LINEAR PROGRAMMING APPLIED TO OPERATIONAL DECISION MAKING IN WEATHER RISK SITUATIONS

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ABSTRACT

The linear programming algorithm is applied to assist decision makers in selecting an optimal course of action in situations involving weather-induced losses or gains.

1. INTRODUCTION

Gleeson [5] has developed two methods for determining the minimum expense strategy for conducting an operation sensitive to a predictand whose occurrence can be divided into a finite number of mutually exclusive and exhaustive classes. Here we have a decision maker with freedom to select from a number of operational decisions $D_1, D_2, \ldots D_n$, each of which is subject to occurrence of various predictands $X_1, X_2, \ldots X_n$ resulting in particular economic gains or losses, a_{ij} (table 1). Gleeson's Method A essentially consists of systematic examination of the payoff matrix, and selecting for each decision that combination of weather frequencies that will minimize the expected economic return E_i. The decision which maximizes the E_i then becomes the optimal decision. Method B utilizes the mathematical theory of games.

Epstein [6] has criticized Gleeson's methods on the grounds that it is unrealistic to suppose that nature can select its own best strategy. However, as noted by Thompson [7], for certain types of decision makers the minimum economic expectation model can be a valid means of selecting the course of action providing maximum benefits.

It is known [1] that any zero-sum two-person matrix game problem can be expressed as a linear program. This equivalence permits formulation of a generalized method based on linear programming considerations which can be used in lieu of Gleeson's Methods A and B. In addition, linear programming techniques are usually more efficient, since the direct solution of large matrix games is a cumbersome process.

TABLE 1.—*Economic gains and losses, a_{ij}, for various decisions D_i and predictand outcomes X_j (after Gleeson [5])*

		Predictand				
		X_1	. . .	X_j	. . .	X_k
Decisions	D_1	a_{11}	. . .	a_{1j}	. . .	a_{1k}
	.	.		.		.
	.	.		.		.
	D_i	a_{i1}	. . .	a_{ij}	. . .	a_{ik}
	.	.		.		.
	.	.		.		.
	D_m	a_{m1}	. . .	a_{mj}	. . .	a_{mk}

2. DEFINITION

In the standard linear programming problem [1, 2] one wishes to minimize a linear function

$$Z=C_1X_1+C_2X_2+\ldots+C_nX_n$$

(called the objective function), subject to a set of m linear inequalities

$$a_{11}X_1+a_{12}X_2+\ldots+a_{1n}X_n\leq b_1$$
$$a_{12}X_1+a_{22}X_2+\ldots+a_{2n}X_n\leq b_2$$
$$\vdots$$
$$a_{m1}X_1+a_{m2}X_2+\ldots+a_{mn}X_n\leq b_m$$

and the further constraints

$$X_1\geq 0,\ X_2\geq 0,\ \ldots,\ X_n\geq 0.$$

It is also customary to write the set of inequalities so that $b_1 \geq 0$, adding or subtracting if necessary a sufficient number of additional positive variables, called "slack variables," to permit this.

The problem is sometimes stated in terms of *maximizing* an objective function, but since $\text{Max}(Z) = -\text{Min}(-Z)$, these are equivalent requirements.

3. SOLUTION OF THE LINEAR PROGRAMMING PROBLEM

The most widely accepted method of solution for linear programming problems is the "simplex" method, originated by Dantzig [4]. While at first glance the simplex algorithm may appear onerous, the method is purely mechanical and imposes no higher mathematical demands on its practitioners than a knowledge of elementary linear algebra. Moreover, the method is readily computerized and "canned" computer programs for solving linear programs with large numbers of variables have been devised and can be purchased if desired.

The basic features of the simplex method can be shown by the following hypothetical problem, based on considerations derived from the work of Thompson and Brier [3].

Example: Given an operation sensitive to a certain weather event and having an associated cost-loss ratio C/L. What values of a, b, c, and d, the frequencies of occurrences and nonoccurrences of the weather event, would provide an operator with the conditions for a least expensive operation?

Here we wish to minimize

$$E=C(b+d)+Lc$$

subject to

$$a+b+c+d=N$$

$$c+d=K \text{ (i.e., climatology is invariant)}$$

$$C/L \geq \frac{c}{a+c}$$

$$C/L \leq \frac{d}{b+d}$$

and

$$a \geq 0,\ b \geq 0,\ c \geq 0,\ d \geq 0.$$

After rewriting the above equations and substituting $X_1=a$, $X_2=b$, $X_3=c$, and $X_4=d$, the linear program becomes,

$$\text{Minimize } E=CX_2+LX_3+CX_4 \quad (1)$$

subject to

$$X_1+X_2+X_3+X_4=N \quad (2)$$

$$X_3+X_4=K \quad (3)$$

$$-CX_1+(L-C)X_3+X_5=0 \quad (4)$$

$$CX_2+(C-L)X_4+X_6=0 \quad (5)$$

where X_5 and X_6 are nonnegative slack variables introduced to eliminate the inequality signs. The equations are arranged in tabular array (A) in table 2. Here $\mathbf{P}_1$, $\mathbf{P}_2$, . . . $\mathbf{P}_6$ are vectors associated with each of the variables in the set of equations above. The simplex procedures require a "basis," i.e. a set of unit vectors equal in number to the equations to be operated upon. By a set of unit vectors is meant a vector set such that

a) One element of the vector is equal to "1";

b) All remaining elements of the vector are zeroes;

c) No two vectors of the set contain the element "1" in the same row.

In array (A) of table 2, vectors $\mathbf{P}_7$ and $\mathbf{P}_8$ have been added to the set of equations in order to provide the necessary basis. The column headed "Cost" in the array refers to the coefficient in the objective function of each variable associated with the basis. Customarily, if the objective function does not contain a certain variable, zero is entered for the cost (e.g., costs for $\mathbf{P}_5$ and $\mathbf{P}_6$ are zero). However, for the costs of those vectors added to augment the matrix in order to provide a basis, we ascribe an indeterminate cost W. Our lack of knowledge of the exact costs for $\mathbf{P}_7$ and $\mathbf{P}_8$ need not concern us here, however, for the simplex algorithm is begun by eliminating these artificial vectors from the basis and replacing them by selecting from the vectors of interest $\mathbf{P}_1$, $\mathbf{P}_2$, . . . $\mathbf{P}_6$.

To do this, select one of the vectors, say $\mathbf{P}_7$, in the first row, and replace it by $\mathbf{P}_1$. However, merely putting $\mathbf{P}_1$ in place of $\mathbf{P}_7$ in the first row does not suffice, since $\mathbf{P}_1$ taken in conjunction with $\mathbf{P}_8$, $\mathbf{P}_5$, and $\mathbf{P}_6$ does not form a basis.

TABLE 2.—*Tabular arrays of results of the simplex computations with solution given in array (C)*

(A)

				C	L	C			W	W
Basis	Cost	$\mathbf{P}_0$	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_7$	$\mathbf{P}_8$
$\mathbf{P}_7$	W	N	1	1	1	1	0	0	1	0
$\mathbf{P}_8$	W	K	0	0	1	1	0	0	0	1
$\mathbf{P}_5$	0	0	$-C$	0	$L-C$	0	1	0	0	0
$\mathbf{P}_6$	0	0	0	C	0	$C-L$	0	1	0	0

(B)

				C	L	C			W
Basis	Cost	$\mathbf{P}_0$	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_8$
$\mathbf{P}_1$	0	N	1	1	1	1	0	0	0
$\mathbf{P}_8$	W	K	0	0	1	1	0	0	1
$\mathbf{P}_5$	0	NC	0	C	L	C	1	0	0
$\mathbf{P}_6$	0	0	0	C	0	$C-L$	0	1	0

(C)

				C	L	C		
Basis	C	$\mathbf{P}_0$	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$
$\mathbf{P}_1$	0	$N-K$	1	1	0	0	0	0
$\mathbf{P}_4$	C	K	0	0	1	1	0	0
$\mathbf{P}_5$	0	$C(N-K)$	0	C	$L-C$	0	1	0
$\mathbf{P}_6$	0	$K(L-C)$	0	C	$L-C$	0	0	1
$Zj-Cj$		CK	0	0	$C-L$	0	0	0

We, therefore, transform into a basis vector by changing $-C$, the third element in the column vector $\mathbf{P}_1$, to a zero. This is effected by multiplying each element of the first row of the array by C and adding each to the corresponding element in the third row, as shown in array (B). The cost for $\mathbf{P}_1$ as determined from the objective function is zero.

We now introduce a suitable vector in lieu of $\mathbf{P}_8$. The vector $\mathbf{P}_2$ will not be satisfactory since its coefficient is zero. Consequently, either $\mathbf{P}_3$ or $\mathbf{P}_4$ are suitable possibilities. We select $\mathbf{P}_4$ and proceed to incorporate it into the basis by first multiplying row 2 by -1 and adding the result to row 1, then multiplying the row by $-L$ and adding to row 3, and finally multiplying by $L-C$ and adding to row 4. The results are shown in array (C). Here, for convenience, the augmented vectors $\mathbf{P}_7$ and $\mathbf{P}_8$ have been dropped from the array, since they have served their purpose and are no longer in the basis.

In array (C), the entry Z_j-C_j refers to the product of the column vector times the cost minus the appropriate coefficient from the objective function. For example, for $\mathbf{P}_3$,

$$Z_3-C_3=0(0)+C(1)+0(L-C)+0(L-C)-L=C-L.$$

The signs of the Z_j-C_j are crucial since 1) only vectors having $Z_j-C_j>0$ may be selected for the basis, and 2) when all $Z_j-C_j(j>0)$ are finally ≤ 0, the simplex procedure has generated the minimum solution, and the iterative process is halted. We note that in array (C), $Z_3-C_3=C-L<0$, since for any problem $C/L<1$; hence all $Z_j-C_j>0(j>0)$, and we have now generated the minimum extreme point solution. In fact, the minimum value of the objective function is given by $Z_0-C_0=CK$. The minimum values appear in $\mathbf{P}_0$ of array (C) and are as follows:

$$X_1=N-K, X_4=K, X_5=C(N-K), \text{ and } X_6=K(L-C).$$

Translating this result into our original problem statement, we find the most favorable milieu in which to conduct a weather sensitive operation is one where

$$a=N-K;\ b=0;\ c=0;\ \text{and } d=K;$$

i.e., "perfect forecasts" are available. The result, of course, obviously could have been foreseen without the necessity for undertaking these calculations, but the example has been instructive as an illustration of the mechanics of the simplex procedure. The technique can now be applied to a more abstruse class of problems.

4. A GENERALIZED METHOD FOR SELECTING OPTIMAL COURSES OF ACTION

Given an arbitrary matrix game as follows:

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Then an equivalent linear programming problem is to find $X_i\geq 0$ and the minimum number V (the game value) such that

$$a_{11}X_1+a_{12}X_2+\dots+a_{1n}X_n\leq V$$
$$a_{21}X_1+a_{22}X_2+\dots+a_{2n}X_n\leq V$$
$$a_{m1}X_1+a_{m2}X_2+\dots+a_{mn}X_n\leq V$$

TABLE 3.—*Payoff matrix for Farmer Smith (after Gleeson [5])*

		Predictand (rainfall)			
		X_1	X_2	X_3	X_4
Decisions	D_1	5	0	−1	−2
	D_2	2	3	1	−1
	D_3	−4	1	5	−1
	D_4	−1	0	1	1
p_j''		0.12	0.46	0.77	0.31
p_j'		0	0.12	0.40	0.04

and

$$X_1+X_2+\dots+X_n=1$$

(i.e., the sum of the probabilities equals one).

Example: The general method of solution will be shown by applying it to one of the examples in Gleeson's paper. The example concerns a Farmer Smith, who is faced with the decision of which weather crop to select for planting next year. His payoff matrix is shown in table 3; here p_j'' and p_j' are the upper and lower confidence limits for the frequencies of the predictand classes X_i (light, moderate, heavy, and excessive rainfall).

Smith, naturally, wishes to select a strategy which will maximize his expected gain regardless of the strategy employed by his fictional opponent, Nature. That is, he wishes to select a decision vector $\mathbf{D}=(d_1, d_2, d_3, d_4)$ which will maximize his minimum economic expectation, returning him at least V units, where V is the value of the gain. The linear program then is to minimize V, subject to

$$5X_1-X_3-2X_4\leq V$$
$$2X_1+3X_2+X_3-X_4\leq V$$
$$-4X_1+X_2+5X_3-X_4\leq V$$
$$-X_1+X_3+X_4\leq V$$
$$0\leq X_1\leq 0.12$$
$$0.12\leq X_2\leq 0.46$$
$$0.40\leq X_3\leq 0.77$$
$$0.04\leq X_4\leq 0.31$$

TABLE 4.—*Tabular arrays of the results of the simplex computation with optimal solution given in array (J)*

(A) $\mathbf{P}_1\rightarrow\mathbf{P}_0$ in basis

B	C	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}
			5		−1	−2	1											W	W	W	W		
P_6	0	0	−3	3	2	1	−1	1															
P_7	0	0	−9	1	6	1	−1		1														
P_8	0	0	−6		2	3	−1			1													
P_9	0	0.12	1								1												
P_{16}	W	0.12		1								−1						1					
P_{11}	0	0.46		1									1										
P_{17}	W	0.40			1									−1					1				
P_{13}	0	0.77			1										1								
P_{18}	W	0.04				1										−1				1			
P_{15}	0	0.31				1											1						
P_{19}	W	1	1	1	1	1															1		
W Values		1.56	1	2	2	2						−1		−1		−1							

TABLE 4.—Continued

(B)	$P_4 \rightarrow P_{18}$		5		−1	−2	1											W	W	W	W		
B	*C*	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}
P_6	0	0.36		3	2	1	−1	1			3												
P_7	0	1.08		1	6	1	−1		1		9												
P_8	0	0.72			2	3	−1			1	6												
P_1	5	0.12	1								1												
P_{16}	W	0.12		1								−1						1					
P_{11}	0	0.46		1									1										
P_{17}	W	0.40			1									−1					1				
P_{13}	0	0.77			1										1								
P_{18}	W	0.04				1										−1				1			
P_{15}	0	0.31				1											1						
P_{19}	W	0.88		1	1	1					−1										1		
$Z_i - C_i$		0.60			1	2					5												
W Values		1.44		2	2	2					−1	−1		−1		−1							

(C)	$P_3 \rightarrow P_6$		5		−1	−2	1											W	W		W		
B	*C*	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}
P_6	0	0.32		3	2		−1	1			3					1							
P_7	0	1.04		1	6		−1		1		9					1							
P_8	0	0.60			2		−1			1	6					3							
P_1	5	0.12	1								1												
P_{16}	W	0.12		1								−1						1					
P_{11}	0	0.46		1									1										
P_{17}	W	0.40			1									−1					1				
P_{13}	0	0.77			1										1								
P_4	−2	0.04				1										−1							
P_{15}	0	0.27														1	1						
P_{19}	W	0.84		1	1						−1					1					1		
$Z_i - C_i$		0.52			1						5					2							
W Values		1.36		2	2						−1	−1		−1		1							

(D)	$P_5 \rightarrow P_7$		5		−1	−2	1											W	W		W		
B	*C*	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}
P_3	−1	0.16		3/2	1		−1/2	1/2			3/2					1/2							
P_7	0	0.08		−8			2	−3	1							−2							
P_8	0	0.28		−3				−1		1	3					2							
P_1	5	0.12	1								1												
P_{16}	W	0.12		1								−1						1					
P_{11}	0	0.46		1									1										
P_{17}	W	0.24		−3/2			1/2	−1/2			−3/2			−1		−1/2			1				
P_{13}	0	0.61		−3/2			1/2	−1/2			−3/2				1	−1/2							
P_4	−2	0.04				1										−1							
P_{15}	0	0.27														1	1						
P_{19}	W	0.68		−1/2			1/2	−1/2			−5/2					1/2					1		
$Z_i - C_i$		0.36		−3/2			−1/2	−1/2			7/2					3/2							
W Values		1.04		−1			1	−1			−4			−1		0							

(E)	$P_2 \rightarrow P_{16}$		5		−1	−2	1											W	W		W		
B	*C*	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}
P_3	−1	0.18		−1/2	1			−1/4	1/4		3/2												
P_5	1	0.04		−4			1	−3/2	1/2							−1							
P_8	0	0.28		−3				−1		1	3					2							
P_1	5	0.12	1								1												
P_{16}	W	0.12		1								−1						1					
P_{11}	0	0.46		1									1										
P_{17}	W	0.22		1/2				1/4	−1/4		−3/2			−1					1				
P_{13}	0	0.59		1/2				1/4	−1/4		−3/2				1								
P_4	−2	0.04				1										−1							
P_{15}	0	0.27														1	1						
P_{19}	W	0.66		3/2				1/4	−1/4		−5/2					1					1		
$Z_i - C_i$		0.38		−9/2				−5/4	1/4		7/2					2.52							
W Values		1.00		3				1/2	−1/2		−4	−1		−1		1							

TABLE 4.—Continued

(F) $P_6 \rightarrow P_{17}$

B	C	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}
			5		−1	−2	1												W		W		
P_3	−1	0.24			1			−1/4	1/2		3/2	−1/2											
P_5	1	0.52					1	−3/2	1/2			−4				−1							
P_8	0	0.64						−1		1	3	−3				2							
P_1	5	0.12	1								1												
P_2	0	0.12		1								−1											
P_{11}	0	0.34										1	1										
P_{17}	W	0.16						1/4	−1/4		−3/2	1/2		−1					1				
P_{13}	0	0.53						1/4	−1/4		−3/2	1/2			1								
P_4	−2	0.04				1										−1							
P_{15}	0	0.27														1	1						
P_{19}	W	0.48						1/4	−1/4		−5/2	3/2				1					1		
$Z_i - C_i$		0.80						−5/4	1/4		7/2	1				1							
W Values		0.64						1/2	−1/2		−4	2		−1		1							

(G) $P_{14} \rightarrow P_{15}$

B	C	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}
			5		−1	−2	1														W		
P_3	−1	0.40			1									−1									
P_5	1	1.48					1		−1		−9	−1	−6		−1								
P_8	0	1.28							−1	1	−3	−1		−4		2							
P_1	5	0.12	1								1												
P_2	0	0.12		1								−1											
P_{11}	0	0.34										1	1										
P_6	0	0.64						1	−1		−6	2		−4									
P_{13}	0	0.37												1	1								
P_4	−2	0.04				1										−1							
P_{15}	0	0.27														1	1						
P_{19}	W	0.32									−1	1		1		1					1		
$Z_i - C_i$		1.47							−1		−4	−1		−5		1							
W Values		0.32									−1	1		1		1							

(H) $P_{12} \rightarrow P_{19}$

B	C	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}
			5		−1	−2	1														W		
P_3	−1	0.40			1									−1									
P_5	1	1.75					1		−1		−9	−1		−6			1						
P_8	0	0.74							−1	1	−3	−1		−4			−2						
P_1	5	0.12	1								1												
P_2	0	0.12		1								−1											
P_{11}	0	0.34										1	1										
P_6	0	0.64						1	−1		−6	2		−4									
P_{13}	0	0.37												1	1								
P_4	−2	0.31				1											1						
P_{14}	0	0.27														1	1						
P_{19}	W	0.05									−1	1		1			−1				1		
$Z_i - C_i$		1.33							−1		−4	−1		−5			−1						
W Values		0.05									−1	1		1			−1				1		

(I) $P_{10} \rightarrow P_{12}$

B	C	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}
			5		−1	−2	1														W		
P_3	−1	0.45			1						−1	1					−1						
P_5	1	2.05					1		−1		−15	5					−5						
P_8	0	0.94							−1	1	−7	3					−6						
P_1	5	0.12	1								1												
P_2	0	0.12		1								−1											
P_{11}	0	0.34										1	1										
P_6	0	0.84						1	−1		−10	6					−4						
P_{13}	0	0.32									1	−1			1		1						
P_4	−2	0.31				1											1						
P_{14}	0	0.27														1	1						
P_{12}	0	0.05									−1	1		1			−1						
$Z_i - C_i$		1.78							−1		−9	4					−6						

TABLE 4.—Concluded

(J) Solution $V=1.38$

B	C	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}
			5		−1	−2	1																
P_3	−1	0.40			1									−1									
P_5	1	1.80					1		−1		−10			−5									
P_8	0	0.79							−1	1	−4			−3			−3						
P_1	5	0.12	1								1												
P_2	0	0.17		1							−1			1			−1						
P_{11}	0	0.29											1	−1			1						
P_6	0	0.54						1	−1		−4			−6			2						
P_{13}	0	0.37												1	1								
P_4	−2	0.31			1												1						
P_{14}	0	0.27														1	1						
P_{10}	0	0.05									−1	1		1			−1						
Z_j-C_j		1.38	0	0	0	0	0	0	−1	0	−10	0	0	−4	0	0	−2						

and

$$X_1+X_2+X_3+X_4=1$$

(i.e., the sum of the frequencies of the elements which comprise the decision strategy equals one).

It should be noted that we have tacitly assumed here that $V\geq 0$, i.e., Smith will not knowingly undertake to engage in a contest which will result in no financial gain to him. However, this restriction is unimportant as will be seen later, and if Smith insists upon undertaking a venture whose outcome is sure to leave him poorer regardless of which course of action he elects to pursue, or if he is uncertain as to whether the "game" is biased in his favor, his least cost strategy may still be determined by linear programming. In any event, a negative V can always be replaced by two nonnegative variables V_1 and V_2 such that $V=V_1-V_2$.

With the addition of the required slack variables, the linear program becomes the following:

Minimize V subject to

$$
\begin{aligned}
5X_1 - X_3 - 2X_4 + X_5 &= V \\
2X_1+3X_2+ X_3 \quad X_4 + X_6 &= V \\
-4X_1+X_2+5X_3-X_4+X_7 &= V \\
-X_1+X_3+X_4+X_8 &= V \\
X_1+X_9 &= 0.12 \\
X_2-X_{10} &= 0.12 \\
X_2+X_{11} &= 0.46 \\
X_3-X_{12} &= 0.40 \\
X_3+X_{13} &= 0.77 \\
X_4-X_{14} &= 0.04 \\
X_4+X_{15} &= 0.31 \\
X_1+X_2+X_3+X_4 &= 1.
\end{aligned}
$$

Before beginning the simplex computations, it is expedient, although not mandatory, to reduce the number of equations to be manipulated by selecting the first equation as the objective function and subtracting it in turn from the next three equations (sufficient artificial variables have been added to produce a basis).

Minimize $5X_1-X_3-2X_4+X_5$ subject to

$$
\begin{aligned}
-3X_1+3X_2+2X_3+X_4-X_5+X_6 &= 0 \\
-9X_1+X_2+6X_3+X_4-X_5+X_7 &= 0 \\
-6X_1+2X_3+3X_4-X_5+X_8 &= 0 \\
X_1+X_9 &= 0.12 \\
X_2-X_{10}+X_{16} &= 0.12 \\
X_2+X_{11} &= 0.46 \\
X_3-X_{12}+X_{17} &= 0.40 \\
X_3+X_{13} &= 0.77 \\
X_4-X_{14}+X_{18} &= 0.04 \\
X_4+X_{15} &= 0.31 \\
X_1+X_2+X_3+X_4+X_{19} &= 1.
\end{aligned}
$$

Note that V has now been eliminated from the constraints, hence the question as to whether or not V is negative need no longer concern us (i.e., all $X_i \geq 0$).

The results of the simplex computation are shown in arrays (A) through (Z) of table 4, with the optimal solution to the defined problem revealed in the final array. The column vector, denoting the optimal strategy of Smith's fictional opponent Nature, is taken from the constant column and is seen to be

$$(X_1, X_2, X_3, X_4) = (0.12, 0.17, 0.40, 0.31).$$

By the primal-dual relationship of linear programming, Smith's strategy is obtained by associating the $Z_j - C_j$ values of the final array, after a change in sign, with the slack variables of the original system, i.e.

$$(X_5, X_6, X_7, X_8) = -(0, 0, -1, 0).$$

That is, Smith's optimum strategy is to plant only Crop C_3. And finally, array (J) reveals the value of the game (to Smith) to be 1.38.

5. CONCLUSION

The mathematical technique known as linear programming has a wide variety of important industrial applications (e.g., expediting traffic flow, determining least cost mixtures of ingredients for blends, improving routing of products through processing centers, etc.). As indicated above, this useful and versatile technique may also be successfully applied by the meteorologist or his client to determine an optimal course of action in situations involving weather-induced losses or gains.

In this paper, linear programming techniques were applied to solve Gleeson's minimum economic expectation model. It should be stressed, however, we are not limited in our applications to a single model. Many desired models, including those with nonlinear constraints or objective function, or those with constraints subject to random variation (stochastic programming) [8] can be accommodated with the broad framework of optimization techniques known as "mathematical programming," of which linear programming can be viewed as a subset.

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