A New Hybrid Sigma-Pressure Vertical Coordinate with Smoothed Coordinate Surfaces

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ABSTRACT: An alternative hybrid sigma-pressure terrain-following coordinate is presented here that provides smoother coordinate surfaces over terrain by allowing a more rapid decay of the influence of smaller-scale topographic structures with height. This is accomplished by first defining a reference surface pressure that includes the influence of the underlying topography. A smoothed version of this reference surface pressure is then created that represents the larger-scale features of the topography, while the deviations from the smoothed profile contain the smaller-scale terrain structures. In the hybrid-sigma coordinate formulation presented here, the influences of these deviations in the reference surface pressure from their smoothed values are removed more rapidly with increasing height, thereby producing smoother coordinate surfaces. Testing this approach using several idealized simulations demonstrates a significant reduction in the artificial circulations compared to those arising with the basic sigma or the conventional hybrid sigma coordinate, confirming the beneficial aspects of the smoothed hybrid coordinate surfaces. The smoothed hybrid sigma-pressure coordinate proposed here provides flexibility in reducing the influence of the terrain on the coordinate surfaces and can be easily substituted for the basic hybrid sigma-pressure coordinate.

SIGNIFICANCE STATEMENT: Terrain-following vertical coordinates are widely employed in atmospheric numerical models because of their advantages in implementing boundary conditions at the surface. However, it is well known that these coordinates can promote artificial circulations due to numerical errors in computing horizontal pressure gradients and advection along the transformed coordinate surfaces. The purpose of this study is to improve the performance of a pressure-based terrain-following (sigma) hybrid vertical coordinate by selectively removing the influences of smaller-scale terrain features, which results in smoother coordinate surfaces. We demonstrate that this enhancement is beneficial in reducing numerical errors that are inherent in computing with terrain-following vertical coordinates.

KEYWORDS: Numerical analysis/modeling; Vertical coordinates; Mountain waves; Model errors

1. Introduction

The advantages of a terrain-following vertical coordinate, based either on height (Gal-Chen and Somerville 1975) or on pressure (Phillips 1957), are clear in implementing the lower boundary conditions. However, the influence of the terrain structure on the terrain-following coordinate surfaces can compromise numerical accuracy, particularly in representing the horizontal pressure gradients along constant height surfaces. Above regions of relatively steep terrain, the small-scale structures in the coordinate surfaces can produce spurious small-scale circulations (Janjić 1989), degrade the computation of advection (Schär et al. 2002), and lead to numerically inconsistent approximations of the horizontal pressure gradients (Mahrer 1984). To alleviate the problems associated with the terrain-following coordinate, various modifications to the basic terrain-following coordinate formulation have been introduced to more rapidly remove terrain influences on the coordinate surfaces with increasing height.

One method of mitigating the disadvantages of the basic terrain-following coordinate is called a hybrid terrain-following coordinate, in which the terrain influences in the coordinate surfaces decrease with increasing height more rapidly than the linear decrease in the basic sigma coordinate. By transitioning to constant height (or pressure) surfaces at a height well below the top of the model domain, the influences of terrain structure on the coordinate surfaces may be significantly reduced (Simmons and Burridge 1981). Further, Schär et al. (2002) proposed a modified terrain-following hybrid coordinate, named the smooth level vertical (SLEVE) coordinate, to provide more flexible control over the reduction of terrain influences on the coordinate surfaces by allowing smaller-scale terrain influences on the coordinate surfaces to be removed more rapidly with height than the pure hybrid coordinate. The idea of SLEVE coordinate is the use of a scale-dependent vertical decay of terrain features, such that small-scale variations decay more quickly with height than larger scale features, resulting in smoother coordinate surfaces at mid and higher levels (Schär et al. 2002). More recently, Klemp (2011) suggested an alternative terrain-following coordinate that smooths the coordinate surfaces of the hybrid coordinate progressively with height, providing more flexibility in selectively removing small-scale structure from the coordinate surfaces.

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The terrain-following coordinate specifications proposed by Schär et al. (2002) and Klemp (2011) are derived for height-based terrain-following coordinate systems only. For a pressure-based terrain following coordinate, Zängl (2003) presented an approach similar to the Schär et al. (2002) method, in which the pseudo sigma coordinate system of MM5 was modified to achieve the coordinate smoothing by separating a reference surface pressure into a smoothed portion and a smaller scale residual that is attenuated more rapidly in the coordinate formulation. However, in this pseudo sigma coordinate system, the surface pressure is based on a reference surface pressure rather than the actual surface pressure, and thus the coordinate surfaces are fixed in time as in a height-based coordinate. Recently, Husain et al. (2020) described a smooth-level coordinate specification for the Global Environmental Multiscale (GEM) model's hybrid log-hydrostatic pressure-based terrain-following vertical coordinate and demonstrated the relative performance in the context of an operational NWP model, in which the model controlled separately the progressive decay of the small- and large-scale orography contributions on the vertical coordinate surfaces.

In this study, an alternative pressure-based terrain-following coordinate employing smoothed coordinate surfaces is proposed that allows more flexible control over terrain influences. The object of this alternative coordinate is to reduce smaller-scale terrain influences on coordinate surfaces with increasing height above the terrain more rapidly than can be achieved with the conventional hybrid sigma-pressure coordinate. The structure of this paper is as follows: the description of the approach is presented in section 2, and the description of the model and test cases follows in section 3. Section 4 provides the results for these test cases and demonstrates the performance for this coordinate in comparison with other terrain-following coordinates. Finally, a summary and discussion are given in section 5.

2. Description of the smoothed hybrid sigma-pressure vertical coordinate

a. Formulation

A hybrid representation of the traditional terrain-following (HYB) coordinate may be written in the form:

$$\pi(i, \eta, t) = B(\eta)[\pi_{\text{sfc}}(i, t) - \pi_{\text{top}}] + [\eta - B(\eta)](\pi_0 - \pi_{\text{top}}) + \pi_{\text{top}}, \quad (1)$$

where π is the hydrostatic pressure of dry air, π_{sfc} is the surface pressure, π_{top} is the top-level pressure, π_0 is a reference sea level pressure (set as 1000 hPa in this study), η represents the transformed vertical coordinate, and i and t denote horizontal and temporal indices, respectively. Here, $B(\eta)$ is the hybrid parameter defining the relative weighting of the terrain-following and pure hydrostatic pressure coordinate. This hybrid representation is used in several models, such as KIM (Korean Integrated Model, Hong et al. 2018; Choi and Hong 2016) and WRF-ARW Model Version 4 (Skamarock et al. 2019; Park et al. 2019). For $B(\eta) = \eta$, this coordinate formulation is the same as the basic sigma (SIG) coordinate written in the following form:

$$\pi(i, \eta, t) = \eta[\pi_{\text{sfc}}(i, t) - \pi_{\text{top}}] + \pi_{\text{top}}. \quad (2)$$

To provide more flexibility in removing small-scale terrain influences on the coordinate surfaces, we alter this hybrid formulation to allow further smoothing of the coordinate surfaces. For a pressure-based coordinate, the terrain structures are projected into the coordinate representation through the surface pressure. By creating a smoothed version of the surface pressure, the deviation part from this smoothed profile contains mainly the smaller-scale terrain structure, and its influence can be removed more rapidly from the coordinate surfaces. However, since the surface pressure is continually changing with time, this approach is problematic. To avoid this complication, we write the surface pressure in terms a reference profile $\overline{\pi_{\text{sfc}}}(i)$ (that contains the influences of terrain) and a perturbation $\pi'_{\text{sfc}}(i, t)$ from that reference such that $\pi_{\text{sfc}}(i, t) = \overline{\pi_{\text{sfc}}}(i) + \pi'_{\text{sfc}}(i, t)$. We then represent the reference surface pressure as

$$\overline{\pi_{\text{sfc}}}(i) = \overline{\pi_{\text{sfc}}}(i)|_{\text{sm}} + \overline{\pi_{\text{sfc}}}(i)|_{\text{dev}}, \quad (3)$$

where $\overline{\pi_{\text{sfc}}}(i)|_{\text{sm}}$ is a smoothed version of the reference surface pressure (that does not vary in time) and $\overline{\pi_{\text{sfc}}}(i)|_{\text{dev}}$ represents the deviation containing most of the smaller scale structure. Since this reference surface pressure will contain a large portion of the actual surface pressure, the smoothed and deviation components of it can be used to preferentially remove the smaller scale terrain influences from the coordinate surfaces. Therefore, a new smoothed hybrid sigma-pressure (SMH) coordinate can be written in the following form:

$$\begin{aligned} \pi(i, \eta, t) = & B(\eta)[\overline{\pi_{\text{sfc}}}(i)|_{\text{sm}} + \pi'_{\text{sfc}}(i, t) - \pi_{\text{top}}] + [\eta - B(\eta)] \\ & \times (\pi_0 - \pi_{\text{top}}) + C(\eta)[\overline{\pi_{\text{sfc}}}(i)|_{\text{dev}}] + \pi_{\text{top}}. \end{aligned} \quad (4)$$

Here, the hybrid parameter $B(\eta)$ regulates the removal rate of the influence of the larger scale features of the terrain while $C(\eta)$ governs the attenuation rate of smaller scale terrain influences, which will be more rapid provided $C(\eta) < B(\eta)$. Again, $C(\eta) = B(\eta)$ recovers the hybrid coordinate and $C(\eta) = B(\eta) = \eta$ reverts to the basic sigma coordinate.

In the SMH coordinate proposed here, we define $B(\eta)$ with a third-order polynomial as in WRF-ARW Model version 4 as proposed by Park et al. (2019):

$$B(\eta) = b_0 + b_1\eta + b_2\eta^2 + b_3\eta^3 \quad (5)$$

subject to the boundary conditions

$$B(1) = 1, \quad \frac{\partial B}{\partial \eta}(1) = 1, \quad B(\eta_b) = 0, \quad \frac{\partial B}{\partial \eta}(\eta_b) = 0, \quad (6)$$

such that

$$\begin{aligned} b_0 &= \frac{2\eta_b^2}{(1 - \eta_b)^3}, & b_1 &= \frac{-\eta_b(4 + \eta_b + \eta_b^2)}{(1 - \eta_b)^3}, \\ b_2 &= \frac{2(1 + \eta_b + \eta_b^2)}{(1 - \eta_b)^3}, & b_3 &= \frac{-(1 + \eta_b)}{(1 - \eta_b)^3}, \end{aligned} \quad (7)$$

where the η_b is a specified value of η at which the coordinate becomes a pure pressure coordinate [$B(\eta) = 0$].

In defining $C(\eta)$ as a polynomial, there are several orders that might be considered. For an example, the first-order polynomial would be

FIG. 1. (a) Example profiles of $A(\eta)$, $B(\eta)$, and the second-order $C(\eta)$ in case of $\eta_b = 0.2$ and $\eta_c = 0.8$ as a function of height for a 288-K isothermal atmosphere in a domain with a 100-hPa upper boundary. (b) Profiles of $C(\eta)$ near the surface for the first-, the second-, and the third-order polynomial.

$$C(\eta) = c_0 + c_1 \eta, \quad (8)$$

with the boundary conditions

$$C(1) = 1, \quad C(\eta_c) = 0, \quad (9)$$

such that

$$c_0 = \frac{-\eta_c}{(1-\eta_c)}, \quad c_1 = \frac{1}{(1-\eta_c)}. \quad (10)$$

Defining $C(\eta)$ as a second-order polynomial we have

$$C(\eta) = c_0 + c_1 \eta + c_2 \eta^2 \quad (11)$$

with the boundary conditions

$$C(1) = 1, \quad C(\eta_c) = 0, \quad \frac{\partial C}{\partial \eta}(\eta_c) = 0, \quad (12)$$

such that

$$c_0 = \frac{\eta_c^2}{(1-\eta_c)^2}, \quad c_1 = \frac{-2\eta_c}{(1-\eta_c)^2}, \quad c_2 = \frac{1}{(1-\eta_c)^2}, \quad (13)$$

where the η_c is a specified value of η at which the small-scale orographic influences have decreased to zero. Using a third-order polynomial for $C(\eta)$ yields the same expressions as (5)–(7), just replacing B , b_i , and η_b with C , c_i , and η_c . It is noteworthy that in employing $C(\eta)$ as a third-order polynomial, the same as $B(\eta)$, the SMH coordinate has a nice property of reverting directly to the original HYB coordinate of (1) for $\eta_c = \eta_b$. Further, the added constraint $(\partial C / \partial \eta)(1) = 1$ relaxes the

restriction on the maximum η_c that maintains the monotonic property of the Jacobian of the transformation (i.e., the requirement that $\partial \pi / \partial \eta > 0$), which is discussed later. Figure 1 illustrates the profiles of $A(\eta) = \eta - B(\eta)$, $B(\eta)$, and $C(\eta)$ used for the experiments with L100 of evenly spaced vertical depth ($\Delta z = 250$ m) in this study, where $\eta_b = 0.2$ and $\eta_c = 0.8$ are specified.

b. Implementation

The governing equations of the model used here utilize perturbation thermodynamic variables from a hydrostatically balanced reference state as prognostic variables (see the appendix). For the perturbation form of the governing equations, the reference pressure $\bar{\pi}(i, \eta)$ can then be defined directly from the SMH vertical coordinate definition in (4):

$$\begin{aligned} \bar{\pi}(i, \eta) = & B(\eta)[\bar{\pi}_{\text{sfc}}(i)|_{\text{sm}} - \pi_{\text{top}}] + [\eta - B(\eta)](\pi_0 - \pi_{\text{top}}) \\ & + C(\eta)[\bar{\pi}_{\text{sfc}}(i)|_{\text{dev}}] + \pi_{\text{top}}. \end{aligned} \quad (14)$$

Other than this modification, no changes in the numerics are required.

With a terrain-following vertical pressure coordinate, the governing equations contain a coordinate metric term μ_d (e.g., see Klemm et al. 2007), which for the coordinate transformation (4), has the following form:

$$\begin{aligned} \mu_d(i, \eta, t) = & \frac{\partial \pi}{\partial \eta}(i, \eta, t) = \frac{\partial B}{\partial \eta}(\eta)[\bar{\pi}_{\text{sfc}}(i)|_{\text{sm}} + \pi'_{\text{sfc}}(i, t) - \pi_{\text{top}}] \\ & + \left[1 - \frac{\partial B}{\partial \eta}(\eta)\right](\pi_0 - \pi_{\text{top}}) + \frac{\partial C}{\partial \eta}(\eta)[\bar{\pi}_{\text{sfc}}(i)|_{\text{dev}}], \end{aligned} \quad (15)$$

where μ_d satisfies the continuity equation:

$$\frac{\partial \mu_d}{\partial t} + \nabla \cdot \mu_d \mathbf{v}_H + \frac{\partial \mu_d \dot{\eta}}{\partial \eta} = 0. \quad (16)$$

Here, $\mathbf{v}_H$ represents the covariant velocities in the horizontal direction and $\dot{\eta}$ is the contravariant vertical velocity. This continuity equation allows the perturbation surface pressure π'_{sfc} to be recovered by integrating (16) from the top of the domain to the surface, and using (15), which yields

$$\int_0^1 \frac{\partial \mu_d}{\partial t} d\eta = \frac{\partial}{\partial t} \int_0^1 \frac{\partial \pi}{\partial \eta} d\eta = \frac{\partial \pi'_{\text{sfc}}}{\partial t} = - \int_0^1 \nabla \cdot \mu_d \mathbf{v}_H d\eta. \quad (17)$$

Thus, the evolving hydrostatic surface pressure is recovered in the same manner as in the SIG (Klemp et al. 2007) and HYB (Choi et al. 2014) coordinate formulations.

To construct a smoothed representation of the surface pressure $\overline{\pi_{\text{sfc}}}(i)|_{\text{sm}}$, we follow Klemp (2011) in recursively applying a local second-order Laplacian:

$$\overline{\pi_{\text{sfc}}}(i)|_{\text{sm}}^{n+1} = \overline{\pi_{\text{sfc}}}(i)|_{\text{sm}}^n + \alpha d^2 \nabla^2 \overline{\pi_{\text{sfc}}}(i)|_{\text{sm}}^n, \quad (18)$$

where α is the dimensionless smoothing coefficient (in this study, $\alpha = 0.25$) and d is a characteristic gridcell dimension ($d = \Delta x$). After applying (18) with a specified number of iterations M ($M = 50$ here), the two parts of the reference surface pressure are defined as follows:

$$\begin{aligned} \overline{\pi_{\text{sfc}}}(i)|_{\text{sm}} &= \overline{\pi_{\text{sfc}}}(i)|_{\text{sm}}^M, \\ \overline{\pi_{\text{sfc}}}(i)|_{\text{dev}} &= \overline{\pi_{\text{sfc}}}(i) - \overline{\pi_{\text{sfc}}}(i)|_{\text{sm}}. \end{aligned} \quad (19)$$

Figure 2 displays the partitioning of the reference surface pressure into the smoothed and the deviation parts for a Schär-type mountain that was used for the resting atmosphere experiment, described in the next section, and illustrates that the smoothed and the deviation parts represent well the large- and small-scale topography, respectively. The magnitudes of both parts are dependent on the smoothing coefficient α and the number of smoothing iterations M . The constraint on the α and the iteration number M can vary depending on the numerical discretization method, but in case of SEM can be analytically evaluated by using eigenvalues of the horizontal 1D Laplacian matrix and their relative character that were investigated in Choi (2018) and Park and Choi (2020). Generally, the way to get the filtering effect by using a local Laplacian operator is to use moderate α and sufficiently large M .

In partitioning the surface pressure and determining the allowable values of η_b and η_c , a necessary condition for the terrain-following coordinate is that the Jacobian of the transformation should be a monotone function (i.e., $\mu_d \equiv \partial \pi / \partial \eta > 0$), as discussed by Gal-Chen and Somerville (1975) and Schär et al. (2002). The constraints imposed by this requirement are hard to cast in a simple form and are more complex than those for the HYB coordinate in (1), which are presented in Park et al. 2019. However, we can get some clues for identifying these constraints as minimum values should be positive in $0 \leq \eta \leq 1$. A minimum could occur at three possible values of η , where $\partial^2 B / \partial \eta^2 = 0$, $\partial^2 C / \partial \eta^2 = 0$, and at $\eta = 1$. Using the

FIG. 2. The partitioning of the reference surface hydrostatic pressure: (a) total surface hydrostatic pressure (π_{sfc}), and the smoothed part ($\pi_{\text{sfc}}|_{\text{sm}}$) produced by the smoothing operator; and (b) the deviation part of the pressure ($\pi_{\text{sfc}}|_{\text{dev}}$) from the smoothed profile.

third-order polynomial for $C(\eta)$, the value of $\mu_d \equiv \partial \pi / \partial \eta$ for $\eta = 1$ is naturally positive due to the definition of the boundary conditions ($\partial B / \partial \eta(1) = 1$ and $(\partial C / \partial \eta)(1) = 1$). The remaining conditions of concern are where $\partial^2 B / \partial \eta^2 = 0$ and $\partial^2 C / \partial \eta^2 = 0$, which occur at $\eta = -(1/3)(b_2/b_3)$ and $\eta = -(1/3)(c_2/c_3)$, respectively. Therefore, the proposed SMH coordinate does not violate the monotonicity if $\min\{\mu_d[i, -(b_2/3b_3)]\} > 0$ and $\min\{\mu_d[i, -(c_2/3c_3)]\} > 0$ for all horizontal index i .

Using the second-order polynomial for $C(\eta)$ described in (11)–(13), the monotonicity condition would be most restrictive for $\eta = 1$, which means that the largest allowable η_c has to be smaller than that for the third-order polynomial (i.e., the level at which the small-scale topographic features in the coordinate surfaces decay totally has to be raised). The dominance of the restriction at $\eta = 1$ is due to the relatively large value of $(\partial C / \partial \eta)(1)$; for $\eta_c = 0.8$, $(\partial C / \partial \eta)(1) = c_1 + 2c_2 \sim 10$. Therefore, employing the third-order polynomial for $C(\eta)$ significantly reduces the constraint of how large η_c can be in maintaining the property $\mu_d \equiv \partial \pi / \partial \eta > 0$. The third-order polynomial allows a larger η_c , but for the

same η_c , the second-order polynomial decreases more rapidly with height.

From a practical standpoint, satisfying the monotonicity condition (although necessary) is not particularly helpful since at this limit the coordinate surfaces have collapsed and cannot represent a viable coordinate transformation. Thus, in seeking an appropriate amount of smoothing for the coordinate surfaces, it is more important to check to ensure coordinate surfaces do not get too close together to maintain numerical stability and accuracy. To monitor the vertical spacing of the coordinate surfaces, an index representing the minimum fractional grid spacing $\gamma_{\min}$ was suggested by Klemp (2011), which can be cast as follows:

$$\gamma_{\min} = \min \left[\frac{\pi_{\text{SMH}}(i, k) - \pi_{\text{SMH}}(i, k-1)}{\pi_{\text{SIG}}(i_0, k) - \pi_{\text{SIG}}(i_0, k-1)} \right], \quad (20)$$

for all i and k , where i and k denote horizontal and vertical indices, respectively; i_0 is a location far away from the mountain; and π_{SMH} and π_{SIG} are the pressure on the SMH and SIG coordinate surfaces defined by (4) and (2), respectively. The $\gamma_{\min}$ will decrease with increasing η_c , which increases the deviation of the reference surface pressure $\pi_{\text{sig}}(i)|_{\text{dev}}$, producing a larger compression of the coordinate surfaces. In the current testing of this study, the $\gamma_{\min}$ was about 0.36 for all experiments, and the appropriate $\gamma_{\min}$ may be in range of 0.2–0.4.

Note that the value of $\eta_c = 0.8$ which is used for all experiments for the Schär-type mountain in this study satisfies the constraint for all three polynomial orders for $C(\eta)$. As expected, the second-order polynomial exhibits the most rapid reduction of the effect of a smaller-scale terrain in the given configuration of $\eta_c = 0.8$, while the third-order polynomial decreases more slowly near $\eta = 1$ (see Fig. 1b).

3. Idealized tests

a. Model description

The model used for this study is the 2D slice version of the dynamical core of KIM, which has employed a hybrid sigma-pressure (HYB) coordinate (Choi et al. 2014) as defined by (1). The model numerics are based on the compressible nonhydrostatic equations for dry atmosphere with the time-split RK3 time integration (Klemp et al. 2007), but use a spectral element method (SEM) for the horizontal discretization and a finite difference method (FDM) for the vertical discretization. A more detailed description of the model can be found in Choi et al. (2014) and Choi and Hong (2016). In this paper, second-order basis polynomials defined at three node points within an element are used for all experiments. Together with the time-split RK3 time integration, the numerics for the model dynamics are second-order accurate.

b. Test case descriptions

To evaluate performance for the smoothed SMH coordinate in comparison to the SIG and HYB coordinates, we present four idealized test case simulations.

1) RESTING ATMOSPHERE EXPERIMENT OVER A SCHÄR-TYPE MOUNTAIN

The resting atmosphere experiment is run with the terrain shape similar to that used by Schär et al. (2002) for their advection test case:

$$h(x) = h^*(x) \cos^2\left(\frac{\pi x}{2\lambda}\right), \quad (21)$$

where

$$h^*(x) = \begin{cases} H \cos^2\left(\frac{\pi x}{2a}\right) & \text{for } |x| \leq a \\ 0 & \text{for } |x| \geq a \end{cases}. \quad (22)$$

This terrain profile represents explicitly a large-scale mountain of half-width $a = 20$ km, a small-scale mountain of half-width $\lambda = 4$ km, and maximum height $H = 1$ km.

The initial state has zero mean wind, and the thermodynamic sounding is specified with constant a Brunt–Väisälä ($N = 0.01 \text{ s}^{-1}$) except for the addition of a low-level inversion layer ($N = 0.02 \text{ s}^{-1}$) for $2 \leq z \leq 3$ km (following Klemp 2011). Inclusion of the inversion is designed to promote larger errors in evaluating the horizontal pressure gradient term in a terrain-following coordinate. Because the approximation of the horizontal pressure gradient in a terrain-following coordinate requires a vertical transformation back to a constant geometric height, the discontinuities in the vertical pressure gradient due to the inversion layer will contribute to larger errors in computing this adjustment.

The domain is defined as $(x, z) \in [-150, 150] \times [0, 25] \text{ km}^2$. The model grid employs $\Delta x = 1$ km and $\Delta z = 250$ m, and is designated ne150np3L100 (ne, the number of elements dividing horizontal domain, np; the number of GLL quadrature points per each element associated with the $(np - 1)$ th-order Lagrange polynomial, L ; the number of the vertical levels) as used in the SEM model (Choi et al. 2014). Here we specify the η levels to be nearly evenly spaced with a vertical depth ($\Delta z = 250$ m), and with the model top at 25 km. A time step $\Delta t = 1$ s was used to minimize the errors related to time integration. The model was run using periodic lateral boundary conditions without any diffusion, viscosity, or sponge-layer zone, to minimize any damping of the artificial motions generated through numerical errors in the model integration.

2) EXPERIMENT WITH WIND SHEAR ABOVE THE TERRAIN

The model configuration for an experiment with a sheared wind uses the same terrain specification, domain size, and resolution as in the resting atmosphere experiment, but now includes a mean wind shear and removes the inversion layer. The constant Brunt–Väisälä frequency is $N = 0.01 \text{ s}^{-1}$, and the wind profile is specified as

$$u(z) = u_0 \begin{cases} 1. & \text{for } z_2 \leq z \\ \sin^2\left(\frac{\pi}{2} \frac{z - z_1}{z_2 - z_1}\right). & \text{for } z_1 \leq z \leq z_2, \\ 0. & \text{for } z \leq z_1 \end{cases} \quad (23)$$

with $u_0 = 10 \text{ m s}^{-1}$, $z_1 = 2 \text{ km}$, and $z_2 = 4 \text{ km}$. Since the wind profile is designed such that there is no wind at or below the mountain-top level at $H = 1 \text{ km}$, the upper-level flow should remain undisturbed as it passes over the mountain.

For this experiment, the sponge layers are included near the lateral and top boundaries. The absorbing zone was 10 km deep below the top at 25 and 5 km wide along the lateral boundaries. Within the sponge layer zone, the prognostic variables were relaxed to the initial state. A constant horizontal numerical diffusion ($\gamma \nabla^4$, $\gamma = 5 \times 10^8 \text{ m}^4 \text{ s}^{-1}$) term was also used here to remove small-scale noise that should be inevitably filtered in any practical model application. Here, using the fourth-order diffusion is necessary for successful simulations for the 3-h simulation time period with the SIG, HYB, and SMH coordinates.

3) MOUNTAIN-WAVE TEST CASE FOLLOWING SCHÄR ET AL. (2002)

The mountain-wave experiment uses a similar but somewhat different profile from the test cases described above:

$$h(x) = H \exp \left[- \left(\frac{x}{a} \right)^2 \right] \cos^2 \left(\frac{\pi x}{\lambda} \right), \quad (24)$$

where $H = 250 \text{ m}$, $a = 5 \text{ km}$, $\lambda = 4 \text{ km}$. This corresponds to the profile in the Schär et al. (2002) mountain-wave test case, which was also simulated by Klemp et al. (2003), Choi et al. (2014), and others. For this experiment, the model is initialized with a stratified atmosphere having a constant Brunt–Väisälä frequency of $N = 0.01 \text{ s}^{-1}$ and a constant mean wind of $\bar{u} = 10 \text{ m s}^{-1}$. The domain is defined as $(x, z) \in [-300, 300] \times [0, 21] \text{ km}^2$. The model employed $\Delta x = 300 \text{ m}$ and $\Delta z = 250 \text{ m}$ (i.e., ne1000np3L80) and was integrated for 10 h without diffusion. Sponge layers were included near the lateral and top boundaries as described for the previous test case. Using a wide domain compared to the terrain profile together with the sponge layers, the experiment is designed to minimize any artificial reflections from the lateral boundaries.

4) RESTING ATMOSPHERE EXPERIMENT OVER A REAL TERRAIN

This experiment is designed to assess the performance for the smoothed SMH coordinate in more realistic model configuration, in which the terrain profile corresponds to a real mountain topography and a realistic vertical coordinate stretching is employed to achieve higher resolution near the surface. The terrain profile is derived from a vertical cross section across the Colorado Rockies at 40°N at a horizontal resolution of 700 m. The stretched vertical coordinate is adopted from the operational KIM and ECMWF IFS L91 models which have employed a hybrid sigma-pressure (HYB) with $\eta_b = 0.073$ and a 1-Pa top-level pressure. The $A(\eta)$ and $B(\eta)$ used for this experiment were specified to be the same as in the operational KIM's and ECMWF IFS's L91 models up to 25-km altitude, so that the model employs $\Delta x = 700 \text{ m}$ and L67 in the domain $(x, z) \in [0,$

$420] \times [0, 25] \text{ km}^2$ (i.e., ne300np3L67). The initial state is the same as the resting atmosphere experiment over a Schär-type mountain but with the inversion layer set at a higher altitude between $3 \leq z \leq 4 \text{ km}$ due to the increased terrain height. The model is run with a time step $\Delta t = 3 \text{ s}$ and a small horizontal numerical diffusion ($\gamma \nabla^4$, $\gamma = 3 \times 10^7 \text{ m}^4 \text{ s}^{-1}$) term was included.

4. Simulation results

a. Resting atmosphere experiment over a Schär-type mountain

The shapes of the coordinate surfaces in the SIG, HYB, and SMH simulations will differ significantly. Using the SIG coordinate, the terrain influence is removed gradually over the entire 25-km vertical extent of the model domain. Thus, influences of the terrain are visible in the coordinate surfaces even near the top of the domain, as shown in Fig. 3a. Permitting the HYB coordinate with $B(\eta)$ defined by (5)–(7) with $\eta_b = 0.2$, the terrain influence in the vertical coordinate is completely removed at about $z = 12 \text{ km}$, as seen in Fig. 3c. For the resting atmosphere test case, a larger value of η_b could be used that would further reduce the artificial circulations. Including the coordinate smoothing, the coordinate surfaces are displayed in Fig. 3e for the SMH simulation using the second-order $C(\eta)$ with $\eta_c = 0.8$. Here, the influence of the smaller-scale terrain structure is rapidly removed with increasing height such that the small-scale terrain influence is limited to the lowest few kilometers $z \sim 2 \text{ km}$.

Figure 4 shows the time evolution of maximum vertical velocities for the models employing the SIG, the HYB with various η_b (i.e., $\eta_b = 0.2, 0.5, 0.8$), and the SMH coordinates with various orders of polynomial $C(\eta)$ for $\eta_b = 0.2$ and $\eta_c = 0.8$. Since the resting atmosphere test in this study did not employ any computational damping that limits the growth of perturbations, the maximum vertical velocities of the artificial circulations can be viewed as the error growth rates for the models. The end point for each curve reflects the time at which the model simulation blows up. Simulating this case with the SMH coordinate produced artificial circulations that are greatly reduced compared to those of the SIG and the HYB coordinate regardless of the order of the polynomial $C(\eta)$. While the HYB coordinate of $\eta_b = 0.2$ provides some improvement over the SIG coordinate, it is not nearly as significant as the improvement with the SMH coordinate, although the HYB results for this test case could be noticeably improved using a larger value of η_b , such as 0.5 and 0.8. However, these larger values of η_b would not likely be feasible for real terrain applications and general NWP models (see Eckermann 2009), so that we use 0.2 as a reference value of η_b unless otherwise noted. For the SMH coordinate the sensitivity of the results to the order of the polynomial $C(\eta)$ is also evident, and indicates that the second-order polynomial for $C(\eta)$ produces the slowest error growth for this value of η_c , and that the error growth is in inverse proportion to the value of $(\partial C / \partial \eta)(1)$

FIG. 3. Coordinate surfaces of η levels and potential temperature contours (contour interval 1 K; from 291 K at the bottom to 312 K at the top) in the resting-atmosphere experiment over a Schär-type mountain for (a),(b) the sigma coordinate (SIG); (c),(d) the hybrid-sigma coordinate (HYB); and (e),(f) the smoothed hybrid-sigma coordinate (SMH) using the second-order $C(\eta)$. The orange and red lines indicate the levels of $\eta_b = 0.2$ and $\eta_c = 0.8$, respectively.

(see Fig. 1b). However, we note that if η_c is increased to 0.85 in the third-order polynomial for $C(\eta)$, the error growth with time (not shown) is essentially the same as that displayed in Fig. 4 for the second-order polynomial with $\eta_c = 0.8$. The experiments presented below use the second-order polynomial unless otherwise noted.

Perturbations in the potential temperature fields are displayed in the right-hand column of Fig. 3 at 5400 s. For the

SIG coordinate simulation, large artificial perturbations are evident in the inversion layer over the terrain with maximum vertical velocities reaching $\sim 10 \text{ m s}^{-1}$, and the model blows up a short time later. In case of the HYB coordinate, the distortion in the potential temperature field at 5400 s (Fig. 3d) are much reduced from those in the SIG simulation. However, the perturbations are growing rapidly (as is evident in Fig. 4) and this simulation blows up shortly after

FIG. 4. Time series of the maximum vertical velocities in the resting-atmosphere experiment over a Schär-type mountain. Note that the models are run without any diffusion such as the sponge-layer zone and viscosity so that the simulations eventually blow up for all of the terrain-following coordinate formulations.

100 min. The most significant reduction of the distortion occurs in the SMH coordinate simulation (Fig. 3f). The artificial perturbations are very small at 5400 s and continue to remain small for as much as twice this time interval. With the second-order polynomial for $C(\eta)$, the time to model blow-up does not occur until about 300 min. We note that in simulating this case with some added constant eddy diffusion (as in practical model applications), simulations with all of the coordinates produced very small values of the maximum perturbation vertical velocities with no model blow-ups (not shown).

b. Experiment with wind shear above the terrain

The sheared wind test case is a modification of the resting atmosphere test case in which there is a mean wind at levels above the terrain. Because there is no wind at or below the altitude of the mountain's summit in this test, the perturbation circulations induced by the simulation are totally artificial. These distortions are produced by numerical errors resulting from the horizontal flow through coordinate surfaces that are not parallel to the flow. Figure 5 displays the vertical velocity fields for the SIG coordinate, HYB coordinate with $\eta_b = 0.2$, and SMH coordinate using the second-order $C(\eta)$ with $\eta_c = 0.8$ at 3 h. In the SIG coordinate simulation, artificial perturbations arise that appear to form a vertically propagating wave train forced by distortions in the shear layer between 2 and 4 km. With the HYB coordinate (Fig. 5b) the amplitudes of the perturbations are noticeably reduced, although a coherent wave train is still visible. In contrast, the SMH coordinate simulation using the second-order $C(\eta)$ (Fig. 5c) exhibits a significant reduction in the perturbations compared to the SIG and HYB simulations (Figs. 5a,b), with the maximum (minimum) of the errors more than an order of magnitude smaller. It is interesting that the distributions of the artificial vertical velocity for both the SIG and the HYB simulations have the

FIG. 5. Vertical velocity fields at $t = 10800$ s in the experiments with the shear wind for the (a) SIG, (b) HYB, and (c) SMH using the second-order $C(\eta)$ coordinates. The minimums and maximums are denoted within angle brackets, and the contour intervals are 0.05 m s^{-1} for SIG and HYB, but are 0.002 m s^{-1} for SMH. The blue solid (red dashed) lines denote positive (negative) values, and zero are omitted.

FIG. 6. The vertical velocity fields in a mountain wave with a constant flow past over a Schär-type mountain after 10 h for the (a) SIG, (b) HYB, (c) SMH using the second-order $C(\eta)$ coordinates, and (d) analytic solution. For all panels, the contour interval is 0.05 m s^{-1} , the blue solid (red dashed) lines denote positive (negative) values, and zeros are emphasized by thick solid lines.

appearance of a well resolved vertically propagating wave train (as in Fig. 2 of Choi et al. 2014). This illustrates that the artificial disturbances do not necessarily produce only small-scale noise. Rather, they can produce what appear to be well-resolved wave structures. Also, we note that simulating this case using the third-order polynomial for $C(\eta)$ produced nearly identical artificial perturbations (not shown) to those obtained using the second-order $C(\eta)$.

c. Mountain-wave test case

Simulations using the SIG, HYB, and SMH coordinates were also conducted for a test case for mountain-induced gravity waves, known as the Schär mountain-wave test. The vertical velocity fields for these three simulations are displayed in Fig. 6 along with the steady-state analytic solution. The results are virtually the same for all three coordinate representations, and all are in good agreement with the analytic solution (Fig. 6d). In the SMH simulation, there is no significant distortion of the steady mountain-wave structure, and it dampens the small-scale noise apparent above the upwind side of the terrain in the SIG and the HYB simulations (Figs. 6a,b). The performance of the SMH is comparable to that of Choi et al. (2014) using the higher-order SEM with the HYB coordinate (their Fig. 5), and that of Klemp et al. (2003) using horizontal finite differencing with the basic sigma coordinate (their Fig. 4). In this case, the vertically propagating mode has a horizontal

wavelength that is well resolved so the sigma-coordinate errors are not significant (Klemp et al. 2003).

d. Resting atmosphere experiment over a real terrain

To evaluate the performance for the smoothed SMH coordinate in more realistic model configuration, the model was run with the stretched coordinate as used by KIM and ECMWF IFS (<https://www.ecmwf.int/en/forecasts/documentation-and-support/91-model-levels>) rather than on the evenly spaced coordinate. Figure 7 describes the characteristics of the coordinate profile used for this experiment, where $\eta_b = 0.073$ corresponds to that used in the operational KIM and the IFS model, and the second-order $C(\eta)$ with $\eta_c = 0.8$ is used as in the other experiments. The coordinate surfaces are densely spaced at low levels over the complex terrain, with a nominal thickness (Δz) of 10 m in the lowest layer.

The contrasting shapes of the coordinate surfaces in the SIG, HYB, SMH simulations are well illustrated in the left panels of Fig. 8. The influence of the smaller-scale terrain structure is rapidly removed from the coordinate surfaces with increasing height in the SMH run. Perturbations in the potential temperature fields are displayed in the right panels of Fig. 8. For the SIG and HYB simulations, the fields are shown at 24 and 33 h, respectively, times just before the simulations blew up due to increased artificial circulations. With the inversion layer at $3 \leq z \leq 4 \text{ km}$, which intersects the terrain, the distortions in the potential temperature of the SIG and HYB simulations are virtually the

FIG. 7. (a) The profiles of $A(\eta)$, $B(\eta)$, and the second-order $C(\eta)$ with $\eta_c = 0.8$ used for the resting-atmosphere experiment over the real terrain, displayed as a function of height with a 25-km upper boundary. The $A(\eta)$ and $B(\eta)$ correspond to the lower 67 levels up to 25 km in the operational KIM and ECMWF IFS L91 models (1-Pa top, $\eta_b = 0.073$). (b) The profile of the nominal vertical grid spacing.

same, with the maximum distortion occurring near a steep isolated peak at $x \sim 200$ km. In case of the SMH coordinate, the model ran successfully for a 100-h simulation with only small artificial perturbations throughout the inversion layer.

5. Summary and discussion

A smoothed hybrid sigma-pressure (SMH) coordinate has been presented that allows for a rapid vertical decay of smaller-scale topographic structures in the coordinate surfaces by including a reference surface pressure in the coordinate definition that contains the influence of the underlying topography. A smoothed version of this reference surface pressure is created that represents the larger scale features of the topography, while the deviations from the smoothed profile contain the smaller-scale terrain structures. The hybrid sigma coordinate is then modified to selectively remove the influence of the smaller scale deviations more rapidly with height. With this approach, smaller-scale structure is effectively removed from the coordinate surfaces more rapidly with increasing height above the terrain as shown by the plots of the coordinate surfaces in Fig. 3 for the resting atmosphere test case. This modification can be easily substituted for the basic hybrid sigma-pressure coordinate since no other changes in the numerics are required.

In the resting atmosphere simulations described above, the SMH coordinate demonstrates a significant reduction in the artificial circulations compared to those arising with the basic sigma (SIG) or hybrid sigma (HYB) coordinates. The improvement of the SMH coordinate is also clear in the sheared-wind experiment. Additional simulations for the Schär mountain-wave test case have been conducted that

further confirm that the SMH coordinate is a practical alternative that maintains good agreement with the analytic solution. Although this SMH coordinate has not yet been implemented in a full model that permits real-weather simulations, we have documented its beneficial behavior for a realistic topographic profile and for densely spaced coordinate surfaces just above the terrain.

The design of the SMH coordinate has similar properties to the SLEVE coordinate specification proposed by Schär et al. (2002), which is applicable to height-based coordinate systems that are fixed in space. A similar approach has been implemented by Zängl (2003), in which coordinate smoothing was included in modifying the pseudo sigma-pressure coordinate system of the MM5 model, in which the coordinate surfaces remained fixed based on a reference pressure field. The SMH coordinate proposed here provides a mechanism for smoothing coordinate surfaces in a terrain-following sigma-pressure coordinate system in which the position of the coordinate surfaces evolve over time in response to changes in the surface pressure. The technique can be readily implemented as a modification to the standard hybrid sigma coordinate and provides flexibility in specifying the scales to be smoothed and the amount of smoothing to be applied. The two parameters η_b and η_c control the rates at which the larger-scale and the smaller-scale terrain influences are removed from the coordinate surfaces, respectively. Although a formal invertibility condition is not readily available for the coordinate representation, a more practical constraint is recommended here that limits the maximum compression of the coordinate surfaces. Since the smoothing of terrain features is applied to a reference representation of the surface pressure, the SMH coordinate system provides a simple way to achieve smoothed coordinate

FIG. 8. As in Fig. 3, but for the resting-atmosphere experiment over the real terrain and using a stretched distribution of vertical levels with high resolution near the surface. For the SIG and HYB simulations, the theta fields are shown at 24 and 33 h, respectively, times just before the simulations blew up. The theta field for the SMH simulation is shown at 100 h, at which time the artificial perturbations remained small. The orange and red lines indicate the levels of $\eta_b = 0.073$ and $\eta_c = 0.8$, respectively.

surfaces without having to continually recompute a smoothed surface pressure field as the pressure is evolving in time.

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Data availability statement. The numerics of the KIM model are documented by Choi and Hong (2016) and the code that supports the findings of this study is available from the corresponding author, S.-J. Choi, upon reasonable request.

APPENDIX

Governing Equations

The governing equations of the model used here are based on Klemp et al. (2007), which are in terms of the flux-form, the dry air, and the terrain-following hydrostatic pressure vertical coordinate. Klemp et al. (2007) described the equations in terms of the SIG coordinate, whereas Choi et al. (2014) uses the HYB coordinate. Here we present the 2D governing equations for a dry atmosphere in which we have implemented the SMH coordinate.

Using the coordinate metric term μ_d as defined in (14), the flux variables are defined as

$$(U, W, \Omega, \Theta) = \mu_d \times (u, w, \eta, \theta), \quad (\text{A1})$$

where u and w are the velocities in the horizontal and vertical directions, respectively, $\eta \equiv \partial\eta/\partial t$ is the contravariant vertical velocity, and θ is the potential temperature. In this context, the coordinate metric term μ_d is considered as the mass of the dry air in the layers.

The flux- and perturbation-form Euler equations for a dry atmosphere are expressed as

$$\begin{aligned} \frac{\partial U}{\partial t} = & - \left[\mu_d \left(\frac{\partial \phi'}{\partial x} + \alpha'_d \frac{\partial p'}{\partial x} + \alpha'_d \frac{\partial \bar{p}}{\partial x} \right) + \frac{\partial \phi}{\partial x} \left(\frac{\partial p'}{\partial \eta} - \mu'_d \right) \right] \\ & - \frac{\partial(Uu)}{\partial x} - \frac{\partial(\Omega \mathbf{v}_H)}{\partial \eta} + F_U, \end{aligned} \quad (\text{A2})$$

$$\frac{\partial W}{\partial t} = g \frac{\partial p'}{\partial \eta} - g \mu'_d - \frac{\partial(Uw)}{\partial x} - \frac{\partial(\Omega w)}{\partial \eta} + F_W, \quad (\text{A3})$$

$$\frac{\partial \mu'_d}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\partial \pi'_d}{\partial \eta} \right) = - \frac{\partial U}{\partial x} - \frac{\partial \Omega}{\partial \eta}, \quad (\text{A4})$$

$$\frac{\partial \phi'}{\partial t} = - \frac{1}{\mu_d} \left(U \frac{\partial \phi}{\partial x} + \Omega \frac{\partial \phi}{\partial \eta} - gW \right), \quad (\text{A5})$$

$$\frac{\partial \Theta}{\partial t} = - \frac{\partial(U\theta)}{\partial x} - \frac{\partial(\Omega\theta)}{\partial \eta}, \quad (\text{A6})$$

where α_d is the inverse density for the dry air, p is the pressure, ϕ is the geopotential, π_d is the hydrostatic pressure of dry air, and F_U and F_W represent forcing terms of the Coriolis and curvature that we ignore here for simplicity. In (A2)–(A6), the governing equations are described with perturbation variables such as $p = \bar{p} + p'$, $\phi = \bar{\phi} + \phi'$, $\alpha_d = \bar{\alpha}_d + \alpha'_d$, and $\mu_d = \bar{\mu}_d + \mu'_d$ where the variables denoted by overbars are reference state variables that satisfy hydrostatic balance.

For completeness, the diagnostic relation for the dry inverse density is given as

$$\frac{\partial \phi'}{\partial \eta} = -\bar{\mu}_d \alpha'_d - \alpha_d \mu'_d, \quad (\text{A7})$$

and the full pressure for dry atmosphere is

$$p = p_0 \left(\frac{R_d \theta}{p_0 \alpha_d} \right)^\gamma, \quad (\text{A8})$$

where $\gamma = c_p/c_v = 1.4$ is the ratio of the heat capacities for dry air, R_d is the gas constant for dry air, p_0 is a reference pressure (typically 10^5 Pa).

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