models, seek a closure for the eddy flux divergence of potential vorticity. Schemes based on the local downgradient transfer of potential vorticity (White and Green, 1981; Marshall, 1981) may not be so grossly in error (in strong, curved flow regimes) as previously suggested (Harrison, 1978; Holland and Rhines, 1980) if the transfer coefficients are reinterpreted as relating the divergent part of the flux to the mean gradient. It is intended to carry out calculations using the eddy statistics from an eddy resolving model, to test whether a diffusive parameterization for the irrotational, divergent potential vorticity flux is appropriate.

REFERENCES

Harrison, D. E., 1978: On the diffusion parameterization of mesoscale eddy effects from a numerical ocean experiment. *J. Phys. Oceanogr.*, **8**, 913–918.

Holland, W. R., 1978: The role of mesoscale eddies in the general circulation—numerical experiments using a wind-driven quasi-geostrophic model. *J. Phys. Oceanogr.*, **8**, 363–392.

——, and P. B. Rhines, 1980: An example of eddy-induced ocean circulation. *J. Phys. Oceanogr.*, **10**, 1010–1031.

Lau, N., 1978: On the three-dimensional structure of the observed transient eddy statistics in the Northern Hemisphere wintertime circulation. *J. Atmos. Sci.*, **35**, 1900–1923.

——, and J. M. Wallace, 1979: On the distribution of horizontal transports by transient eddies in the Northern Hemisphere wintertime circulation. *J. Atmos. Sci.*, **36**, 1844–1861.

Marshall, J. C., 1981: On the parameterization of geostrophic eddies in the ocean. *J. Phys. Oceanogr.*, **11**, 257–271.

Rhines, P. B., 1979: Geostrophic turbulence. *Annual Review of Fluid Mechanics*, Vol. 11, Annual Reviews, 401–441.

——, and W. R. Holland, 1979: A theoretical discussion of eddy-driven mean flows. *Dyn. Atmos. Oceans.*, **3**, 289–325.

White, A. A., and J. S. A. Green, 1982: A nonlinear atmospheric long wave model incorporating parameterization of transient eddies. *Quart. J. Roy. Meteor. Soc.*, **108**, 55–86.

A Weak Formulation of the Shallow-Water Equations for a Rotating Basin

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9 February 1981 and 15 August 1981

ABSTRACT

A weak formulation of the equation for the elevation field arising from the shallow-water equations for a rotating inviscid fluid has been developed.

The difficulties of this problem, which are due chiefly to the fact that the normal velocity along the contour is given by a linear combination of normal and tangential derivatives of the elevation field, are overcome. As a result, many problems, until now treated using the elevation and velocity component variables, might be solved dealing only with the elevation.

1. Introduction

It is well known that the linear shallow-water equations for a rotating basin, in the case of inviscid fluid and time-harmonic motion, can be reduced to a single equation in the elevation field. However, rarely has this equation been approached numerically, because of its boundary conditions which involve a linear combination of normal and tangential derivatives.

For example, if the depth is assumed to be constant, such an equation reduces to a normal Helmholtz equation for which a variational formulation exists. However, this formulation relates the elevation field to its normal derivative, that is no more proportional to the normal velocity field, as in the case in which the earth's rotation can be neglected. Hence, such a formulation is scarcely meaningful from a physical point of view and turns out to be of little help.

Now we will present a weak formulation of the problem which has the advantage of relating the elevation field to the normal velocity along the contour, and hence can be directly used in practical problems.

2. Basic equations

Let ζ be the elevation field, **u** the velocity, g the gravity acceleration, h the depth of the considered basin, f the local Coriolis parameter ($f = 2\Omega \sin\varphi$, where Ω is the angular speed of rotation of the earth and φ is the latitude) and ω, either real or complex, the angular velocity of the motion, assumed to depend on the time τ by the factor $e^{-\iota\omega\tau}$, where ι is the imaginary unit.

Then the linearized shallow-water equations read

$$-\iota\omega\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} + g\nabla\zeta = 0, \tag{1}$$

$$-\iota\omega\zeta + \nabla\cdot(h\mathbf{u}) = 0, \tag{2}$$

0022-3670/81/121680-03$04.75

where $\hat{\mathbf{k}}$ is the unit vector orthogonal to the plane defined by $\zeta = 0$ and upwardly directed, so that $\hat{\mathbf{k}} \times \mathbf{u}$ is the vector $\mathbf{u}$ counterclockwise rotated by 90°.

If we apply to (1) the operator $f \cdot \hat{\mathbf{k}} \times$, and add to the resulting equation the same equation (1) multiplied by $\iota\omega$, we get

$$(\omega^2 - f^2)\mathbf{u} + g(\iota\omega\nabla\zeta + f\hat{\mathbf{k}} \times \nabla\zeta) = 0. \qquad (3)$$

Provided $f \neq \omega$, Eq. (3) allows the evaluation of $\mathbf{u}$ as a function of ζ. If we set

$$\tilde{g} = \frac{g}{\omega^2 - f^2}, \qquad (4)$$

we can rewrite (3) as

$$\mathbf{u} + \tilde{g}(\iota\omega\nabla\zeta + f\hat{\mathbf{k}} \times \nabla\zeta) = 0, \qquad (5)$$

where $\tilde{g}$ is a function of the position.

If we assume that in all the points of the domain (remember also that f is a function of the position) $f \neq \omega$, we can substitute the value of $\mathbf{u}$ given by (4) in (2), to get

$$\iota\omega\zeta + \nabla\cdot[h\tilde{g}(\iota\omega\nabla\zeta + f\hat{\mathbf{k}} \times \nabla\zeta)] = 0. \qquad (6)$$

After a convenient rearrangement of the terms we have

$$\iota\omega\nabla\cdot(h\tilde{g}\nabla\zeta) + \iota\omega\zeta + \nabla\cdot[h\tilde{g}f\hat{\mathbf{k}} \times \nabla\zeta)] = 0. \qquad (7)$$

By multiplying (4), evaluated along the contour, scalarly by the unit vector $\hat{n}$, normal to it and outwardly directed with respect to the domain considered, and setting

$$u = \hat{\mathbf{n}}\cdot\mathbf{u}, \qquad (8)$$

$$\partial_n = \hat{\mathbf{n}}\cdot\nabla, \qquad (9)$$

$$\partial_t = -\hat{\mathbf{n}}\cdot\hat{\mathbf{k}} \times \nabla = \hat{\mathbf{t}}\cdot\nabla, \qquad (10)$$

where

$$\hat{\mathbf{t}} = \hat{\mathbf{k}} \times \hat{\mathbf{n}} \qquad (11)$$

is the unit vector counterclockwise tangent to the contour, the following boundary conditions are derived:

$$\tilde{g}(-\iota\omega\partial_n\zeta + f\partial_t\zeta) = u. \qquad (12)$$

Hence, once u is fixed along the contour of a limited domain, it is possible to solve the original problem by means of (7) and (12) in the only unknown field ζ; then, the velocity field $\mathbf{u}$ can be obtained from (5).

It is to be stressed that when $\omega = 0$, the higher-order derivatives of ζ in (7) disappear, and when $f = \omega$, $\tilde{g}$ diverges, so that in these circumstances the considerations of the following section are no longer valid.

3. The weak formulation

To avoid working with an unusual symbolism, we will assume hereafter that f is constant, so that (7) and (12), written for a domain D of contour C, become

$$\nabla\cdot(h\nabla\zeta) + \frac{\omega^2 - f^2}{g}\zeta + \frac{f}{\iota\omega}\nabla\cdot(h\hat{\mathbf{k}} \times \nabla\zeta) = 0 \quad \text{in D}, \qquad (13)$$

$$-\iota\omega\partial_n\zeta + f\partial_t\zeta = \frac{\omega^2 - f^2}{g}u \quad \text{along C}, \qquad (14)$$

although such a condition is not at all necessary for the derivation of the weak formulation.[1]

Let us define

$$\nabla_m = \nabla + \frac{f}{\iota\omega}\hat{\mathbf{k}} \times \nabla, \qquad (15)$$

$$\partial_m = \hat{\mathbf{n}}\cdot\nabla_m = \partial_n - \frac{f}{\iota\omega}\partial_t, \qquad (16)$$

$$v = -\frac{\omega^2 - f^2}{\iota\omega g}u. \qquad (17)$$

With this notation Eq. (13) becomes

$$\nabla\cdot(h\nabla_m\zeta) + \frac{\omega^2 - f^2}{g}\zeta = 0 \quad \text{in D} \qquad (18)$$

and the boundary condition (14) is

$$\partial_m\zeta = v \quad \text{along C}. \qquad (19)$$

It is easy to show that the weak formulation for problem (18)–(19) is

$$\int_D h\nabla\psi\cdot\nabla_m\zeta - \frac{\omega^2 - f^2}{g}\int_D \psi\zeta = \int_C h\psi v, \qquad (20)$$

where ψ represents any arbitrary limited function defined in D + C.

In fact, the first term of (20) can be worked out as follows:

$$\int_D h\nabla\psi\cdot\nabla_m\zeta = \int_D \nabla\cdot(\psi h\nabla_m\zeta) - \int_D \psi\nabla\cdot(h\nabla_m\zeta)$$

$$= \int_C \psi h\partial_m\zeta - \int_D \psi\nabla\cdot(h\nabla_m\zeta). \qquad (21)$$

Substitution of this expression in (20) yields

$$-\int_D \psi\left[\nabla\cdot(h\nabla_m\zeta) + \frac{\omega^2 - f^2}{g}\zeta\right] = \int_C \psi h(v - \partial_m\zeta). \qquad (22)$$

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[1] Sometimes the last term of Eq. (13) is written

$$\frac{f}{\iota\omega}\nabla h\cdot\hat{\mathbf{k}} \times \nabla\zeta$$

but this formulation does not facilitate any further elaboration of the equation.

By choosing the functions ψ vanishing along the contour, so that the right member of (22) becomes null, and then by applying the usual variational argument, we get Eq. (18). Bearing in mind that now the left member of (22) has already been shown to be zero, then a choice of functions ψ different from zero along the contour also yields Eq. (19).

As can be seen, the proof is rather simple and the formulation (20) is strictly related to the usual weak formulation of the Helmholtz equation; the only relevant change is the substitution of $\nabla\zeta$ with $\nabla_m\zeta$.

On the contrary, this formulation is completely different from that adopted by Le Provost and Poncet (1978), where the real and imaginary parts of Eq. (13) are mixed in a single variational principle.

Notice that by setting $\psi = \delta\zeta$ a variational formulation of the problem does not follow from (20) because the term

$$\int_D h\nabla\delta\zeta\cdot\hat{\mathbf{k}}\times\nabla\zeta \qquad (23)$$

cannot be written as the variation of a given integral quantity.

4. Conclusion

By the present formulation we could solve the problems of the diffraction of a Kelvin wave by a bay or of the propagation of a tide in a gulf [e.g., the problems studied by Miles (1973), Garrett (1975) and Garrett and Greenberg (1977)], taking into account the geometrical configuration of the real basins with great accuracy.

Analogously, the calculation of the normal modes of a lake could be made advantageously by means of this formulation, which is much simpler than that used, for example, by Rao and Schwab (1976), and Rao *et al.* (1976), because it does not require the discretization of the velocity field, thereby reducing by a factor of about 3 the total number of unknowns.

In every case finite-difference schemes could be substituted by finite-element schemes, with all the advantages which are usually attributed to this latter approach. No particular difficulties should arise from the choice of the weighting (ψ) and shape (ζ) functions (Zienkiewicz, 1977). Instead, as previously suggested, numerical difficulties might arise for small values of the angular frequency or when the angular frequency approaches the value of the local Coriolis parameter. An answer to these problems will be available only after having performed particular tests in critical situations.

Acknowledgment. This work has been supported by Italian National Council for Research (C.N.R.).

REFERENCES

Garrett, C. J. R., 1975: Tides in gulfs. *Deep-Sea Res.*, **18**, 493–503.

——, and D. Greenberg, 1977: Predicting changes in tidal regime: The open boundary problem. *J. Phys. Oceanogr.*, **7**, 171–181.

Le Provost, C., and A. Poncet, 1978: Finite element method for spectral modelling of tides. *Int. J. Numer. Methods Eng.*, **12**, 853–871.

Miles, J. W., 1973: Tidal wave diffraction by channels and bays. *Geophys. Fluid Dyn.*, **5**, 155–171.

Rao, D. B., and D. J. Schwab, 1976: Two-dimensional normal modes in arbitrary enclosed basins on a rotating earth; application to Lakes Ontario and Superior. *Phil. Trans. Roy. Soc. London*, **A281**, 63–96.

——, C. H. Mortimer and D. J. Schwab, 1976: Surface normal modes of Lake Michigan: Calculations compared with spectra of observed water level fluctuations. *J. Phys. Oceanogr.*, **6**, 575–588.

Zienkiewicz, O. C., 1977: *The Finite-Element Method*, 3rd ed. McGraw-Hill, 521 pp.