace{1cm} (8)

where variables with an overbar denote the background state and variables with a prime the deviations thereof. The first three terms in Eq. (8), therefore, denote the linear contributions to the advective tendency, and the last term denotes the nonlinear contribution. Using this partitioning, and noting that the background state is identical for the forecast and for the analysis, it is straightforward to rewrite the advective error tendency as...
where $PV'$ and $v'$ refer to the PV anomalies and the associated wind in the analysis, respectively. In this equation, the first two terms denote error evolution due to differences in the linear dynamics, and the last two terms denote error evolution due to differences in the nonlinear dynamics.

We here focus on the near-tropopause dynamics and consider $v'$ and $v^*$ in Eq. (9) as associated with the near-tropopause wind. For nondivergent wind, as the near-tropopause wind, the second and fourth terms in Eq. (9) merely redistribute the error, but do not contribute to a net amplification. In the integrated sense of Eq. (7), the partitioning of the near-tropopause contribution to error growth thus reads

$$n_{TP} = - \int_A PV^* v^* n_{TP} \cdot \nabla_P PV dA = - \int_A PV^* v^* n_{TP} \cdot \nabla_P PV dA$$

$$\quad - \int_A PV^* v^* n_{TP} \cdot \nabla_P PV' dA,$$

where the first and the second terms are the contributions due to differences in the underlying linear and nonlinear dynamics, respectively. Spatially integrated over the Northern Hemisphere, error growth is clearly dominated by differences in the nonlinear dynamics (Fig. 9). The contribution due to differences in the linear dynamics is essentially negligible in our case. It is thus not possible to explain the error evolution based on simple linear arguments for Rossby wave dynamics.

For nonlinear Rossby wave dynamics, we are not aware of an established conceptual framework that could guide a further systematic analysis of the nonlinear processes underlying the error growth. Instead, we here illustrate exemplarily how differences between the near-tropopause PV tendency in the analysis and the forecast can be used to explain a large part of the observed error evolution. Specifically, we consider the localized error pattern in the Atlantic at forecast day 2 and the wavelike error pattern in the Pacific at forecast day 4.

The Atlantic error pattern at forecast day 2 is dominated by an amplitude error of the ridge, which is too weak in the forecast compared to the analysis (Figs. 1a, 10a). This difference is associated with differences in the near-tropopause PV tendency (Figs. 10b,c): the negative tendency on the eastern side of the ridge is of larger amplitude in the analysis than in the forecast. The tendency in the analysis is thus associated with a faster shift (phase speed) of the ridge to the east and an enhanced thinning of the downstream trough, as compared to the forecast. Consistently, at day 3, the ridge in the analysis is farther to the east than in the forecast, leading to a distinct phase error generated between days 2 and 3.
Furthermore, the downstream trough in the analysis is thinner than in the forecast and exhibits a more pronounced positive tilt, indicating that anticyclonic wave breaking occurs earlier in the analysis than in the forecast. Differences in the near-tropopause tendency at day 2 thereby help to explain the observed error evolution in the Atlantic between days 2 and 3.

The Pacific wave pattern in the forecast exhibits a larger amplitude and smaller wavelength at day 4 than the analysis wave pattern, leading to the observed wavelike error pattern on that day (Figs. 5a, 10d). Comparing the near-tropopause PV tendency in the analysis and in the forecast clearly indicates that these differences are associated with large differences in the subsequent evolution of the wave pattern (Figs. 10e,f): while the forecast tendency exhibits a distinct signal of Rossby wave propagation with positive and negative tendencies on the western and the eastern flank of the ridge, respectively, the analysis tendency exhibits instead a patchy signal. The forecast ridge is thus associated with eastward phase propagation between
days 4 and 5, while the analysis ridge is still at a similar location at day 5, but tilted northwestward (gray contours in Fig. 10d). A large part of the observed error evolution between days 4 and 5 can thus be explained by these differences in the nonlinear Rossby wave dynamics.

c. Underlying waviness evolution in the forecast and analysis

In this final subsection, we briefly compare the processes contributing to error amplification with those contributing to the evolution of the underlying Rossby wave patterns. We focus on wave amplitude, or waviness, which we here measure by potential eddy enstrophy (i.e., the square of the PV anomaly). Starting from the PV equation [Eq. (2)], applying some manipulations as in the derivation of the spatially integrated error tendency [Eq. (5)], and using the partitioning of the wind field [Eq. (6)], it is straightforward to derive a tendency equation for waviness that quantifies the contributions from near-tropopause dynamics, tropospheric-deep interaction, upper-tropospheric divergent flow, and non-conservative processes:

\[
\frac{d}{dt} \int_A \frac{PV'^2}{2} dA = - \int_A PV' (v_{nTP} + v_{TPd} + v_{div} + v_{unc}) \cdot \nabla \theta dA + \int_A PV' \cdot v_{div} dA + \int_A PV' \cdot v_{unc} dA
\]

with

\[
NonCons = -\frac{\partial PV}{\partial \theta} + PV \frac{\partial \theta}{\partial p} + \frac{g}{\beta} \left( \nabla \times \frac{\partial \mathbf{v}}{\partial \theta} \right) \cdot \nabla \theta
\]

and residual RES. Here \( PV \) denotes the background PV, and \( PV' \) denotes the PV anomaly (i.e., the deviation from the background PV; see appendix A for details). For our case, the individual contributions to the hemispheric (30°–80°N) waviness evolution in the analysis and in the forecast are shown in Figs. 11a and 11b, respectively.

Tropospheric-deep interaction leads to amplification of the waviness in both the analysis and the forecast (Figs. 11a,b). Our case is thus characterized by baroclinic amplification of the wave patterns. In contrast, tropospheric-deep interaction has only a small influence on the hemispheric error evolution (Fig. 6a). In our case, there is consequently a difference between the processes contributing to the waviness evolution and the processes contributing to the error evolution. Previous studies (e.g., Zhang et al. 2007; Boisserie et al. 2014) often related synoptic-scale error growth to baroclinic instability since the largest error growth was found in baroclinic active regions. Our case indicates, however, that differences in the release of baroclinic instability are not necessarily a primary error growth mechanism and that it is important to distinguish the processes contributing to the error evolution from those contributing to the underlying Rossby wave evolution.

The differences between the individual contributions to the waviness evolution in the forecast and the analysis are not exceeding a magnitude of \( 3 \times 10^8 \text{PVU}^2 \text{m}^{-2} \text{s}^{-1} \) (Fig. 11c), while the largest values of the error tendency are much larger (\( 10 \times 10^8 \text{PVU}^2 \text{m}^{-2} \text{s}^{-1} \); Fig. 6a). Differences in the waviness evolution can thus only account for a small part of the error evolution. Thereby, this comparison supports our result that the error growth in our case is not primarily governed by differences in the (baroclinic) growth rate of the wave patterns. Since near-tropopause dynamics makes the largest contribution to the error growth, we rather suggest that the largest part of the error evolution is related to errors in the phase and the shape of the wave patterns. We hypothesize that errors generated near the tropopause (e.g., due to differences in upper-level divergence associated with latent heat release below) directly project onto the tropopause dynamics and then further amplify due to differences in the nonlinear Rossby wave dynamics (see section 4b). Our results thus indicate that the processes amplifying the errors can differ from the processes generating the initial error.

5. Summary and discussion

This study provides a quantitative analysis of error growth from a PV perspective. A tendency equation for error potential enstrophy is combined with the partitioning of PV tendencies as used in a previous analysis of Rossby wave dynamics (Teubler and Riemer 2016). This allows us to quantify the relative importance of four individual contributions to the PV error evolution near the tropopause, namely, near-tropopause dynamics (influence of upper-level PV anomalies), tropospheric-deep (baroclinic) interaction (influence of lower-tropospheric PV
FIG. 11. Individual contributions (colored lines) to spatially integrated waviness evolution on the 320-K surface as indicated by Eq. (11) for the (a) analysis, (b) forecast. (c) The difference between the spatially integrated waviness evolution in the forecast and the analysis. The integration area is the Northern Hemisphere (30°–80°N). The shading next to the near-tropopause and tropospheric-deep tendencies indicates the uncertainty of the result due to the uncertainty of PPVI.
and PV modification by nonconservative processes. The impact of upper-level anomalies on the tropopause evolution can be related to Rossby wave dynamics (e.g., Hoskins et al. 1985). In the PV framework, baroclinic instability is signified by the mutual amplification of anomalies in the upper and lower troposphere (Eady 1949; Hoskins et al. 1985; Davis and Emanuel 1991; Heifetz et al. 2004) and is thus captured by the tropospheric-deep tendency in our diagnostic. Upper-tropospheric divergent flow is mostly associated with latent heat release below and can thus be interpreted as an indirect impact of diabatic processes, which is of particular importance to ridge building (e.g., Davis et al. 1993; Riemer and Jones 2010; Grams et al. 2011; Riemer et al. 2014; Teubler and Riemer 2016). Indirect diabatic effects may be included also in the near-tropopause and tropospheric-deep tendencies due to the diabatic generation of PV anomalies at earlier times. The partitioning applied in this study therefore allows a quantification of whether baroclinic instability is a dominating error growth mechanism, as assumed in many previous studies.

Error growth is analyzed in a case study of a 6-day forecast for the Northern Hemisphere. The Atlantic and Pacific Rossby wave patterns turn out to be associated with the most pronounced error growth. After 2–3 forecast days, the pattern of the PV error changes from localized error maxima to wavelike error patterns that are located along the dynamical tropopause. The largest error amplification is found within the Rossby wave patterns.

In our case, near-tropopause dynamics clearly dominates the PV error growth near the tropopause, even in the conservative estimate, when all uncertainties of our diagnostic are attributed to the near-tropopause contribution. The near-tropopause error amplification in our case can almost entirely be related to differences in the nonlinear dynamics. During the prominent ridge building in the Atlantic (forecast days 2–3) and in the Pacific (forecast days 4–5), the divergent flow is of comparable importance to error growth as the conservative estimate of near-tropopause dynamics. Within the Pacific wave pattern during forecast days 4–5, tropospheric-deep interaction is also of comparable importance. It is of only minor importance within the Atlantic wave pattern. Significant tropospheric-deep and divergent tendencies can be related to a misrepresentation of the respective surface cyclone development in the forecast.

Considerations with the Eady model (Eady 1949; Davies and Bishop 1994) provide a plausible explanation for the limited importance of tropospheric-deep interaction to the error growth in our case. This idealized model demonstrates that significant tropospheric-deep error growth may occur only when the upper- and lower-level error patterns are in a specific, favorable configuration. However, the error patterns observed in our case do not exhibit the general tendency to develop toward such a configuration. A favorable configuration develops only in a limited region around forecast day 4, which is significantly later than assumed in previous studies (e.g., Zhang et al. 2007; Hohenegger and Schär 2007). Furthermore, the Eady model demonstrates that error growth due to tropospheric-deep interaction decreases with increasing wavenumber of the error pattern. Within the 6-day forecast period investigated here, the spatial scale of the error features is significantly smaller than that of anomalies governing baroclinic development of the midlatitudes. The small importance of tropospheric-deep interaction to error growth is thus also conceptually supported by the idealized considerations with the Eady model.

Our diagnostic reveals a systematic negative residual of considerable magnitude. In accordance with previous authors (e.g., Nie et al. 2016), we argue that this residual can be interpreted as an additional physical process related to the downscale enstrophy cascade of quasi-horizontal flow: the filamentation of PV error features and their effective diffusion at the grid scale. This process constitutes an error sink that cannot be quantified from forecast models’ parameterization schemes. We are currently designing idealized numerical experiments to further investigate the role of the enstrophy cascade in the error evolution.

We emphasize that our analysis quantifies the direct impact of processes on the amplification of existing errors, in contrast to several previous studies, which tracked the accumulation of specific error sources with time using Lagrangian diagnostics (e.g., Martínez-Alvarado et al. 2016). Our approach provides useful insight into the error-growth dynamics, as the processes governing the error growth may differ from the processes generating the error (e.g., Zhang et al. 2007; Hohenegger and Schär 2007; Selz and Craig 2015). Using this approach, we demonstrate that at any given time of the 6-day forecast analyzed here, it is mostly differences in the nonlinear near-tropopause dynamics that amplify existing errors in the tropopause region. Differences in baroclinic instability have instead only a weak (direct) impact on the error growth in our case. Our results are in general agreement with Snyder (1999) and Plu and Arbogast (2005), who showed that nonlinear dynamics can be as important to the error growth as baroclinic and barotropic instabilities. It is clear, however, that a single case study is not sufficient to make
robust, general statements about the processes governing the error growth. Preliminary results from a few additional cases [including the “forecast bust” from Rodwell et al. (2013)], however, indicate that near-tropopause dynamics is, in all of these cases, of large importance to the error growth.

Arguably, the relative importance of the processes governing error growth is regime dependent. A systematic investigation of different synoptic regimes is planned to gain deeper insight into this dependence. Furthermore, we plan to adopt the method presented herein to investigate the divergence of the members in an ensemble forecast system. Thereby, the processes that govern the evolution of the ensemble spread (i.e., forecast uncertainty) could be quantified.

An improved understanding of error growth and the amplification of forecast uncertainty may help to improve the interpretation, and possibly the design, of future forecast systems. Based on our results, we hypothesize that the nonlinear dynamics of high-amplitude Rossby wave patterns is most susceptible to rapid growth of forecast errors and uncertainty. Baroclinic instability and diabatic processes, including the effect of upper-level divergence errors and uncertainty. PV tendencies and is small enough that it does not affect the physical interpretation of our results.

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APPENDIX A

Partitioning of the Individual Processes

For the partitioning of the wind field [Eq. (6)], we employ a Helmholtz partitioning [following Lynch (1989)] to calculate the nondivergent and irrotational wind components. The nondiabatic flow is further partitioned using piecewise PV inversion (PPVI) under nonlinear balance (Charney 1955; Davis 1992). The PPVI is performed on the Northern Hemisphere between 25° and 85°N on 15 evenly spaced pressure levels, with vertical boundary conditions specified at 875 and 125 hPa. Anomalies are defined as deviations from a background state, which is here defined as the 30-day mean (centered at 0000 UTC 14 November 2013) of the analysis. We use a midtropospheric pressure level (600 hPa) as the separation level between upper- and lower-level PV anomalies. The partitioning yields a residual component, which is due to the harmonic flow component; nonlinear effects in the PPVI; and uncertainties in the PPVI’s boundary conditions. This residual does not exhibit a systematic impact on upper-level PV tendencies and is small enough that it does not affect the physical interpretation of our results.

APPENDIX B

Eady Model in the Reformulation of Davies and Bishop (1994)

The Eady model (Eady 1949) in the reformulation of Davies and Bishop (1994) comprises quasigeostrophic channel flow with walls at \( y = (0, \pi/l) \). We set \( l = 6 \) and the height of the domain to \( d = 10^4 \text{m} \). The dynamics of the model are governed by the distribution of \( \theta \) at the upper \((z = d)\) and the lower \((z = 0)\) boundaries.

It is straightforward to derive an error tendency equation for the Eady model analogous to Eq. (4), but for \( \theta \) instead of PV. As the setup of the Eady model is adiabatic and nondivergent, the error tendency equation is simply given by

\[
\frac{\partial}{\partial t} \frac{\theta^*}{2} = -\theta^* \nabla \theta \cdot \frac{\theta^*}{2} \left( \theta^* + \theta \right). \quad (B1)
\]

Again, the second term merely redistributes the error pattern and is therefore not further considered.

Since we are interested in error growth by tropospheric-deep interaction near the tropopause (represented by the upper boundary in the Eady model), it is sufficient to analyze the tropospheric-deep error tendency at \( z = d \):

\[
\frac{\partial}{\partial t} \frac{\theta^*}{2} \bigg| \text{TPd}, z = d = -\theta^* \nabla \theta \cdot \text{TPd}, z = d \cdot \nabla \theta \bigg| z = d. \quad (B2)
\]

At \( z = d \), an idealized analysis state is represented by a zonal background state and a wavenumber-6 pattern:
$$\theta_{z=d} = \frac{\theta_{u}}{g} + \frac{f}{g} \delta \Lambda y + \theta_{0,T} \sin(k_T x + \delta_T) \sin(ly),$$  \hspace{1cm} (B3)$$

where $\Lambda = 3 \times 10^{-3}$ s$^{-1}$ is the uniform vertical shear, $f = 10^{-4}$ s$^{-1}$ is the Coriolis parameter, $\theta_u = 300$ K is the reference temperature, and $g$ is the gravitational acceleration. We choose $\theta_{0,T} = 10$ K and $\delta_T = 0$.

We introduce error patterns at the upper and lower model levels as Eady edge waves:

$$\theta^\ast_T(z) = \frac{\mu^\ast_T}{\mu^\ast_T - 2} \sin(\mu^\ast_T x + \delta^\ast_T) \sin(ly),$$  \hspace{1cm} (B4)$$

$$\theta^\ast_B(z) = \frac{\mu^\ast_B}{\mu^\ast_B - 2} \sin(\mu^\ast_B x + \delta^\ast_B) \sin(ly),$$  \hspace{1cm} (B5)$$

with $\mu^\ast_T = (\mathcal{N} / f)^{1/2}$ and $\mu^\ast_B = (N / f)^{1/2}$ and Brunt–Väisälä frequency $N = 10^{-2}$ s$^{-1}$. By definition, $\theta^\ast_T$ and $\theta^\ast_B$ are zero at the respective other boundary. Since the amplitude and scale of the error patterns observed in our case study are, in general, smaller than those of the analysis pattern, we choose $\theta^\ast_T = 5$ K, $k^\ast_T \approx k_T$, and $k^\ast_B \approx k_T$. The tropospheric-deep contribution to the error wind is given by the wind associated with the lower error wave $\theta^\ast_B$:

$$u^\ast_{TPd,z=d} = \frac{g}{f} \frac{k^\ast_B}{\mu^\ast_B} \frac{1}{\sin(\mu^\ast_B)} \theta^\ast_B \sin(k^\ast_B x + \delta^\ast_B) \cos(ly),$$  \hspace{1cm} (B6)$$

$$v^\ast_{TPd,z=d} = \frac{g}{f} \frac{k^\ast_B}{\mu^\ast_B} \frac{1}{\sin(\mu^\ast_B)} \theta^\ast_B \cos(k^\ast_B x + \delta^\ast_B) \sin(ly).$$  \hspace{1cm} (B7)$$

For Fig. 8a, we consider error growth by tropospheric-deep interaction [spatial integration of Eq. (B2)] as a function of $k^\ast_B$ and $\theta^\ast_B$ for a fixed upper-level error pattern ($k^\ast_T = 8$ and $\delta^\ast_T = 0$). Varying the parameters of the upper error wave reveals that the maximal baroclinic error growth is always attained for $k^\ast_B = k^\ast_T$ and $\delta^\ast_B - \delta^\ast_T = \pi/2$ (not shown). The magnitude of this maximum is displayed in Fig. 8b as a function of the error wavenumber.

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