Vertical Structure of Time-Dependent Flow Dominated by Friction in a Well-Mixed Fluid

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ABSTRACT

Solutions of a linear hydrodynamic equation of motion with linear boundary conditions are obtained to describe the horizontal current, as a function of depth and time, determined by a given history of the wind force and pressure gradient up to that time, at a fixed point in the horizontal plane, in well-mixed water of finite depth. The bottom friction is assumed to be proportional to the bottom current, with zero bottom current and zero bottom friction considered as limiting cases. The general solution is established as an eigenfunction expansion when the eddy viscosity is given as a positive function of depth. Explicit formulas are worked out for viscosity functions that are constant, exponential, or varying as a power of the height from somewhere below the bottom or above the top of the water. For the latter the limit as the viscosity goes to zero at the bottom or top is considered. Numerical results are presented for viscosities that are constant, exponential, linear, or varying as the 3/4 power.

1. Introduction

Herein we obtain solutions of a linear hydrodynamic equation of motion with linear boundary conditions to describe the horizontal current, as a function of depth and time, determined by a given history of the wind force and pressure gradient up to that time, at a fixed point in the horizontal plane, in well-mixed water of finite depth. We assume the eddy viscosity is a given positive function of depth. We assume the bottom friction is proportional to the bottom current; we consider zero bottom current and zero bottom friction as limiting cases.

The general solution is established as an eigenfunction expansion valid for any positive viscosity function. Explicit formulas are worked out for viscosity functions that are constant, exponential, or varying as a power of the height from somewhere below the bottom or above the top of the water. For the latter we consider the limit as the viscosity goes to zero at the bottom or top. Numerical results are presented for viscosities that are constant, exponential, linear, or varying as the 3/4 power. These viscosity functions are shown in Fig. 1.

These formulas were studied first and worked out most completely for constant viscosity (Ekman, 1905; Nomitsu, 1933a; 1933b; 1933c; Nomitsu and Takegami, 1933a; 1933b; Welander, 1957). The steady-state solution for exponential viscosity and zero bottom current was found by Witten and Thomas (1976) in their development of a model for steady currents in Lake Ontario. A power-series solution for steady-state current with zero bottom current was worked out by Fjeldstad (1929) for a viscosity function varying as the 3/4 power which he chose as a best fit to current data. This viscosity function was used recently in a study of coastal currents (Murray, 1975).

A steady-state solution for viscosity that decreases linearly with depth and goes to zero at the bottom was found by Thomas (1975), who used a boundary condition characteristic of a turbulent boundary layer at the bottom. This yields currents that are singular at the bottom. In the limit for viscosity that decreases with depth as a power of the height from the bottom and goes to zero at the bottom, we get different results depending on what that power is. When the decrease of viscosity with depth is slower than linear, we get a non-singular solution no matter what boundary condition we choose for bottom friction. When the decrease of viscosity is linear or faster, we find that zero bottom friction is the only one of our boundary conditions that gives non-singular solutions. We also find singular solutions. When the decrease of viscosity with depth is linear we have a logarithmic singularity that can be identified with a turbulent boundary layer. When the decrease of viscosity is faster, the solutions are more singular. For the linear case we compute singular solutions obtained from Thomas' boundary condition as well as non-singular solutions for zero bottom friction.

Thomas (1975) also shows how the viscosity should depend on the wind force and flow so, when

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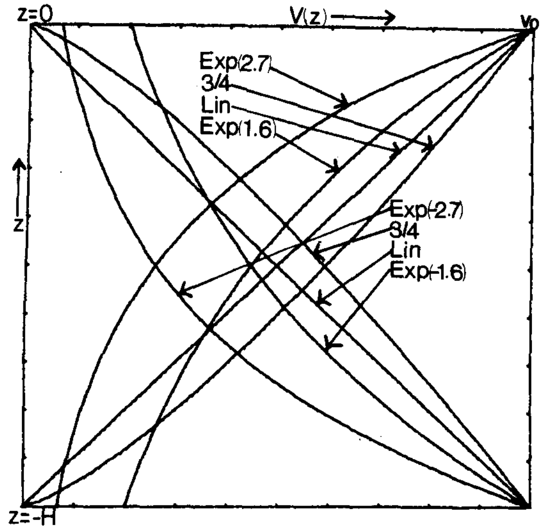

FIG. 1. Viscosity functions $\nu(z)$ used for numerical computations.

time-dependent currents are considered, the viscosity should be made a function of time as well as depth. We have not attempted that.

Madsen (1977) has suggested that the viscosity should increase linearly with depth, at least near the surface, to give the current a logarithmic singularity that can be identified with a turbulent boundary layer near the surface. Therefore, we have used viscosity functions that increase with depth as well as those that decrease. In the limit for viscosity that goes to zero at the surface and increases as a power of the depth, we get different results again depending on what that power is. When the increase of viscosity with depth is slower than linear we get nonsingular solutions. When the increase of viscosity is linear or faster we get singular solutions. When the increase of viscosity is linear we have a logarithmic singularity that can be identified with a turbulent boundary layer. When the increase of viscosity is faster, the solutions are more singular.

Madsen (1977) found the solution for viscosity that increases linearly with depth for infinitely deep water and zero bottom current. We have computed the corresponding solution for water of finite depth, using our various boundary conditions for bottom friction. Our solution does not reproduce Madsen's elegant formula for time-dependent currents because our eigenfunction expansion does not apply to infinitely deep water. We do get Madsen's steady-state solution in the limit of infinitely deep water.

Different eigenfunctions have been used to describe the vertical structure of flow in stratified water (Philander, 1978). Our solution is a more generally applicable form of formulas for time-dependent currents that have been used to calculate bottom friction in storm surges on the Atlantic (Jelesnianski, 1970) and current profiles as a function of depth for storm surges in the Gulf of Mexico (Forristall, 1974). In that work it was assumed that the eddy viscosity is constant and the current goes to zero at the bottom.

With these formulas one can try to split a computation of currents into two steps (Forristall, 1974). First the vertically integrated equation of motion is solved for the transport and water elevation produced by a given wind force. Then the water elevation is used to calculate the hydrostatic pressure gradient which, together with the atmospheric pressure gradient and wind force, is put into the formulas for the current as a function of depth. There seem to have been two main troubles with this approach. One was that the eddy viscosity was usually assumed to be constant. The other difficulty is that when the transport and water elevation are calculated with the vertically integrated equation of motion, the bottom friction can be made to depend only on the transport, which may be a poor substitute for bottom friction depending on bottom current since the transport and bottom current may actually be in opposite directions. Here we show how to deal with the first problem at least to the extent that the eddy viscosity can be chosen to be any positive function of depth.

Laplace-transform methods were used to derive the formulas for recent applications (Jelesnianski, 1970; Forristall, 1974). Our method here is based on a proof that with our boundary conditions a solution of the hydrodynamic differential equation of motion is unique. Knowing that, we need not be interested so much in how a solution is obtained. We need only check that it does satisfy the differential equation and boundary conditions. Thus our method makes it easy for anyone wishing to use these formulas to check to see for himself how they are solutions of the basic hydrodynamic problem.

There is one subtle feature of the formulas that could cause difficulty in checking that they are solutions. We make this explicit by writing the wind-force part of the current as a sum of three terms. The first two cancel in the sense that the second is the negative of a series for the first. Thus, the third term alone may be used for numerical approximation (Jelesnianski, 1970; Forristall, 1974). However, the first two terms are needed to satisfy the differential equation and the boundary condition for the wind force. Thus, it might be better to keep all three terms in a series approximation, particularly if surface currents are important.

The uniqueness proof, other mathematical details, and further numerical results are available in a more extensive report (Jordan and Baker, 1980).

There we also show that a solution is unique if the boundary condition for bottom friction is replaced by the requirement that the current produce a given transport. That can provide a check when current as a function of depth is calculated with our formulas following a calculation of transport and water elevation with the vertically integrated equation of motion.

2. Problem

The problem is simply stated in terms of the differential equation and boundary conditions the current is assumed to satisfy. The horizontal (x and y) current components u and v are represented by the complex variable $w = u + iv$. Distance upward is denoted by z, with $z = 0$ at the surface of undisturbed water, and $z = -H$ at the bottom of the lake. The currents are assumed to satisfy the hydrodynamic equation of motion

$$\frac{\partial w}{\partial t} = -ifw + \frac{\partial}{\partial z}\nu\frac{\partial w}{\partial z} - q, \tag{1}$$

where t denotes time. Here $f = 2\omega \sin\phi$ is the Coriolis parameter, with ω the angular velocity of the earth, and ϕ the latitude. The complex variable

$$q = (1/\rho)(\partial p/\partial x + i\partial p/\partial y)$$

represents the horizontal components $\partial p/\partial x$ and $\partial p/\partial y$ of the pressure gradient, divided by the water density ρ. We assume q does not depend on z. It is a function of t. For example, if we use just hydrostatic water pressure, and neglect changes in atmospheric pressure, we have

$$q = g(\partial h/\partial x + i\partial h/\partial y),$$

where h is the elevation of the water surface, which is a function of x, y and t.

We assume the eddy viscosity ν is a given positive function of z. As examples we will consider cases where ν is a constant, an exponential function, or varying as a power of the height from somewhere below the bottom or above the top of the water. For the latter we will consider the limit where ν goes to zero at the bottom or top.

We neglect horizontal eddy viscosity and nonlinear terms of the equation of motion.

For the boundary condition at the water surface we assume that

$$\left(\nu\frac{\partial w}{\partial z}\right)(z = 0) = F, \tag{2}$$

where $F = (1/\rho)(F_x + iF_y)$ is the complex variable whose real and imaginary parts are the x and y components of the wind force per unit area of water surface, divided by the water density ρ. It is a function of t.

We assume the bottom friction is proportional to the bottom current, which gives us the linear boundary condition

$$\left(\nu\frac{\partial w}{\partial z}\right)(z = -H) = bw(z = -H), \tag{3}$$

with b a non-negative real number. In the limit of infinite b we get the condition

$$w(z = -H) = 0, \tag{4}$$

that the current is zero at the bottom. For $b = 0$ we have the condition

$$\left(\nu\frac{\partial w}{\partial z}\right)(z = -H) = 0, \tag{5}$$

that there is no bottom friction.

We consider currents caused by winds and pressure gradients that start building up from zero at $t = 0$, so $q(t)$ and $F(t)$ are zero for $t \leqslant 0$. Hence we assume that

$$w(t = 0) = 0. \tag{6}$$

The problem, then, is to find a complex solution $w(z,t)$ of the differential equation (1) for $-H \leqslant z \leqslant 0$ and $t \geqslant 0$ that satisfies the boundary conditions (2), (3) or one of its limit forms (4) or (5), and (6), for a given positive function $\nu(z)$, given complex functions $q(t)$ and $F(t)$, and given positive numbers f and b.

A solution of this problem is unique. The proof is straightforward (Jordan and Baker, 1980). Thus, if we find a solution, we know it is the only solution. Hence it does not matter how we find it. (In fact, we guessed the general solution by looking at the form of the solution in particular cases.)

3. Solution

There is a solution of the form

$$w(z,t) = \sum_{n=0}^{\infty} B_n f_n(z) \int_0^t q(t-\tau)e^{-if\tau}e^{-\lambda_n\tau}d\tau \tag{7)(i}$$

$$+ F(t)w_s(z) - F(t)\sum_{n=0}^{\infty} D_n f_n(z) \tag{ii}$$

$$+ \sum_{n=0}^{\infty} (if + \lambda_n)D_n f_n(z) \int_0^t F(t-\tau)e^{-if\tau}e^{-\lambda_n\tau}d\tau. \tag{iii}$$

The λ_n are eigenvalues and the $f_n(z)$ are corresponding eigenfunctions of the operator

$$-\frac{d}{dz}\nu\frac{d}{dz}$$

which we take to act on functions that satisfy the boundary conditions (2) with $F = 0$ and (3), so

$$-\frac{d}{dz}\nu\frac{d}{dz}f_n(z) = \lambda_n f_n(z), \tag{8}$$

$$\left(\nu \frac{df_n}{dz}\right)(z = 0) = 0, \tag{9}$$

$$\left(\nu \frac{df_n}{dz}\right)(z = -H) = bf_n(z = -H). \tag{10}$$

In the limit of infinite b, the case of zero bottom current when the boundary condition (3) is replaced by the limit form (4), Eq. (10) is replaced by

$$f_n(z = -H) = 0. \tag{11}$$

With these boundary conditions, this is a regular (non-singular) self-adjoint Sturm-Liouville operator whose spectrum consists only of eigenvalues (Stone, 1932; Birkhoff and Rota, 1962). (There is no continuous spectrum for a finite interval.) Therefore, the eigenfunctions $f_n(z)$ are complete; they span the space of square-integrable functions on the interval $-H \leq z \leq 0$.

Each eigenfunction $f_n(z)$ is determined to within multiplication by a constant. For each eigenvalue there is only one linearly independent eigenfunction. [If we have two eigenfunctions for the same eigenvalue, we can make them equal, make their difference zero, by multiplying one of them by a constant, because we can choose the constant to make the difference zero at $z = 0$; a solution of the differential equation (8) that satisfies the boundary condition (9) and is zero at $z = 0$ must be zero for all z in the interval $-H \leq z \leq 0$.]

Without loss of generality we can assume each eigenfunction $f_n(z)$ is real; if an eigenfunction were complex, its real and imaginary parts would be eigenfunctions for the same eigenvalue and therefore linearly dependent, one a constant times the other, so the complex eigenfunction can be made real by multiplying it by a constant.

The eigenvalues λ_n are all non-negative real numbers. The proof is straightforward (Jordan and Baker, 1980). We label the eigenvalues as $\lambda_0 < \lambda_1 < \lambda_2 \cdots$. There is an infinite number of them, and $1/\lambda_n \to 0$ as $n \to \infty$ (Birkhoff and Rota, 1962).

In our solution (7), the B_n and D_n are constants chosen so that

$$\sum_{n=0}^{\infty} B_n f_n(z) = -1, \tag{12}$$

$$\sum_{n=0}^{\infty} D_n f_n(z) = w_s(z), \tag{13}$$

for $-H \leq z \leq 0$. We know they exist because the eigenfunctions $f_n(z)$ are complete. Since the eigenfunctions are orthogonal, because different eigenfunctions correspond to different eigenvalues, we have

$$B_n = -\int_{-H}^{0} f_n(z)dz \left[\int_{-H}^{0} f_n(z)^2 dz\right]^{-1}, \tag{14}$$

$$D_n = \int_{-H}^{0} f_n(z)w_s(z)dz \left[\int_{-H}^{0} f_n(z)^2 dz\right]^{-1}. \tag{15}$$

Finally, in our solution (7), $w_s(z)$ is the steady-state solution for the current with $q(t) = 0$ and $F(t) = 1$; so

$$\left(\frac{d}{dz}\nu\frac{d}{dz} - if\right)w_s(z) = 0, \tag{16}$$

$$\left(\nu \frac{dw_s}{dz}\right)(z = 0) = 1, \tag{17}$$

$$\left(\nu \frac{dw_s}{dz}\right)(z = -H) = bw_s(z = -H). \tag{18}$$

In the limit of infinite b, the case of zero bottom current, when the boundary condition (3) is replaced by the limit form (4), Eq. (18) is replaced by

$$w_s(z = -H) = 0. \tag{19}$$

The steady-state solution $w_s(z)$ is unique. This is easy to show. Suppose we have two solutions whose difference is w. It satisfies equations like (16)–(18) for w_s but with 0 instead of 1 on the right side of (17). Then

$$\int_{-H}^{0} \left(\frac{dw}{dz}\right)^* \nu \frac{dw}{dz} dz = -\int_{-H}^{0} w^*\left(\frac{d}{dz}\nu\frac{dw}{dz}\right)dz + w^*\nu\frac{dw}{dz}\Big|_{-H}^{0} = -if\int_{-H}^{0} w^*w dz - b\left|w(z = -H)\right|^2.$$

This shows w is zero. We get this without the last term, the same as when b is zero, in the limit of infinite b, the case of zero bottom current, when Eq. (18) is replaced by (19).

Then it can be proved also that a steady-state solution exists (Coddington and Levinson, 1955). That establishes all the pieces we need to construct the solution (7).

The solution (7) clearly satisfies the boundary conditions (2), (3) or one of the limit forms (4) or (5) as the case may be, and (6). It is easy, and rather fun, to check that it satisfies the differential equation (1).

Part (ii) cancels out of the solution (7) in the sense that the second term of part (ii) is the negative of a series for the first term, according to Eq. (13); but the boundary condition (2) is satisfied entirely by part (ii), and part (ii) is needed to satisfy the differential equation (1). Leaving part (ii) out would be one way to get a numerical approximation (Jelesnianski, 1970; Forristall, 1974). When surface currents are important, it might be better to satisfy the boundary condition (2) by keeping part (ii) in a series approximation.

The role of the eigenfunctions and eigenvalues is particularly evident in the current caused by a constant wind that begins suddenly. Suppose $q(t)$ is zero, $F(t)$ is zero for $t \leqslant 0$, and $F(t) = 1$ for $t > 0$. Then the current obtained from our solution (7) is

$$w(z,t) = w_s(z) - \sum_{n=0}^{\infty} D_n f_n(z) e^{-ift} e^{-\lambda_n t}$$

for $t > 0$.

We will give specific formulas and numerical results, assuming particular viscosity functions. For the cases we consider, the eigenfunctions are Bessel functions. For viscosity varying as a power, the order of the Bessel functions depends on the power as shown in Fig. 2. To make comparison easy, numerical results for the different cases will be shown together. Eigenvalues will be shown in Table 1, and values of B_n and D_n in Tables 2 and 3. Steady-state solutions will be shown in Figs. 3 and 4 and eigenfunctions in Figs. 5 and 6. The convergence of the series in Eqs. (12) and (13) will be shown in Figs. 7–10.

4. Simplification

We now obtain formulas for the constants B_n and D_n and gain some simplification in the general solution and more in the case of zero bottom friction.

Using Eqs. (8)–(11) and (16)–(19) and integrating by parts yields (Jordan and Baker, 1980)

$$(if + \lambda_n) \int_{-H}^{0} f_n w_s dz = f_n(z = 0). \quad (20)$$

From (15) and (20) we have

$$D_n = \frac{1}{if + \lambda_n} \frac{f_n(z = 0)}{\|f_n\|^2}, \quad (21)$$

where

$$\|f_n\|^2 = \int_{-H}^{0} f_n(z)^2 dz. \quad (22)$$

From Eqs. (8)–(10) we also obtain

$$\lambda_n \int_{-H}^{0} f_n dz$$
$$= - \int_{-H}^{0} \frac{d}{dz} \nu \frac{df_n}{dz} dz = \left(\nu \frac{df_n}{dz}\right)(z = -H)$$
$$= bf_n(z = -H), \quad (23)$$

so from (14) we see that

$$B_n = -\left(\nu \frac{df_n}{dz}\right)(z = -H)[\lambda_n \|f_n\|^2]^{-1}$$
$$= -bf_n(z = -H)[\lambda_n \|f_n\|^2]^{-1}. \quad (24)$$

The last term is not applicable in the limit of infinite b, the case of zero bottom current.

In the case of zero bottom friction, when b is zero, there is further simplification. Then Eqs. (8)–(10) yield

$$\lambda_0 = 0,$$
$$f_0(z) = \text{constant}.$$

From Eq. (24) we see that

$$B_n = 0 \quad \text{for} \quad n \neq 0.$$

We can see this also from (14) because f_n is orthogonal to f_0 for $n \neq 0$ and f_0 is constant. Then to satisfy (12) we must have

$$B_0 f_0 = -1.$$

Also, in the case of zero bottom friction, since λ_0 is zero and f_0 is constant, we see from Eq. (21) that

$$D_0 f_0 = \frac{1}{ifH}. \quad (25)$$

There is a zero eigenvalue only if the bottom friction is zero. If λ_0 is zero, we see from (8) that $\nu df_0/dz$ is a constant and from (9) that this constant is zero. Since ν is positive, df_0/dz must be zero so f_0 is a constant. From Eq. (10) we see bf_0 is zero. Since f_0 can not be zero, b must be zero.

5. Constant viscosity

To look at the solution in more detail, we choose some simple functions for the viscosity $\nu(z)$. First, we consider the much-studied case of constant viscosity.

The steady-state solution, which satisfies Eqs. (16)–(18), is

$$w_s(z) = \frac{sh[\gamma(H + z)]}{\nu\gamma ch[\gamma H]} + \frac{ch[\gamma z]}{(\nu\gamma sh[\gamma H] + bch[\gamma H])ch[\gamma H]}, \quad (26)$$

where $\gamma = (if/\nu)^{1/2}$. The eigenfunctions and eigenvalues, which satisfy (8)–(10), are

$$f_n(z) = \cos[\beta_n z], \quad (27)$$
$$\lambda_n = \nu\beta_n^2, \quad (28)$$

where β_n for $n = 0, 1, 2 \ldots$ are the successively increasing positive numbers such that

$$\nu\beta_n \sin[\beta_n H] = b \cos[\beta_n H]. \quad (29)$$

This condition on the β_n ensures that the boundary condition (10) is satisfied. Then

$$B_n = \frac{-2 \sin[\beta_n H]}{\sin[\beta_n H] \cos[\beta_n H] + \beta_n H}, \quad (30)$$

$$D_n = \frac{1}{if + \lambda_n} \frac{2\beta_n}{\sin[\beta_n H] \cos[\beta_n H] + \beta_n H}. \quad (31)$$

Putting all these pieces in Eq. (7), we obtain a formula for the current which has been known for many years (Nomitsu and Takegami, 1933a; 1933b).

We now consider the limit of infinite b, the case of zero bottom current. The steady-state solution, which satisfies the boundary condition (19), is

$$w_s(z) = \frac{sh[\gamma(H + z)]}{\nu\gamma ch[\gamma H]}\,. \qquad (32)$$

This was found by Ekman (1905). Since the real and imaginary parts of w are the x and y components of the current vector, the position of $w(z)$ in the complex plane marks the tip of the current vector at depth z. The path of $w_s(z)$ as z varies from the top to the bottom of the water, for the steady-state solution (32), the "Ekman spiral", is shown in Fig. 3, compared with results for other viscosity functions. Since the real and imaginary parts of $F(t)$ are proportional to the x and y components of the wind force, this solution for $F(t) = 1$ describes currents caused by a constant wind blowing toward the x direction.

In the limit of infinite b, the case of zero bottom current, when Eq. (10) is replaced by (11), Eq. (29) is replaced by

$$\cos[\beta_n H] = 0,$$

so

$$\beta_n = (n + 1/2)\pi/H \qquad (33)$$

for $n = 0, 1, 2. \ldots$ Then our solution (7) reduces to one that was found long ago (Nomitsu, 1933a; 1933b) and was used recently, without part (ii), for numerical approximation (Jelesnianski, 1970; Forristall, 1974).

Values of B_n and D_n are shown in Tables 2 and 3, compared with values for other viscosity functions, and the convergence of $\sum_n B_n f_n(z)$ to -1 and $\sum_n D_n f_n(z)$ to $w_s(z)$ required by Eqs. (12) and (13) is shown in Figs. 7 and 8.

6. Exponential viscosity

Next we consider viscosity that varies exponentially with depth, either decreasing or increasing, so

$$\nu(z) = \nu_0 e^{az} \qquad (34)$$

with positive constant ν_0 and real constant a.

With a change of variable to

$$s = 2|a|^{-1}\nu_0^{-1/2}e^{-az/2}, \qquad (35)$$

with

$$w_s(z) = s\psi(s),$$

$$f_n(z) = s\psi_n(s),$$

the differential equations (16) and (8) for the steady-state solution and eigenfunctions are converted to forms of Bessel's equation for $\psi(s)$ and $\psi_n(s)$, so the general solutions can be seen to be

$$w_s(z) = s[EJ_1(i\sqrt{if}\,s) + GN_1(i\sqrt{if}\,s)], \qquad (36)$$

$$f_n(z) \propto s[J_1(\sqrt{\lambda_n}s) + R_n N_1(\sqrt{\lambda_n}s)], \qquad (37)$$

where J_1 and N_1 are the Bessel and Neumann functions and E, G and R_n are constants.

Using

$$xJ_1'(x) = xJ_0(x) - J_1(x),$$

$$xN_1'(x) = xN_0(x) - N_1(x), \qquad (38)$$

we can write the boundary conditions (17) and (18) for the steady-state solution $w_s(z)$ as

$$-i\sqrt{if}(2/a)[EJ_0(\xi) + GN_0(\xi)] = 1, \qquad (39)$$

with

$$\xi = i(2/|a|)\sqrt{if/\nu_0}, \qquad (40)$$

and

$$if(2/a)[EJ_0(\zeta) + GN_0(\zeta)] = b\zeta[EJ_1(\zeta) + GN_1(\zeta)], \qquad (41)$$

with

$$\zeta = i(2/|a|)\sqrt{if/\nu_0}\,e^{aH/2}. \qquad (42)$$

Similarly, the boundary conditions (9) and (10) for the eigenfunctions $f_n(z)$ become

$$x_n[J_0(x_n) + R_n N_0(x_n)] = 0, \qquad (43)$$

where

$$x_n = (2/|a|)\sqrt{\lambda_n/\nu_0}, \qquad (44)$$

and

$$-(2/a)\lambda_n[J_0(y_n) + R_n N_0(y_n)] = by_n[J_1(y_n) + R_n N_1(y_n)], \qquad (45)$$

where

$$y_n = (2/|a|)\sqrt{\lambda_n/\nu_0}\,e^{aH/2}. \qquad (46)$$

In the limit of infinite b, the case of zero bottom current, when the boundary conditions (18) and (10) are replaced by (19) and (11), Eqs. (41) and (45) are replaced by

$$\zeta[EJ_1(\zeta) + GN_1(\zeta)] = 0, \qquad (47)$$

$$y_n[J_1(y_n) + R_n N_1(y_n)] = 0. \qquad (48)$$

The constants E and G are determined by (39) and (41) or (47). Then Eq. (36) specifies the steady-state solution $w_s(z)$. This was worked out by Witten and Thomas (1976) in the case of zero bottom current. Graphs like the Ekman spiral (described for constant viscosity) are shown in Figs. 3 and 4 compared with solutions for other viscosity functions.

The constant R_n is specified by Eq. (43). (The possibility that λ_0, x_0 and y_0 are zero need not concern us here because we know that occurs only in the case of zero bottom friction which we already considered.) Then (45) or (48) determines the eigenvalues λ_n, and (37) specifies the eigenfunctions $f_n(z)$. By inspecting these equations we can see that the dimensionless eigenvalues $\lambda_n H^2/\nu_0$ depend on only the dimensionless parameters aH and bH/ν_0. [For constant viscosity $\lambda_n H^2/\nu$ is just $(n + 1/2)^2\pi^2$ for zero bottom current, according to Eqs. (28) and (33).]

We have computed examples for zero bottom current. Eigenvalues are shown in Table 1 and an eigenfunction in Figs. 5 and 6 for various values of the parameters and compared with results for other viscosity functions. Values of $B_n f_n(z = 0)$ and $D_n f_n(z = 0)$ are shown in Tables 2 and 3, and the convergence of $\sum_n B_n f_n(z)$ to -1 and $\sum_n D_n f_n(z)$ to $w_s(z)$ required by (12) and (13) is shown in Figs. 7–10.

Table 1 shows that the eigenvalues λ_n are quite different for different viscosity functions. This means that time-dependent currents are different, since each term contains a factor $e^{-\lambda_n \tau}$.

The eigenfunction expansion appears to converge fast enough to be easily used. From Figs. 7–10 and Tables 2 and 3 we can see that the convergence of $\sum_n B_n f_n(z)$ to -1 and $\sum_n D_n f_n(z)$ to $w_s(z)$ is generally fairly good with just five terms, but is somewhat slower when the viscosity increases with depth.

7. Viscosity varying as a power

Finally, we consider viscosity that decreases (or increases) with depth as a power of the height from somewhere below the bottom (or above the top) of the water, so

$$\nu(z) = \nu_0(1 + z/\alpha)^{\mu}, \tag{49}$$

with ν_0 and μ positive constants and α a real constant such that $\alpha > H$ (or $\alpha < 0$).

With a change of variable to

$$s = \left|\alpha(1 - \mu/2)^{-1}\right| \nu_0^{-1/2}(1 + z/\alpha)^{1-\mu/2}, \tag{50}$$

with

$$w_s(z) = s^{\{(1/2)[(1-\mu)/(1-\mu/2)]\}}\psi(s),$$

$$f_n(z) = s^{\{(1/2)[(1-\mu)/(1-\mu/2)]\}}\psi_n(s),$$

the differential equations (16) and (8) for the steady-state solution and eigenfunctions are converted to forms of Bessel's equation for $\psi(s)$ and $\psi_n(s)$, so the general solutions can be seen to be

$$w_s(z) = s^{\{(1/2)[(1-\mu)/(1-\mu/2)]\}}[EJ_\sigma(i\sqrt{if}\, s) + GN_\sigma(i\sqrt{if}\, s)], \tag{51}$$

$$f_n(z) \propto s^{\{(1/2)[(1-\mu)/(1-\mu/2)]\}}[J_\sigma(\sqrt{\lambda_n} s) + R_n N_\sigma(\sqrt{\lambda_n} s)], \tag{52}$$

where J_σ and N_σ are the Bessel and Neumann functions of order

$$\sigma = \frac{1}{2}\left|\frac{1 - \mu}{1 - \mu/2}\right| \tag{53}$$

and E, G and R_n are constants.

Using

$$xJ_\sigma'(x) \pm \sigma J_\sigma(x) = \pm xJ_{\sigma\mp1}(x),$$

$$xN_\sigma'(x) \pm \sigma N_\sigma(x) = \pm xN_{\sigma\mp1}(x), \tag{54}$$

we can write the boundary conditions (17) and (18) for the steady-state solution $w_s(z)$ as

$$\pm(-if)^{\pm\sigma/2}\nu_0^{1\mp\sigma}\left(\frac{\alpha}{1 - \mu/2}\right)\left|\frac{\alpha}{1 - \mu/2}\right|^{-2\pm2\sigma}\xi^{1\mp\sigma} \times [EJ_{\sigma\mp1}(\xi) + GN_{\sigma\mp1}(\xi)] = 1, \tag{55}$$

$$\xi = i(if/\nu_0)^{1/2}\left|\frac{\alpha}{1 - \mu/2}\right|, \tag{56}$$

$$\pm(-if)^{\pm\sigma}\nu_0^{1\mp\sigma}\left(\frac{\alpha}{1 - \mu/2}\right)\left|\frac{\alpha}{1 - \mu/2}\right|^{-2\pm2\sigma}\zeta^{1\mp\sigma} \times [EJ_{\sigma\mp1}(\zeta) + GN_{\sigma\mp1}(\zeta)] = b\zeta^{\pm\sigma}[EJ_\sigma(\zeta) + GN_\sigma(\zeta)], \tag{57}$$

$$\zeta = i(if/\nu_0)^{1/2}\left|\frac{\alpha}{1 - \mu/2}\right|(1 - H/\alpha)^{1-\mu/2}, \tag{58}$$

and we can similarly write the boundary conditions (9) and (10) for the eigenfunctions $f_n(z)$ as

$$\nu_0^{1\mp\sigma}\left(\frac{\alpha}{1 - \mu/2}\right)\left|\frac{\alpha}{1 - \mu/2}\right|^{-2\pm2\sigma}x_n^{1\mp\sigma} \times [J_{\sigma\mp1}(x_n) + R_n N_{\sigma\mp1}(x_n)] = 0, \tag{59}$$

$$x_n = (\lambda_n/\nu_0)^{1/2}\left|\frac{\alpha}{1 - \mu/2}\right|, \tag{60}$$

$$\pm\lambda_n^{\pm\sigma}\nu_0^{1\mp\sigma}\left(\frac{\alpha}{1 - \mu/2}\right)\left|\frac{\alpha}{1 - \mu/2}\right|^{-2\pm2\sigma}y_n^{1\mp\sigma} \times [J_{\sigma\mp1}(y_n) + R_n N_{\sigma\mp1}(y_n)] = by_n^{\pm\sigma}[J_\sigma(y_n) + R_n N_\sigma(y_n)], \tag{61}$$

$$y_n = (\lambda_n/\nu_0)^{1/2}\left|\frac{\alpha}{1 - \mu/2}\right|(1 - H/\alpha)^{1-\mu/2}. \tag{62}$$

In the limit of infinite b, the case of zero bottom current, when the boundary conditions (18) and (10) are replaced by (19) and (11), Eqs. (57) and (61) are replaced by

$$\zeta^{\pm\sigma}[EJ_\sigma(\zeta) + GN_\sigma(\zeta)] = 0, \tag{63}$$

$$y_n^{\pm\sigma}[J_\sigma(y_n) + R_n N_\sigma(y_n)] = 0. \tag{64}$$

For each μ either the upper or lower signs apply in every equation the same as in

$$\pm\sigma = \left(\frac{1}{2}\right)\frac{1 - \mu}{1 - \mu/2}. \tag{65}$$

This function of μ is shown in Fig. 2. Again, the possibility that λ_0, x_0 and y_0 are zero need not concern us here because we know that occurs only in the case of zero bottom friction which we already considered. Some factors involving λ_n have already been cancelled.

In general, these boundary conditions determine

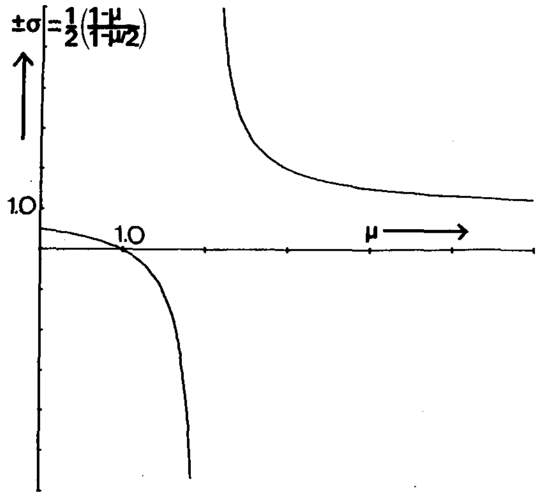

FIG. 2. The order σ of the Bessel functions depending on the power μ of the viscosity function.

the constants E, G and R_n and the eigenvalues λ_n. Then Eqs. (51) and (52) specify the steady-state solution $w_s(z)$ and the eigenfunctions $f_n(z)$. You can work out any example you choose.

8. Viscosity going to zero at the bottom

We chose to consider first $\alpha > H$ and take the limit as $\alpha \to H$ so the viscosity decreases with depth and goes to zero at the bottom. In particular we consider $\mu < 2$. Then $\zeta \to 0$ and $y_n \to 0$ which simplifies the boundary conditions (57) and (61), or (63) and (64), for the bottom friction. (In writing these equations we were careful not to cancel any of these factors that go to zero.)

For $0 < \mu < 1$ we use the upper signs in all our equations. Using the forms of Bessel and Neumann functions for small values of the argument, we find that as $\alpha \to H$ the boundary conditions (57) and (61) become

$$(-if)^{\sigma}\nu_0^{1-\sigma}\left(\frac{\alpha}{1-\mu/2}\right)^{-1+2\sigma}$$

$$\times (E + G\cot[\sigma\pi])2^{\sigma-1}/\Gamma(\sigma) = -bG2^{\sigma}\Gamma(\sigma)/\pi, \quad (66)$$

$$\lambda_n^{\sigma}\nu_0^{1-\sigma}\left(\frac{\alpha}{1-\mu/2}\right)^{-1+2\sigma}(1 + R_n\cot[\sigma\pi])2^{\sigma-1}/\Gamma(\sigma)$$

$$= -bR_n 2^{\sigma}\Gamma(\sigma)/\pi. \quad (67)$$

For $b = 0$, the case of zero bottom friction, we see that G is $-E\tan[\sigma\pi]$ and R_n is $-\tan[\sigma\pi]$ for $n = 1, 2, 3, \ldots$ Then E is determined by (55), and from (51) we have

$$w_s(z) = Es^{\sigma}[J_{\sigma}(i\sqrt{if}\,s) - \tan[\sigma\pi]N_{\sigma}(i\sqrt{if}\,s)]. \quad (68)$$

For $n \neq 0$ the eigenvalues λ_n are determined by Eq. (59) which becomes

$$J_{\sigma-1}(x_n) - \tan[\sigma\pi]N_{\sigma-1}(x_n) = 0, \quad (69)$$

with the eigenfunctions being

$$f_n(z) \propto s^{\sigma}[J_{\sigma}(\sqrt{\lambda_n}s) - \tan[\sigma\pi]N_{\sigma}(\sqrt{\lambda_n}s)]. \quad (70)$$

For the case of zero bottom current, the boundary conditions (63) and (64) become simply

$$G = 0, \quad (71)$$

$$R_n = 0, \quad (72)$$

as $\alpha \to H$. Then E is determined by Eq. (55), and from Eq. (51) we have

$$w_s(z) = E_s{}^{\sigma}J_{\sigma}(i\sqrt{if}\,s). \quad (73)$$

The eigenvalues λ_n are determined by Eq. (59) which is now simply

$$J_{\sigma-1}(x_n) = 0 \quad (74)$$

and from Eq. (52) the eigenfunctions are

$$f_n(z) \propto s^{\sigma}J_{\sigma}(\sqrt{\lambda_n}s). \quad (75)$$

From Eqs. (60), (69), and (74) we see that for zero bottom current or zero bottom friction the dimensionless eigenvalues $\lambda_n H^2/\nu_0$ depend only on the parameter μ and not on ν_0. From equations (59) and (67) we see that for $0 < b < \infty$ they depend also on the dimensionless parameter bH/ν_0.

We computed examples, for zero bottom current, for $\mu = 3/4$ which is the power chosen by Fjeldstad as a best fit to current data (Fjeldstad, 1929). The steady-state solution is shown in Fig. 3, an eigen-

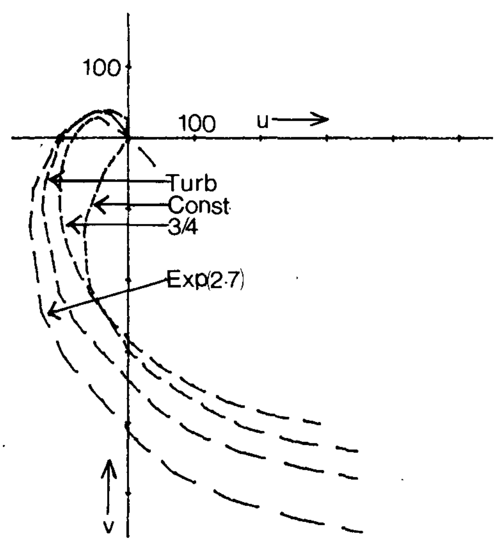

FIG. 3. Steady-state solutions for decreasing exponential (with $aH = 2.7$), 3/4 power, and constant viscosity, with zero bottom current, and decreasing linear viscosity with a turbulent boundary layer at the bottom, all for $\nu_0 = 0.026$ m^2 s^{-1}. Currents (u,v) are shown for depths z varying in 20 equal steps from the top (largest current) to the bottom (zero current). These current velocities are in m s^{-1} and for $F(t) = 1$ which is 10^3–10^4 times larger than typical wind forces.

function in Fig. 5, eigenvalues in Table 1, values of $B_n f_n(z = 0)$ and $D_n f_n(z = 0)$ in Tables 2 and 3, and the convergence of $\sum_n B_n f_n(z)$ to -1 and $\sum_n D_n f_n(z)$ to $w_s(z)$ in Figs. 7–8, compared with solutions for other viscosity functions.

For $\mu = 1$ we get $\sigma = 0$. Then the upper and lower signs yield the same equations because

$$J_{-1}(x) = -J_1(x),$$

$$N_{-1}(x) = -N_1(x).$$

For $1 \leqslant \mu < 2$ we use the lower signs in all our equations. Using the forms of Bessel and Neumann functions for small values of the argument, we find that as $\alpha \to H$ the boundary conditions (57) and (61) imply that b is zero and that

$$G = 0, \tag{76}$$

$$R_n = 0. \tag{77}$$

Then E is determined by (55), and from (51) we have

$$w_s(z) = Es^{-\sigma}J_\sigma(i\sqrt{if}\,s). \tag{78}$$

The eigenvalues λ_n are determined by (59) which becomes

$$J_{\sigma+1}(x_n) = 0, \tag{79}$$

and from (52) the eigenfunctions are

$$f_n(z) \propto s^{-\sigma}J_\sigma(\sqrt{\lambda_n}s). \tag{80}$$

From Eqs. (60) and (79) we see that the dimensionless eigenvalues $\lambda_n H^2/\nu_0$ depend only on the parameter μ and not on ν_0 or b. For $n = 0$ we have the zero eigenvalue and constant eigenfunction characteristic of zero bottom friction. Indeed, for these viscosity functions, the boundary conditions we have written give us no other choice of bottom friction. For $1 \leqslant \mu < 2$ the limit as $\alpha \to H$ is consistent with the boundary conditions (57) and (61) only for $b = 0$, for zero bottom friction. In particular, it is inconsistent with the boundary conditions (63) and (64) for zero bottom current.

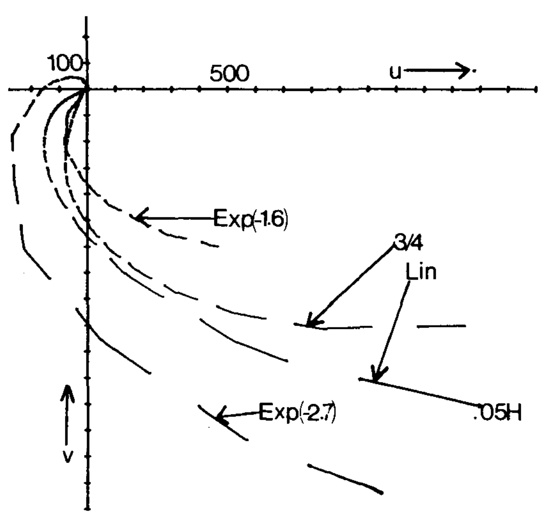

FIG. 4. Steady-state solutions for increasing exponential (with $aH = -1.6$ and -2.7), 3/4 power, and linear viscosity, with ν_0 or κ chosen so that $\nu(z = -H)$ is 0.052 m² s⁻¹ (as shown in Fig. 1), all for zero bottom current. For linear viscosity the current is infinite at the surface so the value one step down is the first shown. Otherwise the currents are shown the same as in Fig. 3.

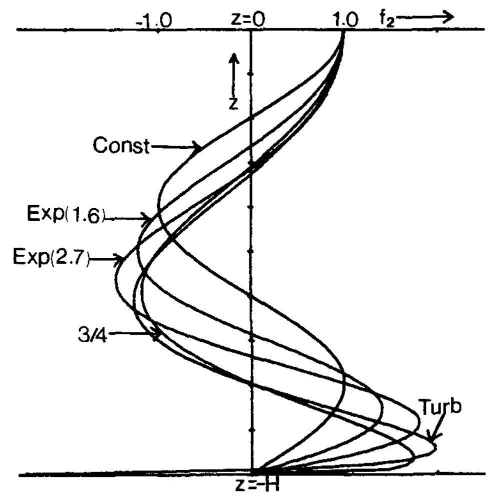

FIG. 5. Eigenfunctions $f_2(z)$ for decreasing exponential (with $aH = 1.6$ and 2.7), 3/4 power, and constant viscosity, with zero bottom current, and decreasing linear viscosity with a turbulent boundary layer at the bottom, all for $\nu_0 = 0.052$ m² s⁻¹. In the linear case the eigenfunction becomes infinite at the bottom.

From Eqs. (51) and (52) we see that the other linearly independent solution, excluded by the boundary conditions we have written, would give

$$w_s(z) \propto s^{-\sigma}N_\sigma(i\sqrt{if}\,s), \tag{81}$$

$$f_n(z) \propto s^{-\sigma}N_\sigma(\sqrt{\lambda_n}s). \tag{82}$$

These are singular as s goes to zero, at $z = -H$. Our previous solutions are not singular. For $0 < \mu < 1$ the solutions (68) and (70) for zero bottom friction have a finite value for $s \to 0$, and, of course, the solutions (73) and (75) for zero bottom current go to zero as $s \to 0$. We get linear combinations of these forms for $0 < b < \infty$. For $1 \leqslant \mu < 2$ the solutions (78) and (80) have a finite value for $s \to 0$. They are the only non-singular solutions available, and they are for zero bottom friction at $z = -H$. Evidently there are no solutions for nonzero bottom friction at $z = -H$. Certainly there are no solutions that go to zero as $s \to 0$, no solutions for zero bottom current at $z = -H$.

For $\mu = 1$, i.e. $\sigma = 0$, linear combinations including the solutions (81) and (82) have a logarithmic singularity at $s = 0$ that can be identified with a turbulent boundary layer at the bottom. For $1 < \mu < 2$ they are more singular. Thomas applied a boundary condition characteristic of the turbulent boundary layer to find a steady-state solution for $\mu = 1$ and $\alpha = H$ (Thomas, 1975). This can be regarded as the

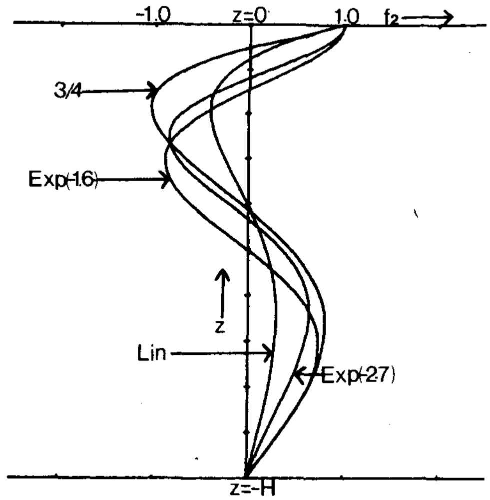

FIG. 6. Eigenfunctions $f_2(z)$ for increasing exponential (with $aH = -1.6$ and -2.7), 3/4 power, and linear viscosity, with ν_0 or κ chosen so that $\nu(z = -H)$ is 0.052 $m^2 s^{-1}$ (as shown in Fig. 1), all for zero bottom current.

condition

$$w(z = -H + \epsilon H) = 0, \quad (83)$$

that the current is zero at a small distance ϵH above the bottom surface $z = -H$. We adopt this, in place of our original boundary condition (4) for zero bottom current, with the understanding that now our solution is for the interval $-H + \epsilon H \leq z \leq 0$. This leads to logarithmic behavior

$$w \propto \ln[(H + z)/\epsilon H] \quad (84)$$

for the current near the bottom. Applied to the steady-state solution (51), this boundary condition gives

$$i\nu_0 e^{-\pi E/G} = \epsilon H^2 f e^{2\gamma}, \quad (85)$$

where $\gamma \approx 0.577$ is an Euler constant. This and Eq. (55) determine E and G. Applied to the eigenfunctions (52), it gives

$$\nu_0 e^{-\pi/R_n} = \epsilon H^2 e^{2\gamma} \lambda_n. \quad (86)$$

This and Eq. (59) determine R_n and λ_n. If one wants to explore other solutions, the same boundary condition could be used for $1 < \mu < 2$, and the more general boundary condition for $0 < b < \infty$ also could be applied at $z = -H + \epsilon H$.

We computed examples for $\mu = 1$, for the zero-bottom-friction and turbulent-boundary-layer cases we have discussed. The steady-state solution is shown in Fig. 3, an eigenfunction in Fig. 5, eigenvalues in Table 1, values of $B_n f_n(z = 0)$ and $D_n f_n(z = 0)$ in Tables 2 and 3, and the convergence of $\sum_n B_n f_n(z)$ to -1 and $\sum_n D_n f_n(z)$ to $w_s(z)$ in Figs. 7–8, compared with solutions for other viscosity functions.

9. Viscosity going to zero at the top

Finally we consider negative α in the limit as α goes to zero so the viscosity increases with depth and goes to zero at the top. Specifically we let

$$\alpha \to 0,$$
$$\nu_0 \to 0,$$
$$\nu_0/|\alpha|^{\mu} \to \kappa,$$

where κ is a positive constant [what Madsen calls κu_* (Madsen, 1977)]. Then from Eq. (49) we get

$$\nu(z) = \kappa(-z)^{\mu}. \quad (87)$$

From Eq. (50) we get

$$s = |1 - \mu/2|^{-1}\kappa^{-1/2}(-z)^{1-\mu/2}. \quad (88)$$

Again we consider $\mu < 2$ in particular. Then s goes to zero at the top of the water, so ξ and x_n go to zero, which simplifies the boundary conditions (55) and (59) for the surface shear stress. (Again, in writing these equations we were careful not to cancel any of these factors that go to zero.) For one factor in all the boundary conditions we obtain the simple finite limit

$$\nu_0^{1\mp\sigma}|\alpha|^{-1\pm 2\sigma} \to \kappa^{1/(2-\mu)}. \quad (89)$$

For $0 < \mu < 1$ again we use the upper signs in all our equations. Using the limit factor (89), and the forms of Bessel and Neumann functions for small values of the argument, we find that the boundary

TABLE 1. Dimensionless eigenvalues $\lambda_n H^2/\nu_0$, or $\lambda_n H^{2-\mu}/\kappa$ for viscosity going to zero at the top.

	n					
	0	1	2	3	4	5
Constant ν; $b = \infty$	2.467	22.206	61.685	120.90	199.85	305.95
3/4 power, $\alpha \to H$; $b = \infty$	0.538	7.106	21.372	43.344	73.036	105.22
Linear ν, $\alpha \to H$; turbulent bottom, $\epsilon = 0.01$	0.301	6.092	17.680	34.675	56.871	84.132
Exponential ν, $aH = 1.6$; $b = \infty$	0.769	9.382	26.596	52.353	87.617	131.001
Exponential ν, $aH = 2.7$; $b = \infty$	0.313	4.658	13.410	26.528	44.034	65.954
3/4 power, $\alpha \to 0$; $b = \infty$	0.538	7.106	21.372	43.344	73.036	105.220
Linear ν, $\alpha \to 0$; $b = \infty$	1.443	7.608	18.715	34.737	55.731	81.640
Exponential ν, $aH = -1.6$; $b = \infty$	7.036	49.023	132.39	257.38	424.03	632.34
Exponential ν, $aH = -2.7$; $b = \infty$	13.166	80.610	212.211	409.071	671.393	999.240

TABLE 2. Values of $B_n f_n(z = 0)$.

	$B_0f_0(0)$	$B_1f_1(0)$	$B_2f_2(0)$	$B_3f_3(0)$	$B_4f_4(0)$
Constant ν; $b = \infty$	−1.27	0.42	−0.25	0.18	−0.14
3/4 power, $\alpha \to H$, $\nu_0 = 0.052$; $b = \infty$	−1.12	0.21	−0.11	0.10	−0.05
3/4 power, $\alpha \to H$, $\nu_0 = 0.026$; $b = \infty$	−1.12	0.19	−0.10	0.08	−0.04
Linear ν, $\alpha \to H$, $\nu_0 = 0.052$; turbulent bottom, $\epsilon = 0.01$	−1.12	0.14	−0.07	0.04	−0.02
Linear ν, $\alpha \to H$, $\nu_0 = 0.026$; turbulent bottom, $\epsilon = 0.01$	−1.10	0.15	−0.08	0.03	−0.02
Exponential ν, $\nu_0 = 0.052$, $aH = 1.6$; $b = \infty$	−1.21	0.30	−0.17	0.12	−0.10
Exponential ν, $\nu_0 = 0.026$, $aH = 2.7$; $b = \infty$	−1.16	0.23	−0.13	0.09	−0.07
3/4 power, $\alpha \to 0$, $\kappa = 0.00104$; $b = \infty$	−1.13	0.31	−0.18	0.12	−0.05
Linear ν, $\alpha \to 0$, $\kappa = 0.00104$; $b = \infty$	−1.60	1.06	−0.85	0.73	−0.64
Exponential ν, $\nu_0 = 0.00105$, $aH = -1.6$; $b = \infty$	−1.38	0.60	−0.37	0.26	−0.21
Exponential ν, $\nu_0 = 0.0017$, $aH = -2.7$; $b = \infty$	−1.20	0.52	−0.37	0.21	−0.13

condition (55) becomes

$$-(-if)^{\sigma/2}(1 - \mu/2)^{1-2\sigma}\kappa^{1/(2-\mu)} \times (E + G\cot[\sigma\pi])2^{\sigma-1}/\Gamma(\sigma) = 1, \quad (90)$$

and the boundary condition (59) becomes

$$1 + R_n \cot(\sigma\pi) = 0 \quad (91)$$

so the eigenfunctions $f_n(z)$ from (52) are simply of the form (70). We obtain the constants E and G from (90) and (57 or 63) where now

$$\zeta = i(if)^{1/2}(1 - \mu/2)^{-1}\kappa^{-1/2}H^{1-\mu/2} \quad (92)$$

from (58), and we use the limit factor (89) in (57). Thus, from Eq. (51), we obtain

$$w_s(z) = s^\sigma[EJ_\sigma(i\sqrt{if}\, s) + GN_\sigma(i\sqrt{if}\, s)]. \quad (93)$$

The eigenvalues λ_n are determined by (61) which, using the form (70) for the eigenfunctions and the limit factor (89), we can write as

$$-\lambda_n^\sigma(1 - \mu/2)^{1-2\sigma}\kappa^{1/(2-\mu)}y_n^{1-\sigma}[J_{\sigma-1}(y_n) - \tan[\sigma\pi]N_{\sigma-1}(y_n)] = by_n^\sigma[J_\sigma(y_n) - \tan[\sigma\pi]N_\sigma(y_n)], \quad (94)$$

TABLE 3. Values of $D_n f_n(z = 0)$.

	$D_0f_0(0)$	$D_1f_1(0)$	$D_2f_2(0)$	$D_3f_3(0)$	$D_4f_4(0)$
Constant ν, $\nu_0 = 0.052$; $b = \infty$	156.1 −i311.1	81.2 −i17.9	30.4 −i2.4	0.7 −i0.1	0.1
Constant ν, $\nu_0 = 0.026$; $b = \infty$	91.1 −i348.3	145.2 −i64.8	61.1 −i10.3	32.3 −i3.1	19.6 −i1.1
3/4 power, $\alpha \to H$, $\nu_0 = 0.052$; $b = \infty$	16.4 −i201.3	114.4 −i69.7	53.8 −i12.6	27.5 −i3.2	16.4 −i1.1
3/4 power, $\alpha \to H$, $\nu_0 = 0.026$; $b = \infty$	8.2 −i185.3	112.3 −i163.3	92.0 −i43.1	52.1 −i12.7	32.3 −i6.0
Linear ν, $\alpha \to H$, $\nu_0 = 0.052$; turbulent bottom, $\epsilon = 0.01$	15.25 −i248.0	105.9 −i85.6	54.5 −i15.2	28.7 −i4.1	17.5 −i1.5
Linear ν, $\alpha \to H$, $\nu_0 = 0.026$; turbulent bottom, $\epsilon = 0.01$	37.6 −i210.3	96.7 −i156.4	89.3 −i49.1	54.0 −i15.1	34.2 −i5.9
Exponential ν, $\nu_0 = 0.052$, $aH = 1.6$; $b = \infty$	49.0 −i314.4	108.2 −i56.2	61.2 −i8.5	23.8 −i2.2	14.4 −i0.8
Exponential ν, $\nu_0 = 0.026$, $aH = 2.7$; $b = \infty$	9.1 −i280.3	75.6 −i162.3	90.2 −i67.3	61.2 −i23.1	39.7 −i8.9
3/4 power, $\alpha \to 0$, $\kappa = 0.00104$; $b = \infty$	368.3 −i387.0	423.6 −i231.1	268.9 −i142.0	115.6 −i71.1	76.4 −i35.3
Linear ν, $\alpha \to 0$, $\kappa = 0.00104$; $b = \infty$	193.5 −i660.8	764.7 −i495.9	649.9 −i171.1	500.0 −i71.1	403.0 −i35.1
Exponential ν, $\nu_0 = 0.0105$, $aH = -1.6$; $b = \infty$	115.4 −i456.0	221.5 −i111.9	100.26 −i18.7	53.07 −i5.11	22.45 −i1.90
Exponential ν, $\nu_0 = 0.0017$, $aH = -2.7$; $b = \infty$	56.1 −i637.1	288.3 −i532.1	326.7 −i229.0	223.8 −i81.4	147.5 −i32.7

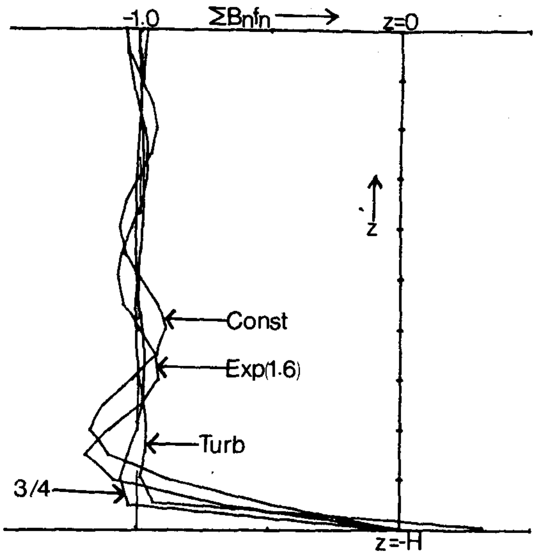

FIG. 7. The first five terms of $\sum_n B_n f_n(z)$ for decreasing exponential (with $aH = 1.6$), 3/4 power, and constant viscosity, with zero bottom current, and decreasing linear viscosity with a turbulent boundary layer at the bottom, all for $\nu_0 = 0.052\ \mathrm{m^2\,s^{-1}}$.

or for zero bottom current, by Eq. (64) which now is

$$J_\sigma(y_n) - \tan[\sigma\pi]N_\sigma(y_n) = 0, \qquad (95)$$

where now, in either case,

$$y_n = \lambda_n^{1/2}(1 - \mu/2)^{-1}\kappa^{-1/2}H^{1-\mu/2}, \qquad (96)$$

from (62). From these equations we can see that the dimensionless eigenvalues $\lambda_n H^{2-\mu}/\kappa$ depend on only the dimensionless parameters μ and $bH^{1-\mu}/\kappa$.

We have computed examples for $\mu = 3/4$ with zero bottom current. The steady-state solution is shown in Fig. 4. Eigenvalues are shown in Table 1 and an eigenfunction in Fig. 6. Values of $B_n f_n(z = 0)$ and $D_n f_n(z = 0)$ are shown in Tables 2 and 3, and the convergence of $\sum_n B_n f_n(z)$ to -1 and $\sum_n D_n f_n(z)$ to $w_s(z)$ in Figs. 9–10.

For $1 \leq \mu < 2$ we use the lower signs in all our equations. Using the limit factor (89) and the forms of Bessel and Neumann functions for small values of the argument, we find that now the boundary condition (55) becomes

$$(-if)^{-\sigma/2}(1 - \mu/2)^{1+2\sigma}\kappa^{1/(2-\mu)} \times G(-2^{\sigma+1}\Gamma(\sigma + 1)/\pi) = 1 \qquad (97)$$

in the limit as $\alpha \to 0$, and the boundary condition (59) becomes

$$R_n = 0 \qquad (98)$$

so the eigenfunctions $f_n(z)$ from (52) are simply of the form (80). We obtain the constants G and E from Eqs. (97) and (57) or (63) using the limit factor (89) and the formula (92) for ζ. Thus, from Eq. (51), we obtain

$$w_s(z) = s^{-\sigma}[EJ_\sigma(i\sqrt{if}\,s) + GN_\sigma(i\sqrt{if}\,s)]. \qquad (99)$$

The eigenvalues λ_n are determined by Eqs. (61) or (64) where again we use the limit factor (89) and the formula (96) for y_n. Again, the dimensionless eigenvalues $\lambda_n H^{2-\mu}/\kappa$ depend on only the dimensionless parameters μ and $bH^{1-\mu}/\kappa$.

We have computed examples for $\mu = 1$ with zero bottom current. The steady-state solution is shown in Fig. 4. Eigenvalues are shown in Table 1 and an eigenfunction in Fig. 6. Values of $B_n f_n(z = 0)$ and

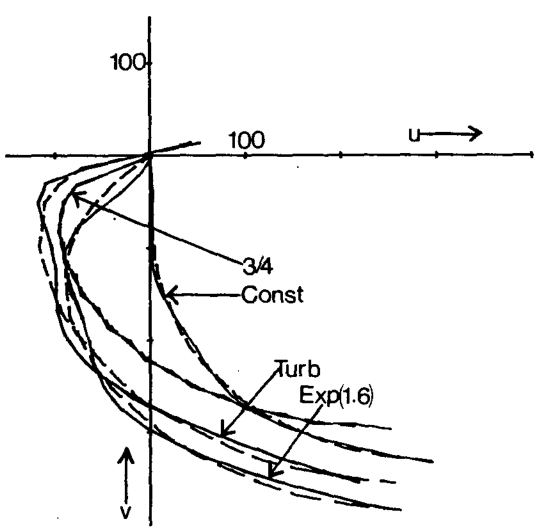

FIG. 8. The first five terms of $\sum_n D_n f_n(z)$ (solid lines) compared with the steady-state solutions (dashed lines) for the same cases as Fig. 7. Currents are shown the same as in Fig. 3.

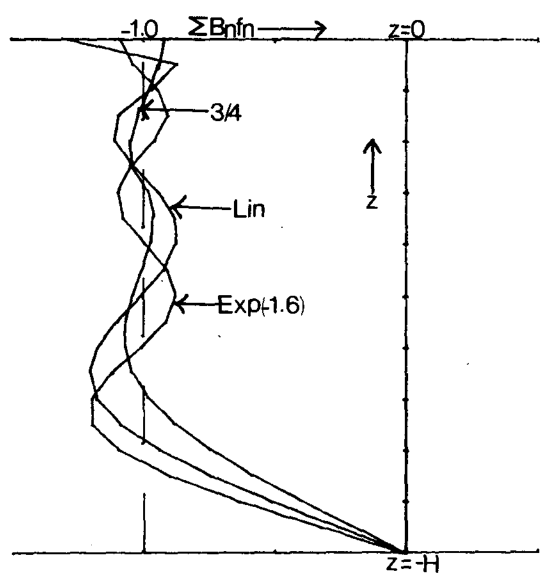

FIG. 9. The first five terms of $\sum_n B_n f_n(z)$ for increasing exponential (with $aH = -1.6$), 3/4 power, and linear viscosity, with ν_0 or κ chosen so that $\nu(z = -H)$ is $0.052\ \mathrm{m^2\,s^{-1}}$ (as shown in Fig. 1), all for zero bottom current.

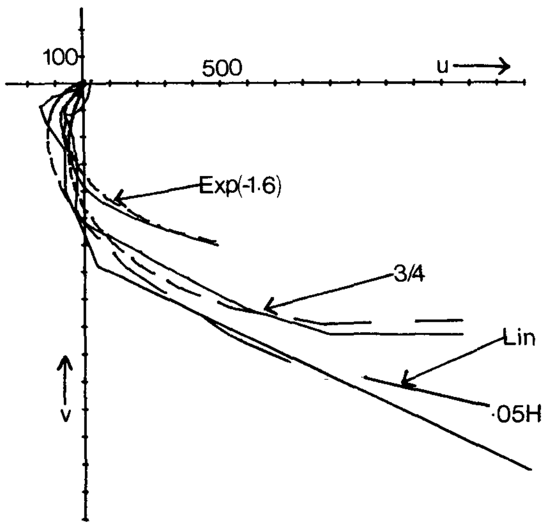

FIG. 10. The first five terms of $\sum_n D_n f_n(z)$ (solid lines) compared with the steady-state solutions (dashed lines) for the same cases as Fig. 9. For linear viscosity the current is infinite at the surface so the value one step down is the first shown. Otherwise the currents are shown the same as in Fig. 3.

$D_n f_n(z = 0)$ are shown in Tables 2 and 3, and the convergence of $\sum_n B_n f_n(z)$ to -1 and $\sum_n D_n f_n(z)$ to $w_s(z)$ in Figs. 9–10.

For $0 < \mu < 1$, the steady-state solution (93) and eigenfunctions of the form (70) are not singular as s goes to zero at the top of the water. For $1 \leqslant \mu < 2$, the eigenfunctions of the form (80) are not singular but the steady-state solution (99) is singular as s goes to zero. For $\mu = 1$, i.e. $\sigma = 0$, this is a logarithmic singularity that can be identified with a turbulent boundary layer at the surface. Therefore, Madsen (1977) has suggested that the viscosity should increase linearly with depth at least near the surface. For $1 < \mu < 2$ the steady-state solution is more singular.

Madsen (1977) found the solution for $\mu = 1$ for infinitely deep water and zero bottom current. We have the solution for water of finite depth. Our solution does not reproduce Madsen's elegant formula for time-dependent currents because our eigenfunction expansion does not apply to infinitely deep water. We do get Madsen's steady-state solution in the limit of infinitely deep water. From (92) we see that $\text{Im}\zeta > 0$ and $|\zeta| \to \infty$ as $H \to \infty$. Then from (63), using the forms of Bessel and Neumann functions for large values of the argument, we get

$$E + iG = 0$$

in the limit as $H \to \infty$. From Eq. (97) we have

$$G = -\pi/\kappa.$$

Thus, we obtain

$$w_s(z) = i(\pi/\kappa)[J_0(i\sqrt{if}\,s) + iN_0(i\sqrt{if}\,s)] \quad (100)$$

in the limit of infinitely deep water. This agrees with Madsen's solution. [What we call κ Madsen calls κu_*. Our solution is for $F(t) = 1$; Madsen's is for $F(t) = i(u^*)^2$.]

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