A New Dynamical Core of the Global Environmental Multiscale (GEM) Model with a Height-Based Terrain-Following Vertical Coordinate

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ABSTRACT

A new dynamical core of Environment and Climate Change Canada's Global Environmental Multiscale (GEM) atmospheric model is presented. Unlike the existing log-hydrostatic-pressure-type terrain-following vertical coordinate, the proposed core adopts a height-based approach. The move to a height-based vertical coordinate is motivated by its potential for improving model stability over steep terrain, which is expected to become more prevalent with the increasing demand for very high-resolution forecasting systems. A dynamical core with height-based vertical coordinate generally requires an iterative solution approach. In addition to a three-dimensional iterative solver, a simplified approach has been devised allowing the use of a direct solver for the new dynamical core that separates a three-dimensional elliptic boundary value problem into a set of two-dimensional independent Helmholtz problems. The issue of dynamics–physics coupling has also been studied, and incorporating the physics tendencies within the discretized dynamical equations is found to be the most acceptable approach for the height-based vertical coordinate. The new dynamical core is evaluated using numerical experiments that include two-dimensional nonhydrostatic theoretical cases as well as 25-km resolution global forecasts. For a wide range of horizontal grid resolutions—from a few meters to up to 25 km—the results from the direct solution approach are found to be equivalent to the iterative approach for the new dynamical core. Furthermore, results from the different numerical experiments confirm that the new height-based dynamical core is equivalent to the existing pressure-based core in terms of solution accuracy.

1. Introduction

The dynamical core of the Global Environmental Multiscale (GEM) model, used operationally by Environment and Climate Change Canada (ECCC) for numerical weather prediction (NWP), employs a log-hydrostatic-pressure-type terrain-following vertical coordinate. The system of nonlinear model equations is linearized around a basic state and is then reduced to an elliptic boundary value (EBV) problem through numerical discretization and elimination of variables (Girard et al.

2014). The existing pressure-type coordinate system then permits the use of a direct solver for the discretized EBV problem to resolve the dynamical component of the flow. The direct solver starts by separating the EBV problem vertically in terms of the vertical eigenvectors of the part of the coefficient matrix that only includes the discretized difference and average operators in the vertical direction (Qaddouri and Lee 2010). For N number of model vertical levels, the resulting N vertically decoupled two-dimensional Helmholtz problems are then separated along the longitude, leading to a system of tridiagonal problems depending only on the latitude. The tridiagonal problems are finally solved using LU decomposition without pivoting. Such an approach is computationally more efficient than most iterative methods, particularly for the spatial resolutions of the current operational NWP systems at ECCC.

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One of the principal incentives for the adoption of the existing pressure-type vertical coordinate in GEM is the computational advantage of the direct solution approach that is permissible with such a coordinate. However, numerical experiments carried out at ECCC have revealed that vertical separability, which is an imperative for the direct solution approach, can become quite restrictive for very high spatial resolutions (e.g., for subkilometer horizontal grid spacing). Furthermore, the reduction of the 2D Helmholtz problems into the 1D tridiagonal problems requires projection of right hand side (rhs) of the 2D problems along the pertinent eigenvectors which is based on Fourier transformation. This necessitates transposing the coefficient matrix that involves global communications, and therefore, becomes inefficient for very large number of processor cores. As a result, the direct solver is found to lose scalability with increasing number of processor cores. Initial research at ECCC as well as published research literature (Müller and Scheichl 2014) reveal that optimized three-dimensional iterative solvers may possess better scalability in these circumstances. The different limitations of the existing direct solver, particularly its potential lack of scalability for future generations of massively parallel supercomputers, therefore warrants the development of more scalable iterative solvers at ECCC. A height-based vertical coordinate is considered to be more amenable to such iterative solvers as the metric terms originating from the vertical coordinate transformation appear explicitly in the discretized EBV.

Another, but more challenging, problem pertaining to the existing GEM dynamical core is the fact that the current model exhibits strong numerical instability over steep orography. Tests conducted at ECCC have determined the maximum permissible terrain slope for maintaining stability to be approximately 45° (Vionnet et al. 2015). This has been considered to be a limitation inherent to the terrain-following coordinate (TFC) systems (Zängl 2012). With a growing demand for very high-resolution operational NWP systems, steep orographic slopes are expected to become more prevalent in the near future. Improving model stability over steep mountains is therefore of critical importance for the future development of subkilometer NWP systems. A number of approaches have been investigated to improve numerical stability with the existing pressure-type vertical coordinate in GEM. These include increased off-centering in the discretized vertical momentum equation, a vertically variable basic state temperature profile, and modifications to the nonhydrostatic contributions in the linear system arising from the discretized GEM formulation. However, none of these approaches

has been found to lead to any meaningful stability improvement for steep orography.

Although the instability induced by steep orography is often characterized as a limitation of the terrain-following nature of the vertical coordinate itself, Smolarkiewicz et al. (2007) were successful in resolving flow around the Pentagon with a model involving height-based TFC where the maximum slope was well above the widely acknowledged 45° threshold. Previous experience with the Mesoscale Compressible Community model at ECCC also suggests that a dynamical core with height-based TFC does not suffer from similar severe orography-induced instability (Girard et al. 2005). Apparently, a dynamical core with a height-based TFC can lead to improved numerical stability through better implicit treatment of the metric terms arising from the vertical coordinate transformation through iterative solvers. More importantly, conventional numerical approximation of the horizontal gradients in a TFC becomes less accurate with increasing terrain slope as well as with increasing vertical resolution close to the model surface (Mahrer 1984). In this context, Zängl (2012) argues that the pressure gradient term, in particular, becomes susceptible to triggering numerical instability when the height difference between two adjacent grid points along a terrain-following vertical level is much larger than the vertical grid resolution adjacent to the level. Numerical approximation of the horizontal gradient terms in the TFC, however, can be significantly improved following the approach proposed by Mahrer (1984). This approach requires determination of the modified horizontal differencing stencils associated with each gridpoint location that minimize the error in the metric corrections for the terrain-following nature of the coordinate. The existing pressure-type TFC varies with time and, therefore, would require determination of the pertinent gridpoint locations in the vertical for the modified differencing stencils at every time step. On the contrary, the height-based TFC is time-invariant and thus would require the determination of these gridpoint locations only once at the beginning of the time integration. Therefore, from a computational efficiency standpoint, a height-based TFC is more suitable for implementing improved numerical approximation of horizontal gradients to address instabilities induced by steep orography.

The aforementioned challenges associated with the existing log-hydrostatic-pressure-type TFC motivated the development of a new dynamical core for the GEM model that utilizes a height-based TFC. The primary objective of the present study is to demonstrate that, for the model configurations where orography-induced

numerical instability is not relevant—that is, for horizontal grid resolutions within the hydrostatic regime—the new dynamical core developed at ECCC with height-based TFC makes predictions that are equivalent to those from the existing model. Another major objective is to propose a strategy that enables the utilization of the direct solver to make the new dynamical core comparable to the existing one in terms of computational performance for spatial resolutions associated with the different GEM-based operational NWP systems. The present study also explores the appropriate strategy for coupling the operational Physics Parameterization Package (PPP) of RPN (Recherche en prévision numérique) with the new height-based dynamical core. Different setups for numerical experiments, covering both hydrostatic and nonhydrostatic scenarios, are utilized to compare the two dynamical cores. The experiments include two-dimensional theoretical cases (Robert 1993; Schär et al. 2002) as well as three-dimensional global forecasts over a Yin–Yang grid (Qaddouri and Lee 2011).

Relevant background information on the GEM dynamical core with the proposed height-based TFC—from the spatial and temporal discretizations to the derivation of the elliptic boundary value problem—is presented in section 2. The different solution methods utilized for the discretized elliptic problem are discussed in section 3. The issue of coupling between the dynamical core and the parameterized physics forcings is presented in section 4. Section 5 contains the comparisons between the existing and the proposed dynamical cores in the context of two-dimensional theoretical benchmark cases as well as three-dimensional deterministic global predictions. The conclusions are then summarized in section 6.

2. Model description

a. Governing equations

The GEM model equations originate from the Euler equations. With the traditional shallow atmosphere approximation (Phillips 1966), the system of equations in a spherical coordinate (λ, ϕ, r) can be expressed as follows:

$$\frac{du}{dt} - \left(f + \frac{\tan\phi}{a}u\right)v + \frac{1}{\rho}\frac{\partial p}{\partial x} = \left(\frac{du}{dt}\right)_{\text{phys}}, \quad (1)$$

$$\frac{dv}{dt} + \left(f + \frac{\tan\phi}{a}u\right)u + \frac{1}{\rho}\frac{\partial p}{\partial y} = \left(\frac{dv}{dt}\right)_{\text{phys}}, \quad (2)$$

$$\frac{dw}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = \left(\frac{dw}{dt}\right)_{\text{phys}}, \quad (3)$$

$$\frac{d\ln\rho}{dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} - \frac{\tan\phi}{a}v = \left(\frac{d\ln\rho}{dt}\right)_{\text{phys}}, \quad (4)$$

$$\frac{d\ln T}{dt} - \kappa \frac{d\ln p}{dt} = \left(\frac{d\ln T}{dt}\right)_{\text{phys}}, \quad (5)$$

where Eqs. (1)–(5) govern the evolutions of the u , v , and w components of velocity, mass, and energy, respectively. The spatial coordinates in the above equations are denoted by (x, y, z) , which are related to the spherical coordinate (λ, ϕ, r) through the differential relations given by

$$dx = a \cos\phi d\lambda, \quad dy = a d\phi, \quad dz = dr, \quad (6)$$

such that u , v , and w are the physical wind components. In Eq. (6), a denotes Earth's radius. The Lagrangian derivative in this case can be expressed as

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}. \quad (7)$$

In addition to the four independent variables (x, y, z, t) , the system of five Eqs. (1)–(5) involves six dependent variables, namely, the velocity components u , v , and w , the temperature T , the pressure p , and the density ρ . Also, in the above equations, f is the Coriolis parameter and $\kappa = R/c_p$, where R is the total gas constant and c_p is the specific heat of moist air at constant pressure. The terms on the rhs of Eqs. (1)–(5) with subscript “phys” denote the various physical forcings. Depending on the equation, these physical forcings may arise from different sources that include friction, diabatic heating and frictional dissipation of kinetic energy. A sixth equation is required to close the system described by the six dependent variables, and is provided by the equation of state, given by

$$p = \rho RT. \quad (8)$$

It is important to note that the atmospheric substance does not only contain dry air but also water vapor and different types of hydrometeors. The displacement and evolution of water vapor and hydrometeors in the atmosphere are governed by their own evolution equations. However, they will also affect the rhs terms through fluxes of water vapor and precipitation which constitute sources of mass. The total air density in the presence of water in its different forms can be expressed as

$$\rho = \rho_d + \rho_w = \rho_d(1 + r_w), \quad (9)$$

where ρ_d is the dry air density, ρ_w is the density of water vapor and hydrometeors, and $r_w = \rho_w/\rho_d$ is the mixing

ratio for the total water content of the atmosphere. The equation of state is then strictly given by

$$p = (\rho_d R_d + \rho_v R_v) T, \quad (10)$$

where ρ_v and R_v are the density and gas constant of water vapor. Equation (10) can then be further rearranged in terms of the total air density ρ as

$$p = \rho R_d T_v, \quad (11)$$

where T_v is the virtual temperature of moist air which is given by

$$T_v = \frac{1 + \frac{R_v}{R_d} r_v}{1 + r_v} T, \quad (12)$$

where $r_v = \rho_v / \rho_d$ is the water vapor mixing ratio. Rewriting the dynamical Eqs. (1)–(5) in terms of T_v is helpful as the equations can then be expressed using only the dry gas constant that does not vary due to the atmospheric water content. It also allows to implicitly account for the effects of water vapor buoyancy and condensed water loading. Furthermore, from the adiabatic point of view, the introduction of T_v has no consequence since the water content is conserved during dynamical transport.

b. Vertical coordinate

The log-hydrostatic-pressure-type terrain-following vertical coordinate of the operational GEM model (Girard et al. 2014) has the following form:

$$\ln \pi = \xi + Bs, \quad (13)$$

where ξ defines the terrain-following vertical coordinate, π denotes the hydrostatic pressure, B is a metric term prescribing the rate of flattening of the vertical coordinate with elevation, and $s = \ln(\pi_s / p_{\text{ref}})$ with π_s being the hydrostatic pressure at the surface and $p_{\text{ref}} = 10^3$ hPa is a reference pressure. The definition of this vertical coordinate follows the concept of the generalized hydrostatic-pressure-type hybrid coordinate proposed by Laprise (1992). Further details regarding the log-hydrostatic-pressure-type vertical coordinate are provided by Girard et al. (2014) and Husain and Girard (2017).

The present study proposes to develop a GEM dynamical core where the vertical coordinate, given by Eq. (13), is replaced by a height-based TFC. The traditional formulation for height-based TFC, as proposed by Gal-Chen and Somerville (1975), can be expressed as

$$\zeta(z) = H \frac{z - z_s}{z_T - z_s}, \quad (14)$$

where z is the true geometric height, z_s and z_T are the surface and the model top level heights, and H is a scaling constant. Equation (14) can be further generalized as

$$z = A + Bz_s, \quad (15)$$

where $A = (z_T/H)\zeta$ and $B = (1 - \zeta/H)$. Assigning $H = z_T$ implies $z_T = \zeta_T$ and, as a result, Eq. (15) becomes

$$z = \zeta + Bz_s, \quad (16)$$

which is similar to Eq. (13) in form. The vertical coordinate for the proposed dynamical core in the present study, however, is modified as

$$z = \zeta + B_1 z_{\text{SL}} + B_2 (z_s - z_{\text{SL}}), \quad (17)$$

which follows the concept of the smooth level vertical (SLEVE)-like coordinate system proposed by Schär et al. (2002), where z_{SL} denotes the large-scale components of the orography. The vertical coordinate defined by Eq. (17) permits separate rates of flattening for the large and small-scale contributions of the orography on the terrain-following vertical coordinate with changing elevation through the metric terms B_1 and B_2 that are defined as

$$B_n = \left(\frac{\zeta_T - \zeta}{\zeta_T - \zeta_s} \right)^{r_n}, \quad (18)$$

where $r_n = [r_{n,\text{max}} - (r_{n,\text{max}} - r_{n,\text{min}})\lambda_k]$ and $\lambda_k = (\zeta_1 - \zeta_k)/(\zeta_1 - \zeta_s)$. The values of $r_{n,\text{min}}$ and $r_{n,\text{max}}$ together determine the rate of flattening of the vertical levels with increasing height. The subscript k of λ indicates the model vertical level number. The value of k decreases with increasing height above the surface such that $k = 1$ indicates the top-most model level. The values of $r_{n,\text{min}}$ and $r_{n,\text{max}}$ are selected to maximize the latter and minimize the former in such a way that allows quick flattening of the terrain effects on the coordinate at the upper levels while ensuring that the vertical levels are not excessively squeezed near the surface over complex terrain.

Henceforth, in this paper, the two dynamical cores with vertical coordinates based on log-hydrostatic-pressure and height are referred to as GEM-P and GEM-H, respectively. Different aspects of the GEM-P dynamical core, including the model formulation, discretization and numerical solution of the discretized problem along with the various modifications to the formulation over the past years, have been discussed in detail in the existing literature (Yeh et al. 2002; Qaddouri and Lee 2011; Girard et al. 2014; Husain and Girard 2017). The following subsections therefore only present the relevant details of the proposed GEM-H dynamical core.

c. GEM-H formulation

The development of the GEM-H formulation requires further modifications to the system of Eqs. (1)–(5). First, virtual temperature, given by Eq. (12), is introduced in the system of equations along with an isothermal basic state temperature T_* such that $T_v = T'_v + T_*$, where T'_v is the temperature deviation. The corresponding basic state pressure p_* is defined hydrostatically as

$$\frac{\partial(\ln p_*)}{\partial z} = -g/(R_d T_*). \quad (19)$$

It is important to note that the standard value of T_* used in GEM is 240 K, whereas the value of p_* at $z = 0$ is set to 10^5 Pa. The equation of state, given by Eq. (11), is then used to eliminate density ρ as a prognostic variable followed by a transformation of the resulting equations from the geometric height coordinate to the terrain-following ζ coordinate. The vertical coordinate transformation leads to the replacements of the independent variables (x, y, z) associated with the z coordinate by (X, Y, ζ) that are defined in the ζ coordinate. As a result, the system of Eqs. (1)–(5) is modified as follows:

$$\frac{dv}{dt} - \left(f + \frac{\tan\phi}{a}u\right)v + \frac{T_v}{T_*} \left(\frac{\partial q}{\partial X} - \frac{J_X}{J_\zeta} \frac{\partial q}{\partial \zeta}\right) = \left(\frac{du}{dt}\right)_{\text{phys}}, \quad (20)$$

$$\frac{dv}{dt} + \left(f + \frac{\tan\phi}{a}u\right)u + \frac{T_v}{T_*} \left(\frac{\partial q}{\partial Y} - \frac{J_Y}{J_\zeta} \frac{\partial q}{\partial \zeta}\right) = \left(\frac{dv}{dt}\right)_{\text{phys}}, \quad (21)$$

$$\frac{dw}{dt} + \frac{T_v}{T_*} \left(\frac{1}{J_\zeta} \frac{\partial q}{\partial \zeta} - g \frac{T'_v}{T_v}\right) = \left(\frac{dw}{dt}\right)_{\text{phys}}, \quad (22)$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{q}{c_*^2} + \ln J_\zeta \right) + \frac{\partial u}{\partial X} + \frac{1}{\cos\phi} \frac{\partial(\cos\phi v)}{\partial Y} + \frac{\partial \dot{\zeta}}{\partial \zeta} - \frac{g}{c_*^2} w \\ = \left[\frac{d \ln(\rho T_v)}{dt} \right]_{\text{phys}}, \end{aligned} \quad (23)$$

$$\frac{d}{dt} \left[\ln \left(\frac{T_v}{T_*} \right) - \frac{q}{c_{\text{pd}} T_*} \right] + \frac{N_*^2}{g} w = \left(\frac{d \ln T_v}{dt} \right)_{\text{phys}}, \quad (24)$$

where $q = R_d T_* \ln(p/p_*)$ is a measure of the pressure deviation from the basic-state pressure p_* , $\dot{\zeta} = d\zeta/dt$ is the vertical motion with respect to the transformed ζ coordinate, $N_*^2 = g^2/(c_{\text{pd}} T_*)$ is the square of the basic-state Brunt–Väisälä frequency, and $c_*^2 = R_d T_*/(1 - \kappa_d)$ is

the square of the speed of sound. In Eq. (23), κ is replaced by $\kappa_d = R_d/c_{\text{pd}}$ as an approximation. It is also important to note that, the physical forcings associated with the modified continuity equation, given by Eq. (23), now includes the same diabatic heating term that appears in the thermodynamic equation, given by Eq. (24).

In the above equations, the terms J_X , J_Y , and J_ζ appear due to the vertical coordinate transformation where $J_X = \partial z/\partial X$, $J_Y = \partial z/\partial Y$, and $J_\zeta = \partial z/\partial \zeta$. It is, however, important to note that the coordinate transformation used to derive the Eqs. (20)–(24) is incomplete and only the following derivative operators have been transformed:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} - \frac{J_X}{J_\zeta} \frac{\partial}{\partial \zeta}, \quad (25)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial Y} - \frac{J_Y}{J_\zeta} \frac{\partial}{\partial \zeta}, \quad (26)$$

$$\frac{\partial}{\partial z} = \frac{1}{J_\zeta} \frac{\partial}{\partial \zeta}, \quad (27)$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial X} + v \frac{\partial}{\partial Y} + \dot{\zeta} \frac{\partial}{\partial \zeta}. \quad (28)$$

As a result of the incomplete transformation, the original vertical velocity w has not been completely eliminated from the system. The system of equations in the ζ coordinate has its own vertical velocity in the form of $\dot{\zeta} = d\zeta/dt$, whereas $w = dz/dt$ remains in the system as a kinematic relation that needs to be dealt with explicitly. Charron et al. (2014) have demonstrated that such an approach is perfectly equivalent to a full coordinate transformation. As the treatment of advection in GEM is based on the semi-Lagrangian approach (Husain and Girard 2017), the kinematic relation defining w is also solved semi-Lagrangially. However, for convenience, the kinematic relation is modified as

$$\frac{d}{dt}(z - \zeta) + \dot{\zeta} - w = 0, \quad (29)$$

where Eq. (29) along with the Eqs. (20)–(24) constitute the complete system of equations for the GEM-H formulation.

One important aspect of any NWP model is how the effects of the physics forcings, as presented in the rhs of the Eqs. (20)–(24), are accounted for as the model equations are integrated at each time step. One way is to resolve the dynamical equations in the absence of any physics forcing and then modify the solution with the parameterized forcing as adjustments outside the dynamics step. Another possible approach is to compute the physics forcing and combine their effects as tendencies within the dynamics step. This important

FIG. 1. (a) Vertical Charney–Phillips grid. (b) Horizontal Arakawa C grid.

aspect of the GEM model with a particular focus on its impact pertaining to the GEM-H dynamical core is discussed in further details in [section 4](#).

d. Spatial grid and discretization

The primary objective of this project has been to implement the option of a dynamical core based on height-type vertical coordinate in addition to the existing pressure-type coordinate in the GEM model. The strategy has been to add the new coordinate option with minimal changes to the dynamical core and the rest of the GEM model source code. Therefore, the spatial grid structures in GEM-H, both in the horizontal and the vertical, are kept the same as those in GEM-P, which implies a staggered Arakawa C grid ([Arakawa 1988](#)) in the horizontal and a staggered Charney–Phillips grid ([Charney and Phillips 1953](#)) in the vertical. The horizontal and vertical grid structures are presented in [Fig. 1](#).

In addition to being similar to GEM-P for the limited-area model (LAM) grids, the global grid system is also kept unchanged in GEM-H, and is therefore, based on a Yin–Yang grid system ([Qaddouri and Lee 2011](#)). The Yin–Yang system combines two overlapping latitude–longitude LAM grids to form a global grid following the Schwarz method for nonmatching domain decomposition ([Qaddouri et al. 2008](#)) and thus avoids pole-related singularity and convergence issues associated with a conventional global lat–lon grid. Further details on the Yin–Yang grid are provided by [Qaddouri and Lee \(2011\)](#).

e. Discretization in time

The general form of an individual equation in the system comprising Eqs. (20)–(24) and (29) can be expressed as

$$\frac{dF_i}{dt} - G_i = P_i, \quad (30)$$

where F_i denotes the advected quantity for an individual equation i within the system, G_i is the associated dynamics source term with linear and nonlinear components, and P_i denotes the corresponding physics forcing. Similarly to GEM-P, treating the advection terms in a semi-Lagrangian way and applying a two-time-level Crank–Nicholson temporal discretization leads to

$$\frac{F_i^A - F_i^D}{\Delta t} - \frac{1+b_i}{2} G_i^A - \frac{1-b_i}{2} G_i^D = \mathbf{s}_c \bar{P}_i, \quad (31)$$

where Δt indicates the time-step length, and the superscripts A and D denote the arrival and departure positions of the air parcels at the current time t and the previous time $(t - \Delta t)$, respectively. The integrals of the source terms G_i for the different dynamical equations are approximated by trajectory averages. The parameter b_i denotes the off-centering weight factor for the averaging of the dynamics source terms. When $b_i = 0$, the averaging of the source term is fully centered, whereas $b_i > 0$ implies additional weight placed on the implicit component of the discretized source term.

Historically, off-centering was implemented in GEM-P primarily to address spurious resonance originating from stationary orographic forcing ([Rivest et al. 1994](#)). However, it also suppresses computational noise and improves numerical stability. These other beneficial impacts have been found to be equally important in the current and previous implementations of GEM-P for the different operational NWP systems at ECCC. As the principal objective of this study is to have a GEM-H dynamical core that is equivalent to GEM-P for the different operational GEM-based NWP systems,

off-centering has been retained in GEM-H. Also, following the latest implementation of GEM-P (Husain and Girard 2017), a differential approach for off-centering has been adopted for GEM-H where b_i varies depending on the dynamical equation denoted by the subscript i . At present, the system of equations, given by Eqs. (20)–(24) and (29), are separated into three groups with an associated off-centering parameter for each group as follows:

- (i) b_m for the horizontal momentum equations [Eqs. (20) and (21)],
- (ii) b_h for the continuity and thermodynamic equations [Eqs. (23) and (24)], and
- (iii) b_{nh} for the vertical momentum and kinematic equations [Eqs. (22) and (29)].

On the rhs of Eq. (31), the term $\overline{P}_i$ denotes the parameterized physics source term and the parameter s_c indicates the mode of coupling between physics and dynamics. Depending on the chosen method for dynamics–physics coupling, the value of s_c can be either 0 or 1. Also, the approach for dynamics–physics coupling determines how the contribution of $\overline{P}_i$ is accounted for in the model. Further discussions regarding the coupling of the parameterized physics forcing with the dynamical core is presented in section 4.

f. Trajectory calculations

Semi-Lagrangian treatment of advection requires the solution of kinematic displacement equations of the following form:

$$\frac{dX}{dt} = u, \quad \frac{dY}{dt} = v, \quad (32a)$$

$$\frac{d\zeta}{dt} = \dot{\zeta}, \quad (32b)$$

to determine the departure positions of the air parcels. In the context of GEM-P, Husain and Girard (2017) have shown that the consistency in the numerical discretizations between the dynamical and trajectory equations is important for accurate solution of the advection problem. To be numerically consistent, similarly to the treatment of the dynamics source term in Eq. (30), the averaging of the velocities in Eq. (32) needs to be done using the trapezoidal rule. Furthermore, the interpolation scheme employed to determine the wind field at the departure positions for the trajectory calculations ideally should be the same as the one applied to determine the source terms in the dynamical equations at the departure positions. In the case of GEM-P, cubic interpolation is used for both the wind field and the dynamical source terms to achieve numerical consistency. Following the conclusions of Husain and Girard

(2017), similar consistent trajectory calculation approach is adopted in GEM-H, that is, trapezoidal rule for evaluating the integral of the source term in Eq. (32) along with cubic interpolation to determine the wind field at the departure positions.

g. The elliptic problem

To solve the system of equations associated with the GEM-H formulation, each equation of the form in Eq. (31) is rearranged to separate the linear and nonlinear components of the implicit part and is expressed as

$$L_i = R_i - N_i, \quad (33)$$

where $L_i = (F_i^A/\tau_i - G_i^A)_{\text{linear}}$ includes the linearized lhs terms, $N_i = F_i^A/\tau - G_i^A - L_i$ contains the remaining nonlinear terms, and $R_i = F_i^D/\tau_i + \beta_i G_i^D + s_c \overline{P}_i$ with $\beta_i = (1 - b_i)/(1 + b_i)$ and $\tau_i = \Delta t(1 + b_i)/2$. It is important to note that, based on the choice of the linear terms in L_i , the nonlinear N_i terms may include some linear residuals as well. Equations (20)–(24) and (29) are rearranged as in Eq. (33), giving

$$L_u = \frac{u}{\tau_m} + \delta_X q - s_i J_X \overline{J_\zeta^{-1} \delta_\zeta q}^{X\zeta}, \quad (34)$$

$$L_v = \frac{v}{\tau_m} + \delta_Y q - s_i J_Y \overline{J_\zeta^{-1} \delta_\zeta q}^{Y\zeta}, \quad (35)$$

$$L_w = \frac{w}{\tau_{nh}} + (s_i J_\zeta^{-1} + s_d) \delta_\zeta q - g \frac{T'_v}{T_v}, \quad (36)$$

$$L_c = \frac{1}{\tau_h} \left(\frac{q}{c_*^2} + \ln J_\zeta \right) + \delta_X u + \frac{1}{\cos \phi} \delta_Y (\cos \phi_v) + \delta_\zeta \dot{\zeta} - \varepsilon \overline{w}^\zeta, \quad (37)$$

$$L_T = \frac{1}{\tau_h} \left(\frac{T'_v}{T_v} - \frac{\overline{q}^\zeta}{c_{pd} T_*} \right) + \mu w, \quad (38)$$

$$L_z = \frac{z - \zeta}{\tau_{nh}} + \dot{\zeta} - w, \quad (39)$$

where the symbol δ_X denotes the finite difference operator along the X direction, and the overline operator $\overline{(\)}^X$ implies spatial averaging in the X coordinate. For convenience of notation, the superscript A has been removed from the F_i^A terms in the above expressions for L_i . Also, the terms g/c_*^2 and N_*^2/g have been replaced by ε and μ , respectively. The corresponding nonlinear components N_i associated with the discretized forms of the Eqs. (20)–(24) and (29) as well as the associated rhs terms R_i are provided in appendix A. The parameters s_i

and $\mathbf{s}_d$ in the above equations denote the choice of the solver for the dynamical core, where the subscripts i and d stand for iterative and direct, respectively. Based on the selected solution approach, these parameters can be either 1 or 0, and are mutually exclusive such that $\mathbf{s}_i = 1 - \mathbf{s}_d$. Further discussion on these solution approaches is presented in section 3. As shown by Côté and Staniforth (1988), the solution of the system of equations of the type Eq. (33) requires nonlinear iterations for convergence, where the nonlinear terms N_i are reevaluated during each iteration using the latest values of the prognostic variables. Furthermore, Crank–Nicholson iterations are required, where the R_i terms are reevaluated during each iteration at the departure positions calculated using the latest velocity estimates. At present, the GEM-based operational NWP systems at ECCC utilize two Crank–Nicholson iterations, and two nonlinear iterations are carried out within each Crank–Nicholson step. As a result, irrespective of the solution approach, the solver is called four times during each dynamical time step.

The discretized equations with the left hand sides (lhs) given by Eqs. (34)–(39) are then reduced into a single elliptic boundary value (EBV) problem through elimination of variables, where the lhs of the final elliptic problem takes the form

$$L_c''' = \delta_X \mathcal{A} + \frac{1}{\cos\phi} \delta_Y [\cos\phi \mathcal{B}] + \delta_\zeta \mathcal{C} - \varepsilon \overline{\mathcal{C}}^\zeta - \gamma q, \quad (40)$$

where $\overline{\mathcal{A}}^\zeta = (\delta_X q - \mathbf{s}_i J_X J_\zeta^{-1} \delta_\zeta q)^{X\zeta}$, $\mathcal{B} = (\delta_Y q - \mathbf{s}_i J_Y J_\zeta^{-1} \delta_\zeta q)$, and $\mathcal{C} = \Gamma[(\mathbf{s}_i J_\zeta^{-1} + \mathbf{s}_d) \delta_\zeta q - \mu \overline{q}^\zeta]$. Also, in Eq. (40), $\gamma = 1/(c_*^2 \tau_h \tau_m)$ and $\Gamma = 1/(\tau_m/\tau_{nh} + N_*^2 \tau_h \tau_m)$. It is important to note that, with $\mathbf{s}_i = 1$, the terms $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ (with $\mu = 0$) are simply the components of the gradients in the terrain-following coordinate system. The sequence of steps involved in deriving the EBV problem (i.e., the final form of L_c''') is provided in appendix B.

h. Initial and boundary conditions

As with the case of any initial value problem, in order to initiate integration in time, the GEM-H dynamical core requires initial values of all the prognostic variables. At present, ECCC's operational data assimilation system provides analyzed initial values for the horizontal wind components u and v , virtual temperature T_v , and surface pressure p_s . The remaining prognostic variables w , ζ , and q are computed in a diagnostic manner at time $t = 0$. The initial value of q is obtained from the analyzed surface pressure p_s with a hydrostatic approximation. Substituting $dw/dt = 0$ in Eq. (22) gives

$$\frac{1}{J_\zeta} \frac{\partial q}{\partial \zeta} = g \frac{T_v'}{T_v} \quad (41)$$

as a hydrostatic approximation. The value of q at the different model levels are then obtained by integrating Eq. (41) where at the surface, due to the hydrostatic approximation, $q_s = R_d T_* \ln(p_s/p_*)$. The initial value of ζ is computed by assuming $\partial\rho/\partial t = 0$ at time $t = 0$ in the continuity equation. In the ζ coordinate, this takes the form

$$\frac{\partial}{\partial X} \left(\rho u \frac{\partial z}{\partial \zeta} \right) + \frac{1}{\cos\phi} \frac{\partial}{\partial Y} \left(\cos\phi \rho v \frac{\partial z}{\partial \zeta} \right) + \frac{\partial}{\partial \zeta} \left(\rho \zeta \frac{\partial z}{\partial \zeta} \right) = 0, \quad (42)$$

which is then discretized to compute the initial value of ζ . Once ζ is known, the initial value of w is obtained from its definition in the ζ coordinate: $w \equiv dz/dx = uJ_X + vJ_Y + \zeta J_\zeta$.

The boundary conditions for the upper and lower boundaries are given by $\zeta_T = \zeta_S = 0$. This implies that the vertical motion in the ζ coordinate vanishes at the surface and the model top which is flat. For LAM problems, GEM-H also requires lateral boundary conditions which are obtained from the driving fields. As the global Yin–Yang system is based on two interacting geometrically identical LAM domains, it therefore similarly requires lateral boundary conditions. In this case, the boundary conditions for one subdomain (Yin or Yang) depend on the solution of the other. Thus the solution of the global problem is obtained by iteratively solving the two subproblems separately and updating the values in the overlapping region until a certain convergence criterion is satisfied (Qaddouri and Lee 2011).

3. Solution of the EBV problem

The EBV problem to be resolved by GEM-H at each model time step can be expressed as

$$\nabla_\zeta^2 \mathbf{q} + \mathbf{M} \mathbf{q} = \mathbf{R}, \quad (43)$$

by replacing L_c''' in Eq. (40) with $(\nabla_\zeta^2 + \mathbf{M})\mathbf{q}$, where $\nabla_\zeta^2 = [\delta_{XX} + (1/\cos\phi) \delta_Y (\cos\phi \delta_Y)]$ is a discretized two-dimensional horizontal operator in the ζ coordinate, $\mathbf{M}$ contains all the remaining terms of L_c''' that include the discretized difference and averaging operators in the horizontal and vertical dimensions, $\mathbf{q}$ is the unknown and $\mathbf{R}$ includes the explicit rhs terms as well as the implicit nonlinear terms. It is important to note that the nonlinear terms in $\mathbf{R}$ require iterations for convergence, irrespective of the solution approach. At present, the

GEM dynamical core uses two iterations for the sufficient convergence of the nonlinear terms and two iterations for trajectories. As a result, the solver is called into action four times during each dynamical step.

Once Eq. (43) is solved to obtain the unknown q , the other prognostic variables are obtained through back substitution as presented in appendix C. As has been mentioned earlier, two general approaches are available for solving the elliptic problem—direct and iterative. The selection of these approaches depends on which terms are included in $\mathbf{M}$, and is determined by the values of the terms $\mathbf{s}_i$ and $\mathbf{s}_d$ in Eqs. (34)–(38).

a. The direct solver

The direct solver works by first decoupling Eq. (43) in the vertical. It is achieved through the expansion of the unknown $\mathbf{q}$ and the rhs $\mathbf{R}$ in terms of the eigenvectors that diagonalizes the operator $\mathbf{M}$ (Qaddouri and Lee 2011). This is only possible when the operator $\mathbf{M}$ does not include contributions from the metric terms arising from the vertical coordinate transformation that involves horizontally variable coefficients imparting horizontal coupling. Therefore, the implementation of direct solver in GEM-H requires a “simplified approach” where all the metric terms of the relevant discretized equations are treated as nonlinear terms. This is achieved by setting $\mathbf{s}_d = 1$. For N_k number of vertical levels used in the model, vertical separation reduces Eq. (43) to a set of N_k independent horizontal Helmholtz problems of the form

$$\nabla_{\zeta}^2 \tilde{\mathbf{q}} + \mathbf{m} \tilde{\mathbf{q}} = \tilde{\mathbf{R}}, \quad (44)$$

where $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{R}}$ are the vertical projections of $\mathbf{q}$ and $\mathbf{R}$, respectively, and $\mathbf{m}$ is the eigenvalue of the operator $\mathbf{M}$. The horizontal solution of the algorithm then proceeds by expanding $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{R}}$ in terms of the eigenvectors that diagonalize the X -component of the two-dimensional operator ∇_{ζ}^2 . For N_i number of grid points along the longitude X , this leads to N_i independent tridiagonal problems of N_j dimension for each model vertical level, where N_j denotes the number of grid points along the latitude Y . The total number of tridiagonal problems to solve is therefore $N_k \times N_i$. Solution to the tridiagonal problems is computed by Gaussian elimination without pivoting, and afterward, the final three-dimensional solution $\mathbf{q}$ is obtained (Qaddouri and Lee 2010).

The “simplified approach” has been primarily implemented in GEM-H to take advantage of the computational performance of the direct solver. The direct solver uses Fast Fourier Transform to compute the horizontal solution $\tilde{\mathbf{q}}$ and outperforms any iterative approach implemented at the ECCC by a substantial

margin for the configurations of the various GEM-based NWP systems running operationally. The “simplified approach” thus makes the application of GEM-H for such configurations comparable to GEM-P in terms of computational performance, and makes the option of a future replacement of the GEM-P core with GEM-H feasible for the operational NWP systems. The simplified approach, however, is applicable as long as the vertical–horizontal coupling, imparted through the metric terms, is not too significant so that these terms can be treated efficiently through nonlinear iterations. This approach works provided the maximum terrain slope is not too steep (less than 40°) which is generally the case for the different operational GEM-based NWP systems. However, with increasing spatial resolution, the slopes in gridscale orography also increase, particularly over complex terrain, which leads to increased vertical–horizontal coupling and, at one point, makes the “simplified approach” inapplicable.

b. The iterative solver

When all the metric terms in $\mathcal{A}$ and $\mathcal{B}$ [see Eq. (40)] are included in $\mathbf{M}$, the resulting vertical–horizontal coupling makes the problem nonseparable. This is done by setting $\mathbf{s}_i = 1$. In such a scenario, the three-dimensional equation of the form in (43) can only be solved at each time step by using an iterative solver. The current iterative solver for the EBV problem in GEM-H is based on the flexible generalized minimal residual (FGMRES) method (Saad 1993; Qaddouri and Lee 2010).

The fully discretized system of equations of the form in (43) can be further generalized as

$$\mathbf{A} \mathbf{q} = \mathbf{R}, \quad (45)$$

where the coefficient matrix $\mathbf{A}$ contains the discretized operator $(\nabla_{\zeta}^2 + \mathbf{M})$. The FGMRES method approximates the solution in a Krylov subspace of small dimension by minimizing the Euclidean norm of its residual. A major advantage of such a method is that instead of explicitly generating the coefficient matrix $\mathbf{A}$, one only needs to compute the vector resulting from the action of the underlying operator $(\nabla_{\zeta}^2 + \mathbf{M})$ on the vector $\mathbf{q}$. Efficient functioning of such an iterative solver, however, requires a preconditioner. At present, the preconditioner is based on the block Jacobi iteration for the EBV in Eq. (43), where all the metric terms in $\mathbf{M}$ are absent. This preconditioner improves the convergence rate of the FGMRES solver. However, the time of execution is still high compared to the fast direct solver. Significant research is currently underway at ECCC to devise iterative solvers that are competitive

with the direct approach and will work for both GEM-P and GEM-H. At present, the current implementation of the iterative solver in GEM-H—although not as efficient—provides the necessary reference for the direct solver approach.

4. Dynamics–physics coupling

Along with the dynamical core, parameterization of the subgrid-scale physical processes constitutes the other fundamental component of any NWP model. Coupling between the dynamical core and the parameterized subgrid-scale physical processes is of critical importance. How to devise the most appropriate coupling strategy is still an unsettled question (Beljaars et al. 2018). This issue is being actively studied at ECCO. However, it is not the objective of this study to delve deep into the fundamental questions around dynamics–physics coupling. Rather, in this section, the issue of coupling the RPN Physics Package with the GEM dynamical core is discussed in order to determine which approach is the most feasible for GEM-H among the options that are available for GEM-P.

It is important to note that, during every model time step in GEM, the RPN Physics Package is executed after the dynamical equations have been resolved by the dynamical core and thus the physics schemes utilize the outputs of the dynamics step as inputs. However, irrespective of the vertical coordinate used in the dynamical core, the physics schemes use a traditional σ coordinate in the vertical, defined as

$$\sigma = \frac{\pi}{\pi_s}, \quad (46)$$

where π is the hydrostatic pressure and the subscript s denotes the surface. Also, the physics schemes work within a one-dimensional configuration where each processor core only has access to the vertical structure of the meteorological fields associated with a single horizontal grid point. The various physical processes are parameterized sequentially where the tendencies estimated by one parameterization scheme affect the ones that follow. The parameterization sequence during each physics step initiates with the radiation scheme and is followed by the parameterizations of the surface processes, gravity wave drag, boundary layer turbulence, convection and gridscale condensation. At the end of the physics step, the tendencies from the different physical parameterization schemes are aggregated to compute the gridscale tendencies for the wind components, temperature, water vapor and the hydrometeors. It is important to note that horizontal diffusion in GEM

is carried out in the split mode between the dynamics and physics steps where the diffusion scheme has access to the three-dimensional outputs of the dynamics substep.

a. Different coupling approaches in GEM

Particularly in the context of GEM-P, two approaches are presently available to couple the RPN Physics Package with the GEM dynamical core. A brief discussion on these methods will be helpful in establishing the rationale behind the approach selected for the GEM-H dynamical core.

- (i) Split method: As has been mentioned earlier, s_c determines the mode of coupling between dynamics and physics. If $s_c = 0$ then the dynamical equations are resolved in the absence of any physical forcing and at the end of the dynamics step their contributions are incorporated as adjustments in the so-called split mode. In the absence of physics forcing, Eq. (31) becomes

$$\frac{F_i^{A*} - F_i^D}{\Delta t} - \frac{1 + b_i}{2} G_i^{A*} - \frac{1 - b_i}{2} G_i^D = 0, \quad (47)$$

where F_i^{A*} is the interim solution of the dynamics step. Once Eq. (47) is resolved, the physics source term is then applied as gridpoint adjustments as follows

$$(\delta F)_{\text{phys}} = F_i^A - F_i^{A*} = \Delta t P_i. \quad (48)$$

Thus, in the split method, the dynamics step predicts an inviscid and adiabatic solution that is modified through adjustments attributable to the parameterized physics forcings in order to obtain the complete solution at the end of each model time step.

- (ii) Tendency method: The second option for dynamics–physics coupling works by directly incorporating the impact of physics forcings as tendencies into the discretized dynamical equations, and hence is referred to as the “tendency method.” This method treats the physics source terms explicitly by setting $s_c = 1$ and replacing $\bar{P}_i$ by P_i^D . As a result, it involves solution of equations of the following form at each model time step:

$$\frac{F_i^A - F_i^D}{\Delta t} - \frac{1 + b_i}{2} G_i^A - \frac{1 - b_i}{2} G_i^D = P_i^D. \quad (49)$$

In this approach, the physics tendency P_i from the previous time step is combined with the rhs term R_i , followed by the determination of R_i^D at the departure positions in a semi-Lagrangian way.

b. Coupling in GEM-H

Although both of the aforementioned coupling methods are available for GEM-P, it is important to note that all the operational NWP systems based on GEM-P at present utilize the split method for dynamics–physics coupling. Nevertheless, there exists strong concern about the split method in general, and a brief discussion highlighting the pertinent issues will be helpful.

In the context of model formulations for both GEM-P and GEM-H, the term F_i does not necessarily match the prognostic model variables. For example, in the case of GEM-P in its hydrostatic mode, as presented by Girard et al. (2014):

$$F = \left\{ u, v, Bs + \ln \left(1 + \frac{\partial B}{\partial \xi} \right), \ln \frac{T_v}{T_*} - \kappa_d Bs \right\}, \quad (50a)$$

$$P = \left\{ \frac{d}{dt}(u, v, \ln \rho, \ln T_v) \right\}_{\text{phys}}, \quad (50b)$$

whereas the prognostic variables are u , v , s , ξ , and T_v . Therefore, in the actual model implementation of the split method, the prediction from the dynamics step is utilized to compute the interim state of the prognostic variables and adjustments are then applied to these variables at the end of the physics step. It is important to note that in this case the only adjustments that have been found to not result in any issue of major concern are $(\delta u)_{\text{phys}}$, $(\delta v)_{\text{phys}}$, and $(\delta T_v)_{\text{phys}}$. Also in GEM-P, adjustments are required for density, as in effect

$$\left(\frac{d \ln \rho}{dt} \right)_{\text{phys}} \equiv \frac{d}{dt} \ln(1 + r_w). \quad (51)$$

Although water vapor and hydrometeors are updated through physical parameterizations, the only variable that could be updated in the split mode appears to be s through a surface pressure adjustment $\delta \pi_s$ that may be computed as

$$\delta \pi_s = \int \delta \ln(1 + r_w) d\pi, \quad (52)$$

which takes into account the net inflow/outflow of mass through Earth's surface at every model grid point. Unfortunately, the vertical distribution of this change in mass through water vapor and precipitation fluxes cannot be correctly accounted for in the split mode, and is found to produce considerable noise in the wind forecast.

Furthermore, in the context of three-time-level discretization, Caya et al. (1998) have shown that the split method can lead to erroneous results for long time steps

that are permissible with the semi-implicit semi-Lagrangian models. Similar conclusions were drawn from a theoretical analysis with the two-time-level scheme by Staniforth et al. (2002). Figure 2a shows the geopotential height contours at 400 hPa from a 72-h global forecast with ECCC's 25-km resolution Global Deterministic Prediction System (GDPS) using the GEM-P dynamical core. The results correspond to split method for dynamics–physics coupling. Although the distribution of geopotential height for the meso and large scales does not reveal any issue of immediate concern, when one looks at the smallest scales (i.e., scales of about a few grid lengths), some spurious computational noise is visible (see Fig. 2b). Historically, the operational NWP systems at ECCC have been utilizing spatial filters over the model-predicted meteorological fields of interest, like the geopotential height, to smooth out any computational noise in the model outputs. As a result, the kind of computational noise shown in Fig. 2b has not been troublesome for the meteorologists using ECCC's operational NWP outputs. Nevertheless, the computational noise associated with the split method remains as a concern. However, as shown in Fig. 2c, when the split method for coupling is replaced by the tendency method, the noise in the geopotential height disappears. It should be noted that, although the tendency approach for coupling can impose stability limitations in terms of the acceptable length of time steps, ECCC's operational NWP system configurations are found to function with the tendency method without requiring any adjustment to the time-step lengths.

Similarly to GEM-P, the F_i terms do not match with the prognostic variables for all of the GEM-H model equations, where

$$F = \left\{ u, v, w, \frac{q}{c_*^2} + \ln J_\xi, \ln \frac{T_v}{T_*} - \frac{q}{c_{pd} T_*} \right\}, \quad (53a)$$

$$P = \left\{ \frac{d}{dt}[u, v, w, \ln(\rho T_v), \ln T_v] \right\}_{\text{phys}}. \quad (53b)$$

Particularly, the presence of the physics forcing term $[d \ln(\rho T_v)/dt]_{\text{phys}}$ in the modified continuity Eq. (23) poses an additional challenge for the GEM-H formulation, as far as the split method is concerned. If one attempts to apply this tendency as a hydrostatic adjustment to pressure in the split mode then of course its effect is not limited to the continuity equation alone. Rather, applying the adjustments to pressure (i.e., changing q in the case of GEM-H) also affects the thermodynamic equation. Such an adjustment leads to over compensation and is found to result in spurious bias in the temperature in upper troposphere and stratosphere,

FIG. 2. (a) Geopotential height (dam) at 400 hPa after 72 h for a 25-km global forecast initiated at 1200 UTC 25 Jan 2015 obtained with GEM-P using the “split method” for dynamics–physiics coupling (contour intervals of 6 dam). (b) The enlarged view over a small section of the global domain (contour intervals of 0.5 dam). (c) As in (b), but with “tendency method” for the dynamics–physiics coupling.

which is unacceptable (not shown). Also, the jet-level wind is found to be adversely affected. As a result, the physics contributions in GEM-H are accommodated through the tendency method by including them as explicit gridscale tendencies in the rhs of the discretized dynamical equations. For a fair comparison between the new and the existing dynamical core, the results for both GEM-P and GEM-H presented in the rest of this paper are obtained with dynamics–physiics coupling based on the tendency approach. It is also important to mention that, even though parameterization of physical processes like boundary layer turbulence can modify vertical motion w , at present the impact of physics on w is neglected. This is the case for both GEM-P and GEM-H.

A hybrid “split-tendency method” has also been tested with GEM-H, where the physics contributions to the thermodynamic and horizontal momentum equations are accounted for through the “split method” while the contributions to the continuity equation is accommodated using the “tendency method.” Such a

hybrid approach with GEM-H produces results that are equivalent to those obtained with the split method for GEM-P. However, questions remain about the numerical consistency of the hybrid split-tendency method.

It is important to mention that the current implementation of the RPN Physics Package, when coupled with the GEM-P (GEM-H) dynamical core through the tendency method, leads to some deterioration in temperature bias in the upper troposphere compared to the split (split-tendency) method. This implies that further research is necessary to have a more consistent dynamics–physiics coupling with the tendency approach. Particularly, the gridscale condensation scheme in GEM, for 10 km or coarser horizontal resolutions, has been found to exhibit large sensitivity with the tendency method which leads to underprediction of clouds (R. McTaggart-Cowan, ECCC, 2018, personal communication). This implies that some process-specific adjustments in the computation of the relevant physics tendencies may be required to improve the overall dynamics–physiics coupling. The challenges imposed by

the process-specific issues are also being explored by other operational NWP centers (Beljaars et al. 2018). Currently, work is underway at ECCO to explore the various issues within the coupling interface as well as the parameterizations of the different physical processes to improve the dynamics–physics coupling in general. Further discussion on this issue is, however, beyond the scope of this paper.

5. Evaluation of GEM-H

One of the most important objectives of this study has been to develop a dynamical core for the GEM model that utilizes a height-based TFC and is capable of producing predictions that are equivalent to the results obtained by the pressure-based dynamical core and with a comparable computational cost. To evaluate the consistency, accuracy and performance of the new dynamical core, a number of numerical experiments covering a wide range of scales—ranging from micro-scales to the meso and synoptic scales—have been carried out. These include two-dimensional theoretical test cases involving bubble convection (Robert 1993) and nonhydrostatic mountain waves (Schär et al. 2002) as well as three-dimensional global NWP. The two-dimensional test cases selected in this paper have become ubiquitous tools in testing the consistency and performance of nonhydrostatic dynamical cores. Also, the availability of the GEM-P core provides the opportunity to have reference solutions for all these cases.

a. Robert's bubble convection case

Robert (1993) presented a two-dimensional theoretical case involving the evolution of a warm bubble within a dry isentropic atmosphere. Initially, the bubble has a diameter of 500 m and is placed 10 m above a flat surface within a $1 \text{ km} \times 1 \text{ km}$ computational domain, and has a uniform potential temperature of 30.5°C . Also, the basic-state atmosphere is at rest under a hydrostatic equilibrium with an isentropic basic-state temperature of 30°C . As the bubble has a potential temperature excess of 0.5°C compared to the surrounding atmosphere, it rises due to the action of the buoyancy force. The absence of any orographic variation makes this experiment an excellent benchmark to test the functioning of advection and buoyancy within a dynamical core during the early stages of its development.

The numerical experiment for bubble convection is carried out with a spatial grid resolution of 10 m and a time step of 5 s. No explicit numerical diffusion is used. The resulting evolution of the bubble, in terms of its potential temperature distribution at two different times

(7 and 10 min), is presented in Fig. 3 for both GEM-P and GEM-H. The bubbles predicted by the two cores initially deform into a somewhat mushroom-like shape (see at $t = 7 \text{ min}$) and then are deformed further (at $t = 10 \text{ min}$). Overall, the predictions from GEM-P and GEM-H are equivalent for the entire range of scales—from the largest to the smallest resolved scales. Such a good resemblance between the two predictions implies negligible impact of the choice of the vertical coordinate and the other modifications in model formulations in the absence of any orographic variation at the model surface. It also indicates that the representation of the advection and buoyancy effects are comparable between the two GEM cores. It is important to note that, due to considerable differences in the model formulations and spatiotemporal discretizations, it is difficult to compare the evolution of the bubbles between two completely separate models in a quantitative manner. Only qualitative comparisons are feasible. Therefore, the lesser resemblance between the results from Robert (1993) and the GEM dynamical cores are not unusual. Although the predictions from the two GEM cores have some large-scale resemblance to the results presented by Robert (1993), significant differences appear at the upper half of the bubble—particularly at $t = 10 \text{ min}$. However, the upper structure of the bubble compares better with the predictions by Smolarkiewicz and Pudykiewicz (1992). Also, because of the implicit dissipation associated with the semi-Lagrangian approach, the GEM solution does not suffer from computational noise like some models based on Eulerian advection (Juang 2000). Overall, as GEM-P is being used operationally at ECCO, the resemblance between GEM-H and GEM-P is of more significant importance, as it confirms consistency of the GEM-H formulation and a neutral impact of the vertical coordinate modification on buoyancy and advection.

b. Schär's mountain case

Schär et al. (2002) presented a linear two-dimensional theoretical test case of mountain waves which is an excellent benchmark for verifying nonhydrostatic dynamical cores, particularly in determining the presence of possible inconsistencies in the numerical details (Husain and Girard 2017; Melvin et al. 2010; Girard et al. 2005; Klemp et al. 2003). The bottom boundary profile of the idealized mountain for this case is defined by

$$z_s = z_0 e^{-(x/a)^2} \cos^2(\pi x/l_x), \quad (54)$$

where $z_0 = 250 \text{ m}$, $l_x = 4 \text{ km}$, $a = 5 \text{ km}$, and π is the conventional mathematical constant. The upstream flow

FIG. 3. Potential temperature (°C) distribution for the rising bubble with: (a) GEM-P at $t = 7$ min, (b) GEM-H at $t = 7$ min, (c) GEM-P at $t = 10$ min, and (d) GEM-H at $t = 10$ min.

conditions are given by uniform upstream wind $U = 10 \text{ m s}^{-1}$, upstream surface temperature $T_{\text{surf}} = 288 \text{ K}$, upstream surface pressure $p_0 = 1000 \text{ hPa}$, and a constant Brunt–Väisälä frequency $N_* = 0.01 \text{ s}^{-1}$. All other conditions for the simulations of this test case are similar to those presented by Husain and Girard (2017).

One major advantage of Schär's mountain case is the availability of a steady-state analytical solution of the corresponding linearized problem as a reference (Schär et al. 2002). The simulated quasi-steady vertical velocity obtained after 4 h of integration are presented in Fig. 4 for both the GEM-P and GEM-H dynamical cores. The analytical solution of the problem, as presented by Schär et al. (2002), is a combination of rapidly decaying small-scale nonhydrostatic mountain waves close to the surface and large-scale hydrostatic waves extending to

much higher altitudes. As can be seen in Fig. 4, the solutions with both types of vertical coordinates generate these two regimes of the mountain waves and are very similar. Results shown in this figure represent simulations that have been carried out without any off-centering in the discretized dynamical equations and with consistent trajectory calculations (i.e., integration of the wind field in the trajectory equations is based on the trapezoidal rule while the wind field at the departure positions is obtained with cubic interpolation). Inconsistent trajectory calculations were found to produce similar distortion in the large-scale hydrostatic waves with GEM-H (not shown) as has been found for GEM-P earlier by Husain and Girard (2017). Although the GEM-H results presented here correspond to the direct solver based on the simplified approach,

FIG. 4. Steady-state vertical velocity contours (at 0.1 m s^{-1} intervals) for Schär's mountain case predicted by: (a) GEM-P and (b) GEM-H. No off-centering is used in the discretized dynamical and trajectory equations.

results with the iterative solver are found to be almost identical (not shown).

Figure 5 reveals that off-centering in the discretized dynamical equations leads to distortions in the vertical velocity distribution, irrespective of the type of the vertical coordinate. The solutions presented here are obtained with uniform off-centering involving $b_m = b_h = b_{nh} = 0.2$, which are the standard values used in the current GEM-based operational NWP systems. The results correspond to $\Delta t = 32 \text{ s}$, for which the maximum Courant number is approximately 0.76. Reducing the time step to even 4 s is unable to remove these distortions. Also, reducing the level of off-centering is found to reduce the level of distortion in the mountain waves,

but the distortions are only completely eliminated when $b_m = b_h = b_{nh} = 0$ (not shown). This conforms to the conclusions drawn by Husain and Girard (2017) in the context of GEM-P. Furthermore, as has been shown by Husain and Girard (2017), consistent trajectory calculations in the presence of off-centering necessitates off-centered averaging applied to the integrals of the source term on the rhs of Eq. (32). With uniform off-centering applied to the discretized dynamical equations, the discretized trajectory equations also require uniform off-centering of the same degree. Figure 6 reveals that in the presence of consistent off-centering in the trajectory calculations, the distortions in the vertical velocity distribution are

FIG. 5. As in Fig. 4, but with uniform off-centering ($b_m = b_h = b_{nh} = 0.2$) used in the discretized dynamical equations. The absence of off-centering in the discretized trajectory equations leads to inconsistent trajectory calculations and distortions in the mountain waves.

FIG. 6. As in Fig. 4, but with consistent off-centering (i.e., off-centering is applied to both the discretized dynamical and trajectory equations).

eliminated for the both dynamical cores. In the presence of differential off-centering (i.e., with different values of b_h and b_{nh} for hydrostatic and nonhydrostatic contributions in the system of dynamical equations), a similar differential approach is required for off-centering in the discretized trajectory equations. As has been shown for GEM-P by Husain and Girard (2017), in order to achieve numerical consistency, the off-centering in the discretized source terms in Eqs. (32a) and (32b) needs to be equal to the values of b_h and b_{nh} used in the discretized dynamical equations, respectively.

c. Global deterministic prediction

A series of 5-day global forecasts, with 25-km horizontal grid spacing, has been carried out covering winter and summer periods for the Northern Hemisphere to compare the predictions by GEM-H and GEM-P from NWP standpoint. Each seasonal period includes 44 cases where the initialization between two consecutive cases is apart by 36 h. The first summer and winter cases start at 0000 UTC 25 June 2014 and 19 December 2014, respectively. The global predictions have been obtained with uniform off-centering in the discretized dynamical equations ($b_m = b_h = b_{nh} = 0.2$). Husain and Girard (2017) have shown that inconsistent off-centering has negligible effects for three-dimensional NWP applications. Therefore, the global forecasts are carried out without any off-centering in the discretized trajectory equations. Simulations have been carried out using 84 vertical levels with the model top set approximately at 0.1 hPa and 65 km for GEM-P and GEM-H, respectively. Furthermore, as has been mentioned in section 4, physics is

coupled with dynamics through the tendency method for both GEM-H and GEM-P for all the tests presented in this section.

First, comparisons are made in the spectral space by comparing the variance spectra associated with the different meteorologically important fields. To compute the spectral variance of any meteorological field, it is first interpolated from the Yin–Yang grid to a global Gaussian grid. Afterward, variance spectra of the field are calculated by decomposing it at a given pressure level using the spherical harmonics. Figure 7 shows the spectral variance of the geopotential height and temperature fields for 120-h forecasts at three different pressure levels averaged over 10 cases from the winter period. The results show spectral similarity between GEM-P and GEM-H for the entire range of scales resolved by the global model—from synoptic to mesoscales. The spectra of kinetic energy and vertical velocity are presented in Fig. 8. The spectral slope of kinetic energy is critical for accurate representation of atmospheric dynamics, and as can be seen in this figure, both GEM-H and GEM-P have the same spectral slope at the synoptic and mesoscales. The vertical motion is also important, particularly for physical processes like convection. Figure 8 shows close spectral similarity between the vertical motions from the two dynamical cores. This implies that changing the vertical coordinate has negligible sensitivity to the extremely important physical processes like convection and convection-driven precipitation. Also, the comparisons in the spectral space confirm that the height-based TFC does not lead to any spurious noise or damping in the meteorological fields for any model-resolved length scale.

FIG. 7. Spectral variance of (left) geopotential height (GZ) and (right) temperature (TT) for 120-h global forecasts with 25-km horizontal grid spacing. Results are presented for three different pressure levels [(top) 100 hPa, (middle) 500 hPa, (bottom) 850 hPa] that are obtained by averaging the spectra for 10 Northern Hemisphere winter forecasts.

Objective forecast scores are computed by comparing the model predictions against radiosonde observations at different pressure levels. The evaluation is based on the bias and standard deviation of error (SDE) in model

predictions for the individual cases as well as for the average of the 44 cases covering each seasonal period. Figure 9 presents the vertical profile of error in the predictions from GEM-P (blue) and GEM-H (red) for

FIG. 8. Spectral variance of (left) kinetic energy (KE) and (right) vertical velocity (WW) for 120-h global forecasts. All other conditions are as in Fig. 7.

the winter period. These figures represent global average scores of 120-h forecast from 44 winter cases for zonal wind (UU), wind speed (UV), geopotential height (GZ), and temperature (TT). An important thing to note while reading this figure is the presence of the

statistical confidence scores at the different pressure levels along the left and right vertical axes of the individual subplots for bias and SDE, respectively. A confidence value (in %) shaded in blue (red) color implies statistically significant improvement obtained with the

FIG. 9. Comparison of 120-h 25-km GDPS forecasts obtained with GEM-P (blue) and GEM-H (red) dynamical cores against radiosonde observations for zonal wind (UU), wind speed (UV), temperature (TT), and geopotential height (GZ). The dashed and solid lines, respectively, indicate bias and standard deviation of error (SDE). The scores are obtained by averaging over 44 Northern Hemisphere winter cases. The red and blue shaded numbers along the left (right) vertical axes within each panel indicate the confidence in percentage in the statistically significant improvements in bias (SDE) for the dynamical core associated with each color. Significance for bias and SDE are computed using t and F test, respectively.

GEM-P (GEM-H) core with respect to the other. The absence of the statistical significance value of a score at a pressure level implies that the difference between the two dynamical cores is statistically insignificant with respect to that score. The confidence scores for the average of SDE and bias are estimated by applying the F test and t test, respectively. Figure 9 reveals that although there are small differences in the bias for the average of the 44 winter cases, there is no statistically significant difference in the SDE. When tested in the absence of physics forcings, no statistically significant difference is found between the two dynamical cores in either bias or SDE (not shown). The meteorological fields are interpolated from the TFC of the dynamics to the σ coordinate for physics through vertical interpolation which can lead to small differences in the vertical for the different definitions of the TFC. Physical

parameterizations can be sensitive to the position of the vertical levels and this is apparently responsible for the small bias differences in Fig. 9. Even though small differences in bias are present, the objective scores from GEM-P and GEM-H can be safely assumed to be statistically equivalent as a whole. The two dynamical cores have also been found to be similarly equivalent for the summer period (not shown).

The geopotential height at 1000 hPa in Fig. 9 shows a negative bias for both dynamical cores. This indicates a loss of mass conservation, which is a consequence of the nonconservative nature of semi-Lagrangian advection. This can be improved by introducing a simple global mass fixer that works by conserving the global mean surface pressure after each dynamics step in the model. The scores in the presence of a global mass fixer are shown in Fig. 10, where the bias at 1000 hPa is eliminated

FIG. 10. As in Fig. 9, but with a global mass fixer in the simulations for both GEM-P and GEM-H to improve mass conservation.

for both dynamical cores. The overall scores for the two cores are again, as expected, found to be equivalent in the presence of a global mass fixer.

Overall, the results presented in the Figs. 3–10 clearly demonstrate that the implementation of GEM-H as presented in this paper produces results that are equivalent to the existing GEM-P dynamical core.

d. Stability over steep terrain

A systematic study demonstrating the benefits of the height-based vertical coordinate over a mass-based coordinate is beyond the scope of this paper. Without implementing any special approach dedicated to improving model stability (e.g., improving the approximation of horizontal pressure gradient) the GEM-H dynamical core at its current form is not expected to lead to any significant stability improvement. Nevertheless, the initial results (not shown) obtained for a two-dimensional test case—involving a steep isolated mountain under an isothermal atmosphere at rest—point to stability improvements with the GEM-H

dynamical core (Husain et al. 2019). This theoretical test case is internally referred to as the “no flow” case. In an ideal situation, when marched forward in time, the atmosphere in the “no flow” case is supposed to maintain zero (or very negligible) motion. However, the difference between temperature of the atmosphere T_0 and the semi-implicit basic-state temperature T_* generates numerical noise that grows with increasing steepness of the mountain. This can lead to spurious vertical motions, and with sufficiently steep terrain, the integration can become unstable. Results from this test case (not shown) reveal that both GEM-P and GEM-H remain stable for terrain slopes below 52° . However, GEM-P generally leads to a more noisy and nonnegligible vertical motion field, particularly as the maximum terrain slope is increased beyond 45° . With increasing terrain slope, the noise level in the vertical motion field resolved by GEM-P increases substantially and becomes unstable as the maximum slope goes beyond 60° . Regarding GEM-H, the simplified approach remains stable up to a maximum terrain slope of 52° , whereas the maximum

slope limit can be extended to almost 70° with the iterative approach. Overall, the height-based GEM-H is found to be less susceptible to noisy solutions and numerical instability over steep terrain when compared to the mass-based GEM-P dynamical core. These are, however, preliminary results. Caution needs to be applied when trying to extend the conclusions drawn from the “no flow” test to three-dimensional NWP simulations with physics. For example, it should be noted that the slope limits can be somewhat dependent on the test case. Therefore, a more systematic detailed study is required to analyze and document the stability response of the two dynamical cores.

6. Summary

A newly developed dynamical core for ECCC's GEM model with a height-based terrain-following vertical coordinate is presented. The operational GEM-P dynamical core, which is based on a log-hydrostatic-pressure-type vertical coordinate, suffers from strong numerical instability when subjected to steep orography which becomes more prevalent with very high-resolution (subkilometer) NWP simulations. Also, the scalability limitation of the direct solver employed in GEM-P is driving extensive research at ECCC aimed at developing optimized three-dimensional iterative solvers for future generations of massively parallel supercomputers. A dynamical core with height-based TCF is expected to be better placed with respect to these ongoing NWP challenges at ECCC. The principal objective of this paper is to provide information pertaining to the different aspects of the new height-based dynamical core that include the changes to the model formulation, numerical discretizations, the solvers for the discretized problems and the strategy for coupling the new dynamical core with the RPN physics package. Another important objective is to demonstrate that the new GEM-H core is capable of making meteorological predictions that are equivalent to the existing GEM-P dynamical core.

Numerical experiments have been conducted throughout the different stages of GEM-H development. Initially, the bubble convection test revealed that the advection and the buoyancy effects in GEM-H are accurately represented and are producing results that are equivalent to GEM-P. When tested for the idealized Schär's mountain case, the nonhydrostatic and hydrostatic components of the mountain waves predicted by GEM-H are found to be very close to the GEM-P predictions as well as the analytical solution. The dynamics source terms in GEM are averaged over the air parcel trajectories using the trapezoidal method and the calculation of the rhs terms in the dynamical

equations are carried out using cubic interpolation. Although it has not been shown explicitly, similarly to GEM-P, in the absence of any off-centering in the discretized dynamical equations, numerical consistency in GEM-H requires a trapezoidal averaging of the source terms in the discretized trajectory equations along with cubic interpolation for the wind fields at the departure positions. Furthermore, in the presence of off-centering in the discretized dynamical equations, Schär's mountain case shows that the discretized sources terms in the trajectory equations also necessitate off-centered averaging for the sake of consistent numerics in both GEM-H and GEM-P. In general, the knowledge acquired over years regarding the different numerical aspects of the GEM-P dynamical core is proven to be equally applicable to GEM-H. Comparisons between GEM-H and GEM-P for global deterministic predictions are also presented. The results are found to be equivalent in the spectral space confirming that the new vertical coordinate does not lead to any spurious noise or damping over the model-resolved scales. When compared against upper-air radiosonde observations, except for small differences in bias, GEM-H and GEM-P are found to produce statistically equivalent results.

The rationale behind the choice of the solution approach for the discretized EBV resulting from the GEM-H formulation as well as the method for dynamics-physics coupling is discussed in detail. Although the general structure of the EBV originating from the GEM-H discretized system of equations require an iterative approach, a simplified approach has been devised where the horizontally variable metric terms—attributable to the vertical-coordinate transformation—are coupled with the nonlinear terms and are treated with nonlinear iterations. This makes the EBV vertically separable and allows the use of a direct solver as in GEM-P. The fact that the GEM-H dynamical core can utilize the direct solver approach for global and regional-scale model resolutions makes it as efficient as the operational GEM-P core. This renders GEM-H feasible for near-future implementations of operational NWP systems at ECCC. Furthermore, a three-dimensional iterative solver based on FGMRES is also developed for situations when the simplified approach is not applicable.

Improving any NWP model is a continuous process. As a stable GEM-H dynamical core is now developed, a number of other relevant issues are currently being studied. The objective is to improve the GEM model in general and the GEM-H dynamical core in particular. One of the most important short-term goal in this regard is to devise a more numerically consistent and accurate coupling between RPN Physics and GEM dynamics

which will benefit both of the dynamical cores. Also, extensive research is being carried out to develop highly optimized three-dimensional iterative solvers that can be competitive against the operationally used direct solver while scaling better for very large number of processor cores for the future generations of massively parallel supercomputers. Currently, the Yin–Yang system uses the Schwarz method where the global solution is produced by iteratively solving two elliptic subproblems for the two subdomains (Yin and Yang) separately and updating the solutions in the overlapping regions until a certain convergence criterion is satisfied. One promising solution to reduce the execution time of the iterative solver for the Yin–Yang system is to remove the Schwarz iterations and to solve the two elliptic subproblems as one by using FGMRES (Zerroukat and Allen 2012). Also, preconditioners based on other types of methods (e.g., incomplete factorization, block Gauss–Seidel or multigrid method) could be used in order to improve the convergence rate of the FGMRES solver. Finally, on the GEM-H front, another important short-term objective is to improve its numerical stability over steep orography by implementing more accurate numerical approximation of the horizontal gradients in the discretized dynamical equations.

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APPENDIX A

Nonlinear and rhs Components of the Discretized Equations

The N_i terms associated with Eqs. (20)–(24) and (29), corresponding to the linear components given by Eqs. (34)–(39), are provided below:

$$N_u = -\left(f + \frac{\tan\phi}{a}u\right)\bar{v}^{XY} + \left(\frac{\bar{T}_v^{X\zeta}}{T_*} - 1\right)\delta_X q - \left(\frac{\bar{T}_v^{X\zeta}}{T_*} - s_i\right)J_X \bar{J}_\zeta^{-1} \delta_\zeta q^{X\zeta}, \quad (\text{A1})$$

$$N_v = \left(f + \frac{\tan\phi}{a}\bar{u}^{XY}\right)\bar{u}^{XY} + \left(\frac{\bar{T}_v^{Y\zeta}}{T_*} - 1\right)\delta_Y q - \left(\frac{\bar{T}_v^{Y\zeta}}{T_*} - s_i\right)J_Y \bar{J}_\zeta^{-1} \delta_\zeta q^{Y\zeta}, \quad (\text{A2})$$

$$N_w = \left[\left(\frac{T_v}{T_*} - s_i\right)J_\zeta^{-1} - s_d\right]\delta_\zeta q - \left(\frac{T_v}{T_*} - 1\right)g\frac{T'_v}{T_v}, \quad (\text{A3})$$

$$N_c = 0, \quad (\text{A4})$$

$$N_T = \frac{1}{\tau_h} \left[\ln\left(\frac{T_v}{T_*}\right) - \frac{T'_v}{T_v} \right], \quad (\text{A5})$$

$$N_z = 0. \quad (\text{A6})$$

The corresponding rhs terms, in the absence of physics forcing, take the following forms:

$$R_u = \frac{u}{\tau_m} - \beta_m \left[-\left(f + \frac{\tan\phi}{a}u\right)\bar{v}^{XY} + \frac{\bar{T}_v^{X\zeta}}{T_*}(\delta_X q - J_X \bar{J}_\zeta^{-1} \delta_\zeta q^{X\zeta}) \right], \quad (\text{A7})$$

$$R_v = \frac{v}{\tau_m} - \beta_m \left[\left(f + \frac{\tan\phi}{a}\bar{u}^{XY}\right)\bar{u}^{XY} + \frac{\bar{T}_v^{Y\zeta}}{T_*}(\delta_Y q - J_Y \bar{J}_\zeta^{-1} \delta_\zeta q^{Y\zeta}) \right], \quad (\text{A8})$$

$$R_w = \frac{w}{\tau_{nh}} - \beta_{nh} \left[\frac{T_v}{T_*} \left(J_\zeta^{-1} \delta_\zeta q - g\frac{T'_v}{T_v} \right) \right], \quad (\text{A9})$$

$$R_c = \frac{1}{\tau_h} \left(\frac{q}{c_*^2} + \ln J_\zeta \right) - \beta \left[\delta_X u + \frac{1}{\cos\phi} \delta_Y (\cos\phi v) + \delta_\zeta \dot{\zeta} - \varepsilon \bar{w}^\zeta \right], \quad (\text{A10})$$

$$R_T = \frac{1}{\tau_h} \left[\ln\left(\frac{T_v}{T_*}\right) - \frac{\bar{q}^\zeta}{c_{pd} T_*} \right] - \beta[\mu w], \quad (\text{A11})$$

$$R_z = \frac{z - \zeta}{\tau_{nh}} - \beta_{nh}[\dot{\zeta} - w]. \quad (\text{A12})$$

APPENDIX B

Deriving the EBV Problem

The lhs of the EBV problem to be solved at every time step in GEM-H is derived by manipulating the

discretized system of equations of the form in Eq. (33). The sequence of operations to derive the EBV is provided below in terms of its linear components:

$$L'_c = \delta_X L_u + \frac{1}{\cos\phi} \delta_Y (\cos\phi L_v) - \frac{1}{\tau_m} \left(L_c - \frac{\ln J_\zeta}{\tau_h} \right), \quad (\text{B1})$$

$$L'_z = L_z - \frac{z - \zeta}{\tau_{nh}}, \quad (\text{B2})$$

$$L'_w = L_w + L'_z / \tau_{nh}, \quad (\text{B3})$$

$$L'_T = L_T + \mu L'_z, \quad (\text{B4})$$

$$L''_T = \Gamma(g\tau_h L'_T + L'_w), \quad (\text{B5})$$

$$L''_c = L'_c + \frac{\varepsilon}{\tau_m} \overline{L'_z}^\zeta, \quad (\text{B6})$$

$$L'''_c = L''_c + \delta_\zeta L''_T - \varepsilon \overline{L''_T}^\zeta. \quad (\text{B7})$$

Similar manipulations are also applied to the nonlinear and rhs components of the discretized equations to obtain $R'_c, N'_c, R'_z, N'_z, R'_w, N'_w, R'_T, N'_T, R''_T, N''_T, R''_c, N''_c, R'''_c$, and N'''_c .

APPENDIX C

Back Substitution to Obtain the Other Prognostic Variables

The solution of the EBV computes the unknown $\mathbf{q}$. The rest of the variables are then calculated from the discretized dynamical equations through back substitution in the following sequence:

$$u = \tau_m [R_u - N_u - (\delta_X q - \mathbf{s}_i J_X \overline{J_\zeta^{-1} \delta_\zeta q}^{X\zeta})], \quad (\text{C1})$$

$$v = \tau_m [R_v - N_v - (\delta_Y q - \mathbf{s}_i J_Y \overline{J_\zeta^{-1} \delta_\zeta q}^{Y\zeta})], \quad (\text{C2})$$

$$\dot{\zeta} = \tau_m \{R''_T - N''_T - \Gamma[(\mathbf{s}_i J_\zeta^{-1} + \mathbf{s}_d) \delta_\zeta q - \mu \overline{q}^\zeta]\}, \quad (\text{C3})$$

$$w = \dot{\zeta} - L'_z, \quad (\text{C4})$$

$$T_v = gT_* \left[g - \frac{\dot{\zeta}}{\tau_{nh}} + (\mathbf{s}_i J_\zeta^{-1} + \mathbf{s}_d) \delta_\zeta q - R'_w - N'_w \right]^{-1}. \quad (\text{C5})$$

REFERENCES

- Arakawa, A., 1988: Finite-difference methods in climate modeling. *Physically-Based Modelling and Simulation of Climate and Climatic Change*, M. E. Schlesinger, Ed., Springer, 79–168, <https://doi.org/10.1007/978-94-009-3041-4>.
- Beljaars, A., and Coauthors, 2018: The numerics of the physical parametrization in the ECMWF model. *Front. Earth Sci.*, **6**, 137, <https://doi.org/10.3389/feart.2018.00137>.
- Caya, A., R. Laprise, and P. Zwack, 1998: Consequences of using the splitting method for implementing physical forcings in a semi-implicit semi-Lagrangian model. *Mon. Wea. Rev.*, **126**, 1707–1713, [https://doi.org/10.1175/1520-0493\(1998\)126<1707:COUTSM>2.0.CO;2](https://doi.org/10.1175/1520-0493(1998)126<1707:COUTSM>2.0.CO;2).
- Charney, J., and N. A. Phillips, 1953: Numerical integration of the quasi-geostrophic equations for barotropic and simple baroclinic flows. *J. Meteor.*, **10**, 71–99, [https://doi.org/10.1175/1520-0469\(1953\)010<0071:NIOTQG>2.0.CO;2](https://doi.org/10.1175/1520-0469(1953)010<0071:NIOTQG>2.0.CO;2).
- Charron, M., A. Zadra, and C. Girard, 2014: Four-dimensional tensor equations for a classical fluid in an external gravitational field. *Quart. J. Roy. Meteor. Soc.*, **140**, 908–916, <https://doi.org/10.1002/qj.2185>.
- Côté, J., and A. Staniforth, 1988: A two-time-level semi-Lagrangian semi-implicit scheme for spectral models. *Mon. Wea. Rev.*, **116**, 2003–2012, [https://doi.org/10.1175/1520-0493\(1988\)116<2003:ATTLSL>2.0.CO;2](https://doi.org/10.1175/1520-0493(1988)116<2003:ATTLSL>2.0.CO;2).
- Gal-Chen, T., and C. Somerville, 1975: On the use of a coordinate transformation for the solution of the Navier–Stokes equations. *J. Comput. Phys.*, **17**, 209–228, [https://doi.org/10.1016/0021-9991\(75\)90037-6](https://doi.org/10.1016/0021-9991(75)90037-6).
- Girard, C., R. Benoit, and M. Desgagné, 2005: Finescale topography and the MC2 dynamics kernel. *Mon. Wea. Rev.*, **133**, 1463–1477, <https://doi.org/10.1175/MWR2931.1>.
- , and Coauthors, 2014: Staggered vertical discretization of the Canadian Environmental Multiscale (GEM) model using a coordinate of the log-hydrostatic-pressure type. *Mon. Wea. Rev.*, **142**, 1183–1196, <https://doi.org/10.1175/MWR-D-13-00255.1>.
- Husain, S., and C. Girard, 2017: Impact of consistent semi-Lagrangian trajectory calculations on numerical weather prediction performance. *Mon. Wea. Rev.*, **145**, 4127–4150, <https://doi.org/10.1175/MWR-D-17-0138.1>.
- , —, A. Qaddouri, and A. Plante, 2019: Global Environmental Multiscale model with a new terrain-following vertical coordinate based on height. *Workshop on PDEs on the Sphere*, Montréal, QC, Canada, Canadian Meteorological Centre, Environment and Climate Change Canada, http://collaboration.cmc.ec.gc.ca/science/pdes-2019/pdfs/PDEs-2019_paper_11.pdf.
- Juang, H.-M. H., 2000: The NCEP mesoscale spectral model: A revised version of the nonhydrostatic regional spectral model. *Mon. Wea. Rev.*, **128**, 2329–2362, [https://doi.org/10.1175/1520-0493\(2000\)128<2329:TNMSMA>2.0.CO;2](https://doi.org/10.1175/1520-0493(2000)128<2329:TNMSMA>2.0.CO;2).
- Klemp, J. B., W. C. Skamarock, and O. Fuhrer, 2003: Numerical consistency of metric terms in terrain-following coordinates. *Mon. Wea. Rev.*, **131**, 1229–1239, [https://doi.org/10.1175/1520-0493\(2003\)131<1229:NCOMTI>2.0.CO;2](https://doi.org/10.1175/1520-0493(2003)131<1229:NCOMTI>2.0.CO;2).
- Laprise, R., 1992: The Euler equations of motion with hydrostatic pressure as an independent variable. *Mon. Wea. Rev.*, **120**, 197–207, [https://doi.org/10.1175/1520-0493\(1992\)120<0197:TEEOMW>2.0.CO;2](https://doi.org/10.1175/1520-0493(1992)120<0197:TEEOMW>2.0.CO;2).
- Maher, Y., 1984: An improved numerical approximation of the horizontal gradients in a terrain-following coordinate system. *Mon. Wea. Rev.*, **112**, 918–922, [https://doi.org/10.1175/1520-0493\(1984\)112<0918:AINAOT>2.0.CO;2](https://doi.org/10.1175/1520-0493(1984)112<0918:AINAOT>2.0.CO;2).
- Melvin, T., M. Dubal, N. Wood, A. Staniforth, and M. Zerroukat, 2010: An inherently mass-conserving iterative semi-implicit semi-Lagrangian discretization of the non-hydrostatic vertical-slice equations. *Quart. J. Roy. Meteor. Soc.*, **136**, 799–814, <https://doi.org/10.1002/qj.603>.

- Müller, E. H., and R. Scheichl, 2014: Massively parallel solvers for elliptic partial differential equations in numerical weather and climate prediction. *Quart. J. Roy. Meteor. Soc.*, **140**, 2608–2624, <https://doi.org/10.1002/qj.2327>.
- Phillips, N. A., 1966: The equations of motion for a shallow rotating atmosphere and the “traditional approximation.” *J. Atmos. Sci.*, **23**, 626–628, [https://doi.org/10.1175/1520-0469\(1966\)023<0626:TEOMFA>2.0.CO;2](https://doi.org/10.1175/1520-0469(1966)023<0626:TEOMFA>2.0.CO;2).
- Qaddouri, A., and V. Lee, 2010: The elliptic solvers in the Canadian limited area forecasting model GEM-LAM. *Modeling, Simulation, and Optimization—Tolerance and Optimal Control*, S. Cakaj, Ed., InTech, 109–124, <https://doi.org/10.5772/9038>.
- , and —, 2011: The Canadian Global Environmental Multiscale model on the Yin-Yang grid system. *Quart. J. Roy. Meteor. Soc.*, **137**, 1913–1926, <https://doi.org/10.1002/qj.873>.
- , L. Laayouni, S. Loisel, J. Côté, and M. J. Gander, 2008: Optimized Schwarz methods with an overset grid for the shallow-water equations: Preliminary results. *Appl. Numer. Math.*, **58**, 459–471, <https://doi.org/10.1016/j.apnum.2007.01.015>.
- Rivest, C., A. Staniforth, and A. Robert, 1994: Spurious resonant response of semi-Lagrangian discretizations to orographic forcing: Diagnosis and solution. *Mon. Wea. Rev.*, **122**, 366–376, [https://doi.org/10.1175/1520-0493\(1994\)122<0366:SRROSL>2.0.CO;2](https://doi.org/10.1175/1520-0493(1994)122<0366:SRROSL>2.0.CO;2).
- Robert, A., 1993: Bubble convection experiments with a semi-implicit formulation of the Euler equations. *J. Atmos. Sci.*, **50**, 1865–1873, [https://doi.org/10.1175/1520-0469\(1993\)050<1865:BCEWAS>2.0.CO;2](https://doi.org/10.1175/1520-0469(1993)050<1865:BCEWAS>2.0.CO;2).
- Saad, Y., 1993: A flexible inner-outer preconditioned GMRES algorithm. *SIAM J. Sci. Comput.*, **14**, 461–469, <https://doi.org/10.1137/0914028>.
- Schär, C., D. Leuenberger, O. Fuhrer, D. Lüthi, and C. Girard, 2002: A new terrain-following vertical coordinate formulation for atmospheric prediction models. *Mon. Wea. Rev.*, **130**, 2459–2480, [https://doi.org/10.1175/1520-0493\(2002\)130<2459:ANTFVC>2.0.CO;2](https://doi.org/10.1175/1520-0493(2002)130<2459:ANTFVC>2.0.CO;2).
- Smolarkiewicz, P. K., and J. A. Pudykiewicz, 1992: A class of semi-Lagrangian approximations for fluids. *J. Atmos. Sci.*, **49**, 2082–2096, [https://doi.org/10.1175/1520-0469\(1992\)049<2082:ACOSLA>2.0.CO;2](https://doi.org/10.1175/1520-0469(1992)049<2082:ACOSLA>2.0.CO;2).
- , R. Sharman, J. Weil, S. G. Perry, D. Heist, and G. Bowker, 2007: Building resolving large-eddy simulations and comparison with wind tunnel experiments. *J. Comput. Phys.*, **227**, 633–653, <https://doi.org/10.1016/j.jcp.2007.08.005>.
- Staniforth, A., N. Wood, and J. Côté, 2002: Analysis of the numerics of physics-dynamics coupling. *Quart. J. Roy. Meteor. Soc.*, **128**, 2779–2799, <https://doi.org/10.1256/qj.02.25>.
- Vionnet, V., S. Bélair, C. Girard, and A. Plante, 2015: Wintertime subkilometer numerical forecasts of near-surface variables in the Canadian Rocky Mountains. *Mon. Wea. Rev.*, **143**, 666–686, <https://doi.org/10.1175/MWR-D-14-00128.1>.
- Yeh, K.-S., J. Côté, S. Gravel, A. Méthot, A. Patoine, M. Roch, and A. Staniforth, 2002: The CMC-MRB Global Environmental Multiscale (GEM) model. Part III: Nonhydrostatic formulation. *Mon. Wea. Rev.*, **130**, 339–356, [https://doi.org/10.1175/1520-0493\(2002\)130<0339:TCMGEM>2.0.CO;2](https://doi.org/10.1175/1520-0493(2002)130<0339:TCMGEM>2.0.CO;2).
- Zängl, G., 2012: Extending the numerical stability limit of terrain-following coordinate models over steep slopes. *Mon. Wea. Rev.*, **140**, 3722–3733, <https://doi.org/10.1175/MWR-D-12-00049.1>.
- Zerroukat, M., and T. Allen, 2012: On the solution of elliptic problems on overset/Yin–Yang grids. *Mon. Wea. Rev.*, **140**, 2756–2767, <https://doi.org/10.1175/MWR-D-11-00272.1>.