A Two-Stage Fourth-Order Multimoment Global Shallow-Water Model on the Cubed Sphere

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ABSTRACT: A new multimoment global shallow-water model on the cubed sphere is proposed by adopting a two-stage fourth-order Runge–Kutta time integration. Through calculating the values of predicted variables at half time step $t^{n+\frac{1}{2}}$ by a second-order formulation, a fourth-order scheme can be derived using only two stages within one time step. This time integration method is implemented in our multimoment global shallow-water model to build and validate a new and more efficient numerical integration framework for dynamical cores. As the key task, the numerical formulation for evaluating the derivatives in time has been developed through the Cauchy–Kowalewski procedure and the spatial discretization of the multimoment finite-volume method, which ensures fourth-order accuracy in both time and space. Several major benchmark tests are used to verify the proposed numerical framework in comparison with the existing four-stage fourth-order Runge–Kutta method, which is based on the method of lines framework. The two-stage fourth-order scheme saves about 30% of the computational cost in comparison with the four-stage Runge–Kutta scheme for global advection and shallow-water models. The proposed two-stage fourth-order framework offers a new option to develop high-performance time marching strategy of practical significance in dynamical cores for atmospheric and oceanic models.

KEYWORDS: Advection; Shallow-water equations; Coordinate systems; Grid systems; Nonlinear models

1. Introduction

The atmospheric dynamics are governed by Navier–Stokes equations representing the conservation laws of mass, momentum, and energy. Since the analytic solutions to Navier–Stokes equations are unknown, numerous numerical models have been designed to approximate the spatial and temporal differentials in governing equations. The performance of time integration strategies plays an important role in atmospheric general circulation models (AGCMs). One of the major issues regarding time integration strategy is that the physically insignificant fast waves limit the maximal time step due to the computational stability of explicit time integration algorithms, which is usually too small in comparison with time scales of the physically dominant variations (Durran 2010). Thus, the computational efficiency of the simple explicit time integration strategies cannot meet the requirements of practical applications. Some special techniques, such as explicit time-splitting method (Klemp and Wilhelmson 1978), implicit-explicit method (Weller et al. 2013), semi-implicit semi-Lagrangian method (Staniforth and Côté 1991) among others, have been developed for AGCMs considering the overall accuracy, efficiency, and stability of spatial and temporal discretizations. A review of the current and emerging methods for time integration in AGCMs can be referred to Mengaldo et al. (2019).

Runge–Kutta schemes were originally derived to solve ordinary differential equations (ODEs), and now widely used for the time marching of partial differential equations (PDEs), where high-order spatial discretization is separately used for the spatial derivative operators. This class of numerical methods, known as the method of lines (MOL) (Schiesser 1991), have got great success in computational fluid dynamics (Shu 1988; Cockburn and Shu 1989; Kennedy et al. 2000), and an increasing popularity in atmospheric and oceanic modeling (Mengaldo et al. 2019). The existing implementation
of high-order Runge–Kutta schemes in the MOL framework needs to compute the intermediate guesses of the predicted physical fields between two time levels. These intermediate guesses are usually obtained through the first-order substep time integration approximations. As a result, a Runge–Kutta scheme can achieve $p$th-order accuracy with at least $p$ stages where the spatial-discretization operator has to be repeatedly computed at each stage.

In Chan and Tsai (2010), a type of Runge–Kutta schemes with fewer stages were proposed by evaluating the intermediate guesses with higher-order approximations. Li et al. (Li and Du 2016; Li 2019; Cheng et al. 2019) designed a two-stage fourth-order scheme (RK24 scheme hereafter for brevity) for the time integration of the hyperbolic systems, where the original PDEs are used to represent the temporal derivatives in terms of the spatial discretizations, which is also known as the Cauchy–Kowalewski procedure or the Lax–Wendroff procedure (Toro 2009). According to the tests of Li et al. (Li and Du 2016; Li 2019; Cheng et al. 2019), while maintaining numerical accuracy, the computational efficiency of the numerical model can be significantly improved in a model using the Hermite weighted essentially nonoscillatory (WENO) method for the spatial reconstruction. It is noted that a similar idea was also implemented in the ADER scheme (Titarev and Toro 2002), where computation is largely simplified by a linearization procedure.

A fourth-order global shallow-water equation (SWE) model has been proposed in Chen and Xiao (2008) on the cubed-sphere grid with the application of multimoment finite-volume scheme. To achieve the desired accuracy in time, the widely used four-stage fourth-order Runge–Kutta scheme (RK44 scheme hereafter) is adopted for accomplishing the time integration as described in Eqs. (45) and (46) of Chen and Xiao (2008). In this paper, we present a new version of this SWE model with the implementation of the two-stage RK24 time integration scheme. Efforts have been made to design proper numerical formulations to evaluate the derivative in time by using the fourth-order multimoment spatial-discretization operator, which is the key task of building a two-stage fourth-order formulation. Being the first step to design a more efficient numerical framework for high-order atmospheric simulations, we proposed a RK24 time integration formulation for global advection and shallow-water equations on the cubed-sphere grid in this work. Although the proposed two-step RK24 method requires more operations during each substage to formulate the Cauchy–Kowalewski procedure, it saves about 30% of the total computational time compared to the four-step RK44, as shown in the numerical results for both advection equation and SWE presented in this paper.

The rest of this paper is organized as follows. In section 2, the two-stage RK24 scheme is briefly reviewed and its application in the one-dimensional multimoment scheme is described in detail. The proposed algorithm is extended to the SWE model on the cubed sphere in section 3. The benchmark tests are checked in section 4 for validation of the proposed two-stage fourth-order global SWE model and a short summary is given in section 5.

2. Two-stage multimoment scheme in one dimension

a. Two-stage fourth-order time marching scheme

To introduce the two-stage fourth-order time marching scheme, we consider the following ODEs:

$$q_i = \mathcal{U}(q_i),$$

(1)

$$q_{i=0} = q_0,$$

where $q_i$ is the predicted variables, $q_0$ is the initial condition, and $\mathcal{U}$ represents the operator for spatial discretization of partial differential equations.

To estimate the values of predicted variables at next time step ($q^{n+1}$), we integrate Eq. (1) as

$$q^{n+1} = q^n + \int_{t_n}^{t_{n+1}} \mathcal{U}(q) dt.$$

(2)

The integration in Eq. (2) is approximated by numerical quadrature. To achieve high-order accuracy, the values of predicted variables $q$ at an intermediate time instant $t^* = t^n + \alpha \Delta t (0 < \alpha < 1)$, that is, $q^*$, is adopted. Using Taylor series expansion of variable $q$ with respect to time $t$ up to second-degree term, we can approximate $q^*$ as

$$q^* = q^n + \alpha \Delta t \mathcal{U}(q^n) + \frac{\alpha^2 \Delta t^2}{2} \mathcal{U}_t(q^n).$$

(3)

Then the solution at next time step is the combination of $q$, $L(q)$, and $L_t(q)$ at $t^n$ and $t^*$:

$$q^{n+1} = q^n + \Delta t \left[ \gamma_1 \mathcal{U}(q^n) + \beta_1 \mathcal{U}(q^*) \right] + \frac{\Delta t^2}{2} \left[ \gamma_2 \mathcal{U}_t(q^n) + \beta_2 \mathcal{U}_t(q^*) \right].$$

(4)

As shown in Li and Du (2016), the two-stage time integration given by Eqs. (3) and (4) is able to achieve fourth-order accuracy with following coefficients:

$$\alpha = \frac{1}{2}, \quad \gamma_1 = 1, \quad \gamma_2 = \frac{1}{3}, \quad \beta_1 = 0, \quad \beta_2 = \frac{2}{3}. $$

(5)

In comparison with the model using RK44 scheme, the key task to accomplish the two-stage fourth-order time integration scheme is to construct the numerical algorithm to evaluate the derivative term $\mathcal{U}_t(q)$. Using the chain rule and original ODEs [Eq. (1)], we have

$$\mathcal{U}_t(q) = \mathcal{U}_q q_i = \mathcal{U}_q [\mathcal{U}(q)].$$

(6)

b. Two-stage fourth-order multimoment scheme

The fourth-order spatial-discretization formulations based on multimoment concept with two kinds of moments was developed in Chen and Xiao (2008). To use the two-stage time marching strategy, we describe here how to accomplish $\mathcal{U}_q [\mathcal{U}(q)]$ in Eq. (6). We first consider one-dimensional scalar hyperbolic equation as

$$\phi_t + f_x = 0,$$

(7)
where \( \phi(x, t) \) and \( f(\phi) \) are predicted variable and corresponding flux function, respectively.

The definition of degrees of freedom (DOFs) is shown in Fig. 1. Two kinds of DOFs are adopted, that is, the pointwise values defined at cell interface \([\phi_{i-1/2}(t)}\) and center \([\phi_i(t)}\). Instead of explicitly updating the volume-integrated average \([\overline{\phi}_i(t)}\), the framework of multimoment constrained finite-volume method (MVC) (Li and Xiao 2009) is adopted here for its straightforward extension to multidimensional models. Actually, the following spatial-discretization-formulations-based three-point MCV formulations are identical to that developed in Chen and Xiao (2008), if the volume-integrated average is recovered from local DOFs by three-point Simpson’s rule as

\[
\overline{\phi}_i(t) = \frac{1}{6} \left[ \phi_{i-1/2} + 4\phi_i + \phi_{i+1/2} \right].
\]

The methodology to derive the spatial discretization \( \mathcal{Q}(\phi) \) is described in detail in Chen and Xiao (2008) and Li and Xiao (2009). For sake of brevity, we directly give the formulations as follows without details of derivation process.

At cell interface, the derivative of flux function is determined by solving the derivative Riemann problem as

\[
\mathcal{Q}_{i-1/2} = -\hat{f}_{i-1} = \frac{1}{2} \left( \hat{f}^+_{i-1} + \hat{f}^-_{i-1} \right) - \frac{1}{2} a_i \left( \phi_{i-1/2}^+ - \phi_{i-1/2}^- \right),
\]

where \( \hat{f}_{i-1/2} \) is a numerical approximation of the derivative of the flux function, \( \hat{f}_{i-1/2} \), at the cell boundary \( x_{i-1/2} \), the superscripts ‘+’ and ‘−’ denote the derivatives evaluated from the spatial reconstructions of left and right cells and parameter \( a \) is determined by the adopted approximate Riemann solver.

The fourth-order accuracy is achieved by evaluating the derivatives of flux function as

\[
\hat{f}^+_{i-1/2} = \frac{1}{3\Delta x} \left( f_{i-1/2}^* - 6f_{i-1} + 3f_{i-1/2}^* + 2\hat{f}_{i-1} \right),
\]

\[
\hat{f}^-_{i-1/2} = \frac{1}{3\Delta x} \left( -2f_{i-1} - 3f_{i-1/2}^* + 6\hat{f}_{i-1} - f_{i-1/2}^* \right),
\]

where \( \Delta x \) is grid spacing of a uniform grid and \( \hat{f} \) is value of flux function at solution point approximated by known pointwise DOF.

Similarly, the derivatives of predicted variable, that is, \( \hat{\phi}_{i-1/2} \) and \( \hat{\phi}_{i-1/2} \), can be calculated through exchanging \( \hat{f} \) with \( \phi \) in Eq. (10).

At cell center, the time tendency of the DOF is derived with the constraint condition on volume-integrated average to assure the numerical conservation as

\[
\mathcal{Q}_i = \frac{3}{2} \frac{\hat{f}_{i-1/2} - \hat{f}_{i+1/2}}{\Delta x} + \frac{1}{2} \left( \mathcal{Q}_{i-1/2} + \mathcal{Q}_{i+1/2} \right).
\]

By rewriting Eqs. (9) and (11), the spatial discretization can be expressed at any solution point as

\[
\phi_{i-1/2} = \phi_i - \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} f(x, t) \ dx,
\]

\[
\phi_{i+1/2} = \phi_i + \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} f(x, t) \ dx.
\]
With the spatial discretization expressed in Eq. (12), we can then derive \( \mathcal{U}_q \) as

\[
\mathcal{U}_q = M^f \text{diag} \left[ f_1(f_1), f_2(f_2), \ldots, f_{N_f}(f_{N_f}) \right] + M^q, \tag{17}
\]

where the components of vector \( M^q \), in other words the parameters \( a_1 \) and \( a_2 \), are assumed here to be fixed at corresponding solution points.

Above formulas can be straightforwardly extended to the one-dimensional hyperbolic system for variables \( q \) and corresponding flux vector \( f(q) \), which are vectors of dimension \( N_q \), the number of equations. Similar as Eq. (12), the spatial discretization is expressed as

\[
\mathcal{U} = M^f F + M^q Q, \tag{18}
\]

where \( F = [f_1, f_2, \ldots, f_{N_f}] \), \( Q = [q_1, q_2, \ldots, q_{N_q}] \), and the dimensions of vectors \( F, Q \) are \( N_p \times N_q \).

Accordingly, row vector \( M^f \) becomes a \( N_q \times (N_q \times N_p) \) matrix having the form of

\[
M^f = \left[ m_1^f, m_2^f, \ldots, m_{N_q}^f \right], \tag{19}
\]

where \( I \) is an identity matrix of dimension \( N_q \), \( m_1^f, \ldots, m_{N_q}^f \) are elements of \( M^f \) given in Eqs. (13) and (15).

Matrix \( M^q \) has the same form as in Eqs. (14) and (16), except parameter \( a \) being a matrix of dimension \( N_q \) for the hyperbolic systems.

The operator \( \mathcal{U}_q \) also has the similar form as Eq. (17), except the element along principle diagonal, that is, \( I_q(q_k) (k = 1, \ldots, N_q) \), are \( N_q \times N_q \) Jacobian matrices of flux vector of hyperbolic systems at involved solution points.

With known DOFs, two types of spatial discretization \( \mathcal{U} \) and \( \mathcal{U}_q \) are accomplished using above formulations in sequence at each stage. The time integration is then carried out to update the predicted variables to next time step using Eqs. (3) and (4).

### 3. Global SWE model with two-stage time marching

The gnomonic cubed-sphere grid (Sadourny 1972) is adopted to represent the spherical geometry to avoid polar problems on longitude–latitude (lon–lat) grid in our global SWE model. Shown in Table 1, the cubed-sphere grid has the desirable uniformity in grid element size for different resolutions. The cubed-sphere consists of six patches with identical coordinate system. To extend the high-order multimoment scheme described in previous section to the 2D structured mesh on each patch, the shallow-water equations are first recast in local coordinate system in the so-called vector-invariant form as (Nair et al. 2005a)

\[
q + e_\xi + f_\eta = s, \tag{20}
\]

where the local coordinate system is \( (\xi, \eta) = (Ra, R\beta) \) with \( \alpha \) and \( \beta \) being the central angles of gnomonic projection, \( R \) is the radius of Earth, predicted variables \( q = [H h, u, v]^T \) include height of fluid \( (h) \), covariant velocity components \( (u, v) \), and \( J \) is Jacobian of transformation.

![Image of Table 1](image.png)

**Table 1.** Minimum, maximum area, and the largest area ratio for different grid resolutions.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Min area (km²)</th>
<th>Max area (km²)</th>
<th>Min/max</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 32 )</td>
<td>70,843.882</td>
<td>97,731.952</td>
<td>0.725</td>
</tr>
<tr>
<td>( N = 64 )</td>
<td>17,501.852</td>
<td>24,447.690</td>
<td>0.716</td>
</tr>
<tr>
<td>( N = 128 )</td>
<td>4349.117</td>
<td>6112.843</td>
<td>0.712</td>
</tr>
<tr>
<td>( N = 256 )</td>
<td>1083.974</td>
<td>1528.268</td>
<td>0.709</td>
</tr>
</tbody>
</table>

The flux functions and source term are

\[
e = \begin{bmatrix} Jh \hat{u} \\ Jh \hat{v} \\ 0 \end{bmatrix}, \quad f = \begin{bmatrix} Jh \hat{v} \\ 0 \end{bmatrix}, \quad \text{and} \quad s = \begin{bmatrix} 0 \\ J\hat{u}(\xi + f) \\ -J\hat{u}(\xi + f) \end{bmatrix}, \tag{21}
\]

where \( (\hat{u}, \hat{v}) \) are contravariant velocity components, \( f \) is the Coriolis parameter, \( \Psi \) is geopotential, \( \xi \) is the relative vorticity, and \( E_k \) is the kinetic energy.

On the cubed sphere, those quantities are defined as \( [u, v]^T = G^T[u, v]^T \), where \( G^T \) is contravariant metric tensor, \( \Psi = g(h + h_s) \) with \( g \) and \( h_s \) being the gravity acceleration and height of bottom mountain, and \( \xi \) and \( E_k \) are specified as \( \xi = 1/(Jf(\eta - u) \) and \( E_k = (1/2)(\hat{u} \hat{u} + \hat{v} \hat{v}) \).

The details of projection relations and transformation laws can be found in Nair et al. (2005b,a). The spatial discretization of shallow-water equations in spherical geometry, including the treatment of patch boundaries, is described in detail in Chen and Xiao (2008). For sake of brevity, we only focus in this section on description of modifications for adopting the two-stage fourth-order time integration [Eq. (4)].

The multimoment algorithm developed in previous section can be easily extended to solve the global shallow-water equations on the cubed sphere by applying the 1D formulations in different directions, respectively (Li and Xiao 2009). In \( \xi \) direction, we solve a hyperbolic system as

\[
(q)_{\xi}^f + e_\xi s = s_\xi, \tag{22}
\]

where the source term in \( \xi \) direction is specified as

\[
s_\xi = \begin{bmatrix} 0 \\ \hat{u} / \hat{v} \end{bmatrix}, \tag{23}
\]

Here the source term due to the Coriolis force are calculated in \( \xi \) direction. Actually, it can be evaluated during spatial discretization in either \( \xi \) or \( \eta \) direction as it does not include the partial differential operator.

Due to the existence of source term, expression of the spatial discretization shown in Eq. (18) has an additional term represent the discretization of source term, which can be expressed as

\[
\mathcal{U}^\xi = M^{\xi f} E + M^{\xi q} Q + M^{\xi q} Q. \tag{24}
\]
where the first two terms have the same form as what we have described in previous section for 1D hyperbolic system and the parameter (matrix) $a$ are $([|u| + \sqrt{G^1 gh}]$ by adopting the local Lax–Friedrichs Riemann solver (Nair et al. 2005a; Chen and Xiao 2008).

The last term in Eq. (24) represents the discretization of source term. The Coriolis force can be evaluated at each solution point directly by known pointwise values of DOFs. The derivative of $v$ with respect to $\xi$ used to calculate relative vorticity is computed using a centered scheme as follows.

At cell interface $\xi = \xi_{i-1/2}$,

$$\tilde{v}_{i-1/2} = \frac{1}{2} \left( \frac{v_{i-1/2} + v_{i+1/2}}{\Delta \xi} \right),$$

where $\tilde{v}_{i-1/2}$ are estimated using Eq. (10) by exchanging $f$ with $v$ and here we omit the subscript in $\eta$ direction since the same formulas are applied along different grid lines with constant $\eta$.

At cell center $\xi = \xi_i$, similar formulation as derived above for evaluating the derivative of flux function in Eq. (11) is adopted as

$$v_i = \frac{3}{2} \frac{v_{i+1} - v_{i-1}}{\Delta \xi} - \frac{1}{4} \left( v_{i+1/2} + v_{i-1/2} \right).$$

With Eqs. (25) and (26), we can easily derive the expression of matrix $M_q^{\xi}$.

Same as in previous section, we can derive $\mathcal{L}^\xi_q$ as

$$\mathcal{L}^\xi_q = M_q^{\xi} \text{diag} \left[ e_q(q_1), e_q(q_2), \ldots, e_q(q_N) \right] + M_q^{\eta} + M_q^{\xi},$$

where matrices $M_q^{\xi}$ and $M_q^{\eta}$ are again fixed at each solution point and the Jacobian matrix of flux vector $e(q)$ having the form of

$$e_q(q) = \begin{bmatrix} G^{11}u + G^{12}v & JG^{11}h & JG^{12}h \\ -G^{11}u + G^{12}v & G^{21}u + G^{22}v \end{bmatrix}. \tag{28}$$

Following the similar procedure, we can derive the spatial discretization in $\eta$ direction, i.e., $\mathcal{L}^\eta_q$ and $\mathcal{L}^{\eta_q}$ as follows:

$$\mathcal{L}^\eta_q = M_q^{\eta} F + M_q^{\eta} Q + M_q^{\xi}, \quad \mathcal{L}^{\eta_q} = M_q^{\eta} \text{diag} \left[ f_q(q_1), f_q(q_2), \ldots, f_q(q_N) \right] + M_q^{\eta} + M_q^{\xi}, \tag{29}$$

where

$$\mathcal{L}(q) = \mathcal{L}^\xi_q + \mathcal{L}^\eta_q, \quad \mathcal{L}_q(q) = \mathcal{L}^{\eta_q}[\mathcal{L}(q)] + \mathcal{L}_q^{\eta_q}[\mathcal{L}(q)]. \tag{30}$$

4. Numerical tests and results

In this section, numerical tests for both advection and shallow-water equations are conducted to verify the accuracy and efficiency of proposed two-stage fourth-order global
model on the cubed sphere in comparison with the existing RK44 method. The source code for the global advection model using RK24 time integration is available in the online supplemental material. The normalized errors are calculated for quantitatively evaluating the numerical results, which are defined following Williamson et al. (1992) as

\[
I_1 = \frac{I(|q - q_e|)}{I(q_e)}, \quad I_2 = \sqrt{\frac{I((q - q_e)^2)}{I((q_e)^2)}}, \quad I_\infty = \frac{\max|q - q_e|}{\max|q_e|},
\]

where \( I \) denotes the global integration, and \( q \) and \( q_e \) stand for the numerical result and exact solution, respectively.

### Table 2: The wall clock time (s) needed to finish the 12- and 5-day simulations of the Gaussian–Hill test and global steady zonal flow test, respectively, using RK24 and RK44 schemes. The wall clock time is measured at a PC with 3.20 GHz Intel i7–8700 processor and Ubuntu 18.04 OS.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>( T_{\text{RK24}} )</th>
<th>( T_{\text{RK44}} )</th>
<th>( \frac{(T_{\text{RK44}} - T_{\text{RK24}})}{T_{\text{RK44}}} \times 100% )</th>
<th>( T_{\text{RK24}} )</th>
<th>( T_{\text{RK44}} )</th>
<th>( \frac{(T_{\text{RK44}} - T_{\text{RK24}})}{T_{\text{RK44}}} \times 100% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 32 )</td>
<td>14.614</td>
<td>20.140</td>
<td>27.44%</td>
<td>105.966</td>
<td>161.934</td>
<td>34.56%</td>
</tr>
<tr>
<td>( N = 64 )</td>
<td>115.244</td>
<td>168.063</td>
<td>31.43%</td>
<td>841.753</td>
<td>1303.241</td>
<td>35.41%</td>
</tr>
<tr>
<td>( N = 128 )</td>
<td>925.403</td>
<td>1319.576</td>
<td>29.87%</td>
<td>7215.387</td>
<td>11 292.293</td>
<td>36.10%</td>
</tr>
<tr>
<td>( N = 256 )</td>
<td>8090.023</td>
<td>12 161.758</td>
<td>33.48%</td>
<td>64 974.192</td>
<td>97 770.584</td>
<td>33.54%</td>
</tr>
</tbody>
</table>

**FIG. 3.** Numerical errors and convergence rates of the (a),(b) global steady zonal geostrophic flow and (c),(d) unsteady solid body rotation tests (\( \alpha = \pi/4 \)). The results of (left) RK24 and (right) RK44 schemes are shown.
The relative conservation errors of mass ($\psi = h$), energy ($\psi = (h/2)(gh + \bar{u}\bar{u} + \bar{w}\bar{w})$), and potential enstrophy [$\psi = (\zeta + f)^2/2h$] are calculated as

$$\text{Err}_{\psi}(\psi, t) = \frac{I[\psi(t)] - I[\psi(0)]}{I[\psi(0)]}.$$  \hfill (35)

**a. Comparison with the RK44 scheme**

The RK42 method has been quantitatively compared with the RK44 method regarding accuracy and efficiency for both global advection and SWE models.

1) **ADVECTION MODEL**

Advection test of transporting a smooth profile with a given velocity field is adopted to check the numerical accuracy in terms of numerical error and convergence rate of the proposed two-stage RK24 global model and the RK44 model. Advection of a Gaussian hill on a unit sphere (Zhang and Nair 2012) was solved with the initial condition given as

$$h(x, y, z, 0) = h_{\text{max}} \exp\left\{-h_0[(x-x_c)^2 + (y-y_c)^2 + (z-z_c)^2]\right\},$$  \hfill (36)

where $h_{\text{max}} = 100$, $(x, y, z)$ is Cartesian coordinates of a point on sphere surface and $(x_c, y_c, z_c)$ is the initial location of hill center at $(3\pi/2, 0)$ in lon–lat coordinates.

The solid rotation flow field was given in lon–lat grid $(\lambda, \theta)$ as (Williamson et al. 1992)

$$u_\lambda = u_0(\cos\theta \cos\alpha + \sin\theta \cos\lambda \sin\alpha), \quad u_\theta = -u_0 \sin\lambda \sin\alpha,$$  \hfill (37)

where $u_0 = 2\pi R/(12 \text{ days})$ and $\alpha = \pi/4$ produces a northeast flow, which is usually considered as the most challenging path for the cubed-sphere grid since it passes two complete patch boundaries.

The resolution of the cubed-sphere grid is denoted by an integer $N$, which is the number of elements along each direction on every patch. Thus, the grid spacing along equator is $90^\circ/N$ (or $45^\circ/N$ in terms of DOFs). This test is carried out on gradually refined grids with $N = 32$ to $N = 256$. The uniformity of mesh spacing in different grids is determined using the ratios of the minimum and maximum areas of mesh cells as shown in Table 1. All normalized errors and convergence rates for both RK24 and RK44 schemes are given in Fig. 2. There are only slight differences in the numerical errors of the two schemes, and both schemes have nearly identical convergence rate close to the theoretical fourth-order accuracy. The comparison of computational efficiency in terms of the wall clock time is shown in Table 2. The RK24 scheme, with fewer substeps, is more efficient. For a grid with a resolution of practical interest, RK24 saves about 30% computational time compared to the RK44 method.

2) **SWE MODEL**

Comparisons between RK24 and RK44 schemes have been carried out for the SWE global model as well. We solved the SWE model for the global steady zonal geostrophic flow test (Williamson et al. 1992) (Case 2) and the unsteady solid body

![Fig. 4. Numerical results of advection of cosine bell ($\alpha = \pi/4$) using RK24 scheme on $N = 32$ grid at (a) $T/4$, (b) $T/2$, and (c) $T$, where $T$ represents one complete period. The solid line and red square indicate the numerical and exact solutions, respectively.](image-url)
rotation test Läuter et al. (2005) (Example 3) on gradually refined grids. The details of the setup of these two tests are given in Williamson et al. (1992) and Läuter et al. (2005).

The numerical results measured by normalized errors of the predicted depth of fluid for both RK24 and RK44 schemes are shown in Fig. 3. Again, both schemes generate slight difference in numerical errors and nearly identical convergence rates close to fourth order as expected. It is observed that the errors in the steady case are smaller compared to the unsteady case, but the convergence rate remains nearly fourth order as well in the unsteady case. Regarding computational efficiency, RK24 shows even larger advantage over RK44 in solving the nonlinear system equations. Shown in Table 2, RK24 saves 36.10\% of the wall clock time in comparison with RK44 on N = 128 grid for the global steady zonal geostrophic flow test.

We also examined the numerical accuracy of RK24 by solving the unsteady SWE with a very small time step interval $\Delta t$, which is about 1/40 of that used in Fig. 3 for each grid resolution. The normalized errors and convergence rates of the results at 6 h are shown in Table 3. It is observed that all conclusions on numerical accuracy gained from Fig. 3 with large $\Delta t$ hold for solutions using small $\Delta t$.

b. Advection of cosine bell

Advection of cosine bell (Williamson et al. 1992) was extensively used in literature to check the performance of global models. In this test, the number of the time steps is set to 256 and $\Delta t$ is 4050.00 s. The numerical results of advection in northeast direction [using velocity specified in Eq. (37)] are given in Fig. 4 at different time instants within one complete period. It can be observed that our two-stage model produces adequate numerical solutions without notable dissipation and dispersion errors. The normalized errors of advection of cosine bell in different directions after one period are displayed in Table 4 and corresponding time evolution of errors are given in Fig. 5. There is no significant difference in numerical errors among the results of different flow directions. The case of $\alpha = \pi/4$ shows errors slightly smaller than the cases of $\alpha = 0$ and $\alpha = \pi/2$. The errors produced by flows along the equator and crossing two poles are identical due to symmetry of the cubed sphere, which verifies the correctness of computer codes. And no obvious increase of errors when the cosine bell passes the patch boundaries.

c. Zonal flow over an isolated mountain

Case 5 of Williamson et al. (1992) test suite is a zonal flow over an isolated mountain, which is a widely used benchmark test to evaluate the performance of a numerical model in solving problems including topographic effect.

The center of the mountain is located at $(\lambda_c, \theta_c) = (3\pi/2, \pi/6)$ and this test case is integrated for 15 model days with $\Delta t = 100.00$ s on a N = 32 grid. It is observed that RK24 results shown in Fig. 6 are visually identical to that given in Chen and Xiao (2008) with four-stage traditional RK method and are quite similar to those of the spectral transform method with much higher-resolution T213 [Fig. 5.1 of Jakob-Chien et al. (1995)].

d. Rossby–Haurwitz wave

The Rossby–Haurwitz wave test [case 6 in Williamson et al. (1992)] provides a testbed for global medium-range
simulations. The numerical results by the spectral method on the high-resolution T213 grid [Fig. 5.5 of Jakob-Chien et al. (1995)] are widely accepted as the reference solutions due to its accuracy for multiscale phenomena.

Numerical results for integrating the SWE model up to days 14 on grid \( N = 48 \) are shown in Fig. 7. It is found that RK24 results shown in Figs. 7a and 7c, after integration for 7 and 14 days, agree well with the reference solution and almost identical to the results of RK44 scheme. The numerical solutions over the Northern and Southern Hemispheres demonstrate perfect symmetry. The evolution of relative conservation errors of total energy and potential enstrophy are given in Fig. 8 on grid \( N = 32 \) to be consistent with the configuration in existing studies. The magnitudes of relative conservation errors of total energy and potential enstrophy of RK24 are comparable to Jakob-Chien et al. (1995) on grid T63. Again, the difference of the results between RK24 and RK44 is not significant.

e. Barotropic instability test

The barotropic instability test was proposed in Galewsky et al. (2004). This test is considered as a very challenging test case for numerical models on the cubed sphere, since the relative vorticity field strongly deforms, due to a perturbation in height field, within a very narrow belt-type zone along four patch boundaries. On coarse grid, the numerical result will be dominated by grid imprinting errors. It is of great interests to check if the numerical results will finally converge to the reference solution on T341 grid (Galewsky et al. 2004) with refined resolutions. The numerical results of relative vorticity fields on gradually refined grids are shown in Fig. 9 at day 6. The 4-wave phenomenon is observed on grid \( N = 32 \), which obviously reflects the structure of the cubed-sphere grid. It is also observed that some small-scale structures around 60°N are different in comparison with those in Chen and Xiao (2008) on this coarsest grid. With refined resolutions, the grid imprinting...
error has been successfully suppressed on grid $N = 64$ and no noticeable differences can be found by comparing the results on grids $N = 96$ and $N = 128$. Except the result on grid $N = 32$, the proposed model well reproduces the structures of relative vorticity field compared with the reference solution. Meanwhile, the numerical noises due to the very steep distribution at day 6 are obviously reduced in our results owing to the inherent numerical viscosity of upwind schemes. Again, the numerical results by multimoment models using different time marching schemes are visually identical on the finest grid.

5. Conclusions

The objective of this paper is to present the formulation and verification of a two-stage fourth-order multimoment scheme for shallow-water equations on the cubed-sphere grid. The key task to accomplish the time integration specified by Eqs. (3) and (4) is to derive the formulations to estimate the second-order derivative in time of the computational variables by the first-order temporal derivative of the multimoment spatial-discretization operator through the Cauchy–Kowalewski procedure (or the Lax–Wendroff procedure).

In practice, the time derivative of the spatial operator is recast as a product of the spatial operator and the derivative of the spatial operator with respect to the predicted variables. The latter is calculated straightforwardly by differentiating the multimoment spatial discretization with respect to the predicted variables. It results in a simple formulation without second- or higher-order derivatives of the predicted variables or flux functions with respect to the spatial coordinates.

The numerical results verify the fourth-order convergence rate and comparable accuracy in comparison with the existing global SWE model using traditional four-stage Runge–Kutta scheme. As expected, using this two-stage fourth-order time integration does not degrade the accuracy of our global SWE model. The two-stage fourth-order scheme saves about 30% of the computational cost in comparison with the four-stage Runge–Kutta scheme for global advection and shallow-water models. This study presents a new numerical framework to efficiently solve complex atmospheric flows.

As for the practical models, the time marching strategy is usually more complicated instead of simple application of explicit Runge–Kutta schemes. The proposed RK24 method may provide a more efficient numerical framework. For example, considering the atmospheric general circulation models, the horizontally explicit/vertically implicit (HEVI) strategy is getting popular recently as the flow mechanisms and corresponding computational setup have very different characteristics in horizontal and vertical directions (Gardner et al. 2018). With fewer substeps in RK24 method, the overall computational overheads may be greatly reduced due to fewer calls of time-consuming iteration solver for nonlinear equation set. In the future, it is worth further investigations on designing the high-order implicit-explicit (IMEX) schemes based on explicit RK24 method and carefully analysis on their accuracy, stability and efficiency properties.

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APPENDIX

The Coefficient Matrices in RK24 SWE Scheme

In $\xi$ direction, the coefficient matrices $M^f$ and $M^q$ at cell boundary $[i - (1/2), j]$ can be easily derived from Eqs. (13), (14), and (19) and given as follows:
and

\[
\mathbf{M}'_{ij} = \mathbf{a}_{ij} \left[ \begin{array}{cccccccc}
-\frac{1}{6} & 0 & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{6} & 0 & 0 & \frac{4}{3} & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{6} & 0 & 0 & \frac{4}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right].
\]

(A1)

where

\[
\mathbf{a}_{ij} = \left[ \begin{array}{cccccccc}
\tilde{u}_{i-\frac{1}{2}}, h_{i-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{u}_{i-\frac{1}{2}}, h_{i-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{u}_{i-\frac{1}{2}}, h_{i-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{u}_{i-\frac{1}{2}}, h_{i-\frac{1}{2}} & 0 & 0 & 0 & 0 \\
\end{array} \right].
\]

(A3)

The coefficient matrices of source term in Eqs. (24) and (27) can be derived as

\[
\mathbf{M}_1' = \mathbf{a}_{ij} \left[ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{b_1}{6} & 0 & 0 & -\frac{4b_1}{3} & 0 & 0 & 0 \\
0 & 0 & \frac{b_2}{6} & 0 & 0 & -\frac{4b_2}{3} & 0 & 0 \\
0 & 0 & 0 & \frac{b_1}{6} & 0 & 0 & -\frac{4b_1}{3} & 0 \\
0 & 0 & 0 & 0 & \frac{b_2}{6} & 0 & 0 & -\frac{4b_2}{3} \\
\end{array} \right].
\]

(A4)

and

\[
\mathbf{M}_2' = \mathbf{a}_{ij} \left[ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{b_1}{6} & 0 & 0 & -\frac{4b_1}{3} & 0 & 0 & 0 \\
0 & 0 & \frac{b_2}{6} & 0 & 0 & -\frac{4b_2}{3} & 0 & 0 \\
0 & 0 & 0 & \frac{b_1}{6} & 0 & 0 & -\frac{4b_1}{3} & 0 \\
0 & 0 & 0 & 0 & \frac{b_2}{6} & 0 & 0 & -\frac{4b_2}{3} \\
\end{array} \right].
\]

(A5)
where $b_1 = \tilde{u}_{i-(1/2),j}$, $b_2 = -\tilde{u}_{j-(1/2),i}$, and $b_3 = u_{ij-(1/2)}$.

In $\eta$ direction, the coefficient matrices $\mathbf{M}_i^\eta$ and $\mathbf{M}_j^\eta$ at cell boundary $[i, j - (1/2)]$ can be easily obtained by changing $\Delta \xi$ and $\mathbf{a}_i^\eta$ in Eqs. (A1) and (A2) to $\Delta \eta$ and

\[
\mathbf{a}_i^\eta = \begin{bmatrix}
\sqrt{|\tilde{v}_{i, j-(1/2)}|} + \sqrt{\frac{g G_{ij}^2}{\gamma} h_{i-(1/2)}} & 0 & 0 \\
0 & \sqrt{|\tilde{v}_{i, j-(1/2)}|} + \sqrt{\frac{g G_{ij}^2}{\gamma} h_{j-(1/2)}} & 0 \\
0 & 0 & \sqrt{|\tilde{v}_{i, j-(1/2)}|} + \sqrt{\frac{g G_{ij}^2}{\gamma} h_{j-(1/2)}}
\end{bmatrix}, \tag{A6}
\]

respectively. For the source term,

\[
\mathbf{M}_i^\eta = \frac{1}{\Delta \eta} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_4 & 0 & 0 & -4b_4 & 0 & 0 & 0 & 0 & \frac{4b_4}{3} & 0 \\
b_5 & 0 & 0 & -4b_5 & 0 & 0 & 0 & 0 & \frac{4b_5}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{A7}
\]

and

\[
\mathbf{M}_j^\eta = \frac{1}{\Delta \eta} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_4 & 0 & 0 & -4b_4 & 0 & -b_6 G_{ij}^{12} & 0 & \frac{4b_4}{3} & 0 \\
0 & b_5 & 0 & 0 & -4b_5 & 0 & b_6 G_{ij}^{12} & 0 & \frac{4b_5}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_5
\end{bmatrix}, \tag{A8}
\]

where $b_4 = -\tilde{v}_{j-(1/2),i}$, $b_5 = \tilde{u}_{j-(1/2),i}$, and $b_6 = u_{ij-(1/2)}$.

Similarly, the coefficient matrices in both $\xi$ and $\eta$ at cell center $(i, j)$ can be easily obtained from Eqs. (15), (16), and (26).

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