Vertical-Structure Functions for Time-Dependent Flow in a Well-Mixed Fluid with Turbulent Boundary Layers at the Bottom and Top

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ABSTRACT

The elements of an eigenfunction expansion for time-dependent currents as a function of depth are worked out for viscosity that is given as a parabolic function of depth that goes to zero at both the bottom and top of the water. This yields currents with logarithmic behavior characteristic of turbulent boundary layers at both the bottom and top. Also, solutions are obtained for the two viscosity functions that are half a parabola, going to zero at either the bottom or top but not both. In all cases the solutions are Legendre functions. In some cases the eigenfunctions are Legendre polynomials.

1. Introduction

We recently developed an eigenfunction expansion that gives the solution of linear equations for time-dependent currents as a function of depth when the viscosity is given as a function of depth (Jordan and Baker, 1980). Among the viscosity functions we considered were linear functions that go to zero at the bottom or top of the water and give currents with logarithmic behavior characteristic of a turbulent boundary layer at the bottom (Thomas, 1975) or top (Madsen, 1977). Reid suggested that logarithmic behavior characteristic of turbulent boundary layers at both the bottom and top could be obtained with a parabolic viscosity function that goes linearly to zero at both the bottom and top (Reid, 1979, private communication). We work that out here. At the same time we consider the two viscosity functions that are half a parabola, going to zero at either the bottom or top but not both. In all cases the solutions are Legendre functions. In some cases the eigenfunctions are Legendre polynomials.

2. General solution

We let the viscosity as a function of depth be

$$\nu(z) = -\kappa(z + \delta)(z + \alpha), \tag{1}$$

where κ and α are positive constants, with $\alpha > H$, and δ is a negative constant. The coordinate z is distance upward, with $z = 0$ at the surface and $z = -H$ at the bottom of the water, so $\nu(z)$ is positive in the range $-H \leqslant z \leqslant 0$ where there is water, and $\nu(z)$ goes to zero at $z = -\alpha$, which is below the bottom and at $z = -\delta$, which is above the surface. We shall eventually consider the limit where $\alpha \to H$ and $\delta \to 0$, as well as other cases.

We change to the variable

$$s = 1 + 2(z + \delta)/(\alpha - \delta), \tag{2}$$

which goes from -1 to 1 as z goes from $-\alpha$ to $-\delta$. Then the differential equations for the steady-state solution and eigenfunctions [Eqs. (16) and (8) of Jordan and Baker (1980)] are converted to Legendre's equation, so the general solutions can be seen to be

$$w_s(z) = EP_\sigma(s) + GQ_\sigma(s), \tag{3}$$

with

$$\sigma(\sigma + 1) = -if/\kappa \tag{4}$$

for the steady-state solution $w_s(z)$, and

$$f_n(z) \propto P_{\sigma(n)}(s) + R_n Q_{\sigma(n)}(s), \tag{5}$$

with $\sigma(n) \geqslant 0$ such that

$$\sigma(n)[\sigma(n) + 1] = \lambda_n/\kappa \tag{6}$$

for the eigenfunction $f_n(z)$ associated with the eigenvalue λ_n, where P_σ and Q_σ are Legendre functions of the first and second kind and E, G and R_n are constants.

The boundary conditions at the surface [Eqs. (17) and (9) of Jordan and Baker] are

$$(\kappa/2)(\alpha - \delta)(1 - s^2)dw_s/ds = 1 \quad \text{at} \quad z = 0, \tag{7}$$

$$(1 - s^2)df_n/ds = 0 \quad \text{at} \quad z = 0, \tag{8}$$

and the boundary conditions for bottom friction [Eqs. (18) and (10) of Jordan and Baker] are

$$(\kappa/2)(\alpha - \delta)(1 - s^2)dw_s/ds = bw_s \quad \text{at} \quad z = -H, \tag{9}$$

$$(\kappa/2)(\alpha - \delta)(1 - s^2)df_n/ds = bf_n \quad \text{at} \quad z = -H. \tag{10}$$

In the limit of infinite b we have the case of zero bottom current in which Eqs. (9) and (10) are simply

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TABLE 1. Numerical results for $\nu(z) = -0.208\ \mathrm{m^2\ s^{-1}}\ z(z + H)/H^2$.

	n					
	0	1	2	3	4	5
$\sigma(n)[\sigma(n) + 1]$ for turbulent bottom	0.2256	2.863	7.723	14.75	23.92	35.20
$\sigma(n)[\sigma(n) + 1]$ for $b = 0$	0	2.000	6.000	12.00	20.00	30.00
$B_n f_n(0)$ for turbulent bottom	−1.09	0.308	−0.208	0.161	−0.131	0.110
$D_n f_n(0)$ for turbulent bottom	17.3 − i225.9	263.2 − i117.3	180.3 − i29.8	130.1 − i11.3	102.4 − i5.4	84.0 − i3.1
$D_n f_n(0)$ for $b = 0$	−i192.0	256.2 − i163.5	191.6 − i40.7	138.6 − i14.7	107.7 − i6.8	87.9 − i3.7

replaced by the condition that $w_s(z)$ and $f_n(z)$ are zero at $z = -H$. For solutions that are singular at $z = -H$ we use the boundary condition that $w_s(z)$ and $f_n(z)$ are zero at $z = -H + \epsilon H$, with ϵ a small positive number; then the whole solution [Eq. (7) of Jordan and Baker] is for the interval $-H + \epsilon H \leq z \leq 0$. For the viscosity functions we consider, this gives currents having the logarithmic behavior

$$w \propto \ln[(H + z)/\epsilon H] \tag{11}$$

characteristic of a turbulent boundary layer at the bottom. In general, the boundary conditions determine the constants E, G and R_n and the eigenvalues λ_n, for whatever values of α and δ you choose.

3. Viscosity going to zero at the bottom and top

We consider the limit as $\delta \to 0$ in which $\nu(z)$ is zero at the water surface $z = 0$. Then

$$s \to 1 + 2z/\alpha \tag{12}$$

and $s \to 1$ at $z = 0$. Using properties of Legendre functions at $s = 1$ (Bateman, 1953) we find that the boundary conditions (7) and (8) at the water surface give

$$G = 2/\kappa\alpha, \tag{13}$$

$$R_n = 0. \tag{14}$$

The eigenfunctions $f_n(z)$ are proportional to $P_{\sigma(n)}(s)$, which are not singular as $s \to 1$ at the water surface, but $w_s(z)$ contains $Q_\sigma(s)$ which has a logarithmic singularity as $s \to 1$. This gives currents having the logarithmic behavior

$$w(z,t) - w_0(t) \propto \ln(-z/z_0) \tag{15}$$

characteristic of a turbulent boundary layer at the surface with roughness length z_0 and surface current w_0 at $z = -z_0$ (Madsen, 1977).

In addition we consider the limit as $\alpha \to H$ in which $\nu(z)$ is zero at the bottom surface $z = -H$. Then

$$s \to 1 + 2z/H \tag{16}$$

and $s \to -1$ at $z = -H$. Using properties of Legendre functions at $s = -1$ (Bateman, 1953) we find this limit is consistent with the bottom-friction boundary conditions (9)–(10) only for $b = 0$, for zero bottom friction. In particular, it is inconsistent with zero bottom current at $z = -H$. For $b = 0$, the boundary condition (9) gives

$$E(2/\pi)\sin(\sigma\pi) + G\cos(\sigma\pi) = 0 \tag{17}$$

which determines E, with $G = 2/\kappa H$ from Eq. (13). Then

$$w_s(z) \to -G\pi/2 \sin(\sigma\pi) \quad \text{as} \quad s \to -1 \tag{18}$$

at the bottom surface $z = -H$. For any other ratio of E and G, the steady-state solution $w_s(z)$ has a logarithmic singularity at $z = -H$. For $b = 0$, the boundary condition (10) is that $\sigma(n)$ is an integer so, from (6) the eigenvalues are

$$\lambda_n = \kappa n(n + 1) \tag{19}$$

for $n = 0, 1, 2, 3 \cdots$, and the eigenfunctions are Legendre polynomials

$$f_n(z) \propto P_n(s). \tag{20}$$

For $n = 0$ we have the zero eigenvalue and constant eigenfunction characteristic of zero bottom friction. If $\sigma(n)$ is not an integer, the eigenfunction $f_n(z)$ has a logarithmic singularity at the bottom surface $z = -H$. Thus, in all solutions except this one for zero bottom friction, both $w_s(z)$ and $f_n(z)$ have a logarithmic singularity at $z = -H$. For these singular solutions we use the boundary condition that $w_s(z)$ and $f_n(z)$ are zero at $z = -H + \epsilon H$; then we get currents with the logarithmic behavior (11) characteristic of a turbulent boundary layer at the bottom. For the steady-state solution $w_s(z)$, this gives

$$[E(1/\pi)\sin(\sigma\pi) + G(1/2)\cos(\sigma\pi)] \times [\ln\epsilon + \gamma + 2\psi(\sigma + 1)] + E\cos(\sigma\pi) - G(\pi/2)\sin(\sigma\pi) = 0, \tag{21}$$

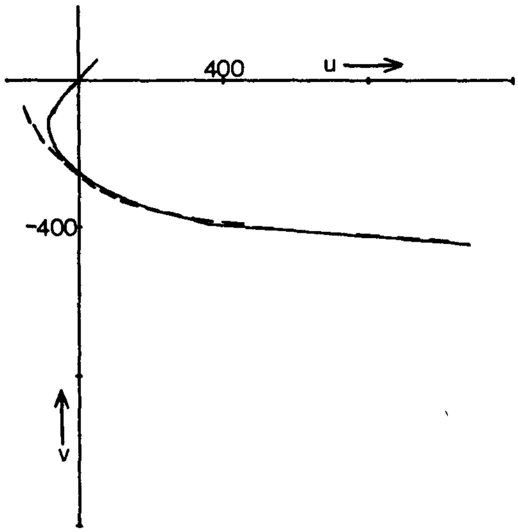

FIG. 1. Steady-state solutions for zero bottom friction (dashed line) and a turbulent boundary layer at the bottom (solid line), both for $\nu(z) = -0.208$ m^2 s^{-1} $z(z + H)/H^2$. Currents (u, v) were calculated for depths z varying in 20 equal steps from the top (largest current) to the bottom (smallest current). These current velocities are in m s^{-1} and for $F(t) = 1$ which is 10^3 to 10^4 times larger than typical wind forces.

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant and ψ the derivative of the logarithm of the gamma function (Bateman, 1953). This determines E, with $G = 2/\kappa H$ again from Eq. (13). For the eigenfunctions $f_n(z)$, this boundary condition gives

$$\ln\epsilon + \gamma + 2\psi[\sigma(n) + 1] + \pi \cot[\sigma(n)\pi] = 0 \quad (22)$$

which determines $\sigma(n)$, and thus the eigenvalues λ_n from (6) and the eigenfunctions $f_n(z)$ from (5) with $R_n = 0$ from (14).

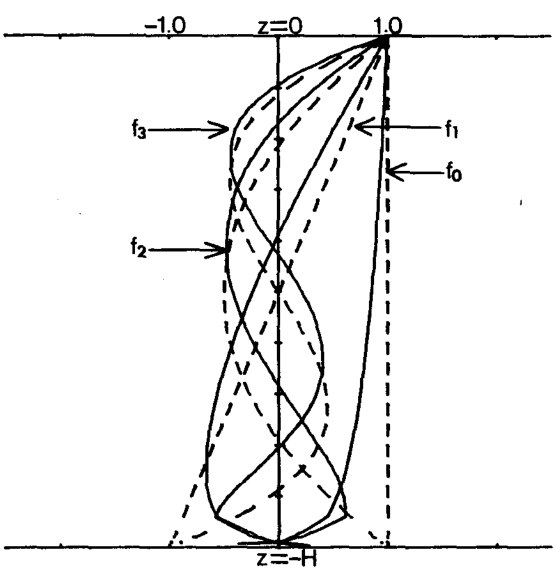

FIG. 2. Eigenfunctions for the same cases as Fig. 1.

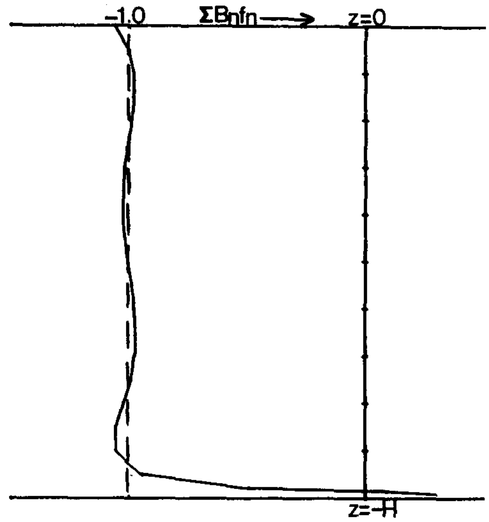

FIG. 3. The first five terms of $\sum_n B_n f_n(z)$ for a turbulent boundary layer at the bottom, for the same viscosity function as Fig. 1.

Since the eigenfunctions for zero bottom friction are the easily computed Legendre polynomials, it is interesting to see how well the solutions for zero bottom friction approximate those for a turbulent boundary layer at the bottom, at depths not too near the bottom. We did numerical computations for both cases, taking κ to be 0.208 m^2 s$^{-1}/H^2$ to get the maximum value of $\nu(z)$ to be 0.052 m^2 s^{-1}, and letting ϵ be 0.01 for the boundary layer. Eigenvalues and values of $B_n f_n(z = 0)$ and $D_n f_n(z = 0)$ are shown in Table 1, steady-state solutions in Fig. 1, and eigenfunctions in Fig. 2. The convergence of

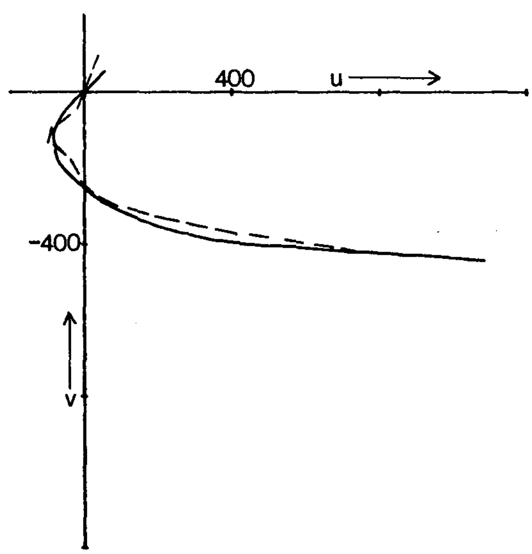

FIG. 4. The first five terms of $\sum_n D_n f_n(z)$ (dashed line) compared with the steady-state solution (solid line) for a turbulent boundary layer at the bottom, the same as in Fig. 1.

$$\sum_n B_n f_n(z) \text{ to } -1 \quad \text{and} \quad \sum_n D_n f_n(z) \text{ to } w_s(z)$$

is shown in Figs. 3 and 4 for a turbulent boundary layer at the bottom. It is fairly good with just five terms, so the eigenfunction expansion appears to converge fast enough to be easily used.

4. Viscosity going to zero at the top only

Still considering the limit for $\delta \to 0$ in which $\nu(z)$ is zero at the water surface $z = 0$, we obtain another interesting case by letting $\alpha = 2H$. Then $\nu(z)$ increases with depth all the way down and levels off to its maximum value at the bottom surface $z = -H$. We have

$$s = 1 + z/H \tag{23}$$

with $s = 0$ at $z = -H$. Using properties of Legendre functions at $s = 0$ (Bateman, 1953) we find that the bottom-friction boundary condition (9) becomes

$$\kappa H[E(2/\pi)\sin(\sigma\pi/2) + G\cos(\sigma\pi/2)] \times [\Gamma(1 + \sigma/2)/\Gamma(\tfrac{1}{2} + \sigma/2)]^2 = b[E(1/\pi)\cos(\sigma\pi/2) - G(\tfrac{1}{2})\sin(\sigma\pi/2)], \tag{24}$$

which determines E, with $G = 1/\kappa H$ from Eq. (13). This is particularly simple for zero bottom friction ($b = 0$), when the left side is zero, or zero bottom current ($b \to \infty$), when the right factor is zero. The bottom-friction boundary condition (10) becomes

$$\kappa H 2\sin(\sigma(n)\pi/2)[\Gamma(1 + \sigma(n)/2)/\Gamma(\tfrac{1}{2} + \sigma(n)/2)]^2 = b\cos(\sigma(n)\pi/2), \tag{25}$$

which determines $\sigma(n)$, and thus the eigenvalues λ_n from (6) and the eigenfunctions $f_n(z)$ from (5) with $R_n = 0$ from (14). For zero bottom friction ($b = 0$) we see that $\sigma(n)$ is an even integer so the eigenvalues are

$$\lambda_n = \kappa 2n(2n + 1) \tag{26}$$

and the eigenfunctions are Legendre polynomials

$$f_n(z) \propto P_{2n}(s) \tag{27}$$

for $n = 0, 1, 2, 3 \cdots$. For zero bottom current ($b \to \infty$) we see that $\sigma(n)$ is an odd integer so the eigenvalues are

$$\lambda_n = \kappa(2n + 1)(2n + 2) \tag{28}$$

and the eigenfunctions are Legendre polynomials

$$f_n(z) \propto P_{2n+1}(s) \tag{29}$$

for $n = 0, 1, 2, 3 \cdots$.

5. Viscosity going to zero at the bottom only

We obtain yet another interesting case by letting $\delta = -H$ and considering the limit as $\alpha \to H$. Then $\nu(z)$ is zero at the bottom surface $z = -H$ and increases with z all the way up, levelling off to its maximum value at the water surface $z = 0$. We have

$$s = z/H \tag{30}$$

with $s = 0$ at $z = 0$ and $s = -1$ at $z = -H$. Using properties of Legendre functions at $s = 0$ (Bateman, 1953) we find that the surface boundary conditions (7) and (8) become

$$\kappa H[E\pi^{-1/2}2\sin(\sigma\pi/2) + G\pi^{1/2} \times \cos(\sigma\pi/2)]\Gamma(1 + \sigma/2)/\Gamma(\tfrac{1}{2} + \sigma/2) = 1, \tag{31}$$

$$(2/\pi)\sin[\sigma(n)\pi/2] + R_n\cos[\sigma(n)\pi/2] = 0. \tag{32}$$

As before, we find the limit as $\alpha \to H$ is consistent with the bottom-friction boundary conditions (9)–(10) only for $b = 0$. Then (9) gives Eq. (17) again, and E and G are determined by Eqs. (17) and (31). The value of $w_s(z)$ at $z = -H$ is again given by (18). For any ratio of E and G except that given by (17), the steady-state solution $w_s(z)$ has a logarithmic singularity at $z = -H$. For $b = 0$, the bottom-friction boundary condition (10) and the surface boundary condition (32) imply that R_n is zero and $\sigma(n)$ is an even integer, so the eigenvalues are given by Eq. (26) and the eigenfunctions are the Legendre polynomials (27). For any R_n and $\sigma(n)$ except those satisfying the boundary condition (10) for $b = 0$, the eigenfunction $f_n(z)$ has a logarithmic singularity at $z = -H$. Again, in all solutions except this one for zero bottom friction, both $w_s(z)$ and $f_n(z)$ have a logarithmic singularity at the bottom. For these singular solutions we again use the boundary condition that $w_s(z)$ and $f_n(z)$ are zero at $z = -H + \epsilon H$; again we get currents with the logarithmic behavior (11) characteristic of a turbulent boundary layer at the bottom. For the steady-state solution $w_s(z)$, this gives Eq. (21) with $\ln\epsilon$ replaced by $\ln(\epsilon/2)$. This and Eq. (31) determine E and G. For the eigenfunctions $f_n(z)$, this boundary condition gives

$$\ln(\epsilon/2) + \gamma + 2\psi[\sigma(n) + 1] + \pi\cot[\sigma(n)\pi/2] = 0 \tag{33}$$

which determines $\sigma(n)$. Then R_n is determined from Eq. (32), the eigenvalues from Eq. (6), and the eigenfunctions from Eq. (5).

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