Wave–Turbulence Interactions in the Upper Ocean. Part I: The Energy Balance of the Interacting Fields of Surface Wind Waves and Wind-Induced Three-Dimensional Turbulence

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ABSTRACT

We analyze in detail the budget of total and fluctuating energy in the surface layer of the ocean. We suggest a rational scheme for separating the budget of turbulence from that of random wind-generated surface waves, and suggest in particular a form for the interaction term appearing in the turbulent energy equation. This is derived from an analysis of the potential velocity field excited by surface motion, and its interaction with the turbulence. Finally, we explore some implications of the proposed interaction both for the surface wave energetics, as well as for the turbulence structure.

1. Introduction

When wind blows over the sea surface it generates surface waves, a mean shear current and turbulent motions associated with the processes of wave breaking and shear instability. This wind-induced three-dimensional turbulence can be important in limiting the growth of surface wind waves—a fact which is still not widely recognized. Because the upper-ocean turbulence depends on the statistical characteristics of the surface wind waves (at the very least due to wave breaking) the growth of wind-generated waves and the turbulent boundary layer in the water cannot in principle be considered independently of each other. Unfortunately the present spectral wave prediction models, as well as models of the deepening of the turbulent layer, do not reflect this feature. Consequently, it is important to find the proper foundation for the parametrization of the energy balance between surface wind waves and turbulent vertical mixing in the upper ocean. The aim of the present note is two-fold: first, to derive a system of equations for the description of the turbulent boundary layer in the water during wind-wave growth, and second, to point out ideas on which we can base the parameterization of the dissipation of surface wave energy due to wave turbulence interactions.

2. Conservation of total and fluctuating energy in the upper ocean

Following Phillips (1977) we derive the equations for the balance of mean horizontal momentum and mean total energy by integrating the horizontal momentum equation ($\alpha, \beta = 1, 2$),

$$\partial_t(\rho u_\alpha) + \partial_\beta(\rho u_\alpha u_\beta + p\delta_{\alpha\beta} - \mu\partial_\beta u_\alpha) + \partial_z(\rho u_\alpha w - \mu\partial_z u_\alpha) = 0, \quad (1)$$

and the mechanical energy equation,

$$\partial_t\left(\frac{1}{2}\rho u_i^2 + \rho g z\right) + \partial_j\left[u_j\left(p + \frac{\rho u_i^2}{2} + \rho g z\right)\right] = u_i f_i, \quad (2)$$

from the free surface, $z = \zeta(x, y, t)$, to the bottom (or an undisturbed streamline), $z = -d$.

In (2) f_i is the frictional force per unit volume, $f_i = 2\mu\partial_j e_{ij} = \mu\partial_j^2\mu_i$, where μ is the viscosity of the fluid, and $e_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$. Using the kinematic conditions at the surface

$$w(\zeta) = \partial_t\zeta + u_\alpha(\zeta)\partial_\alpha\zeta, \quad (3)$$

the integral forms of (1) and (2) can be written as

$$\partial_t \int_{-d}^{\zeta} (\rho u_\alpha)dz + \partial_\beta \int_{-d}^{\zeta} (\rho u_\alpha u_\beta + p\delta_{\alpha\beta} - \mu\partial_\beta u_\alpha)dz = (p\partial_\alpha\zeta - \mu\partial_\beta u_\alpha\partial_\beta\zeta + \mu\partial_z u_\alpha)_{z=\zeta}, \quad (4)$$

$$\partial_t \int_{-d}^{\zeta} \left(\frac{\rho u_i^2}{2} + \rho g z\right)dz + \partial_\alpha \int_{-d}^{\zeta} u_\alpha\left(p + \frac{1}{2}\rho u_i^2 + \rho g z\right)dz - \partial_\alpha \int_{-d}^{\zeta} 2\mu(u_i e_{i\alpha})dz = (-p\partial_t\zeta)_{z=\zeta} + [2\mu(u_i e_{i3}) - 2\mu(u_i e_{i\alpha})\partial_\alpha\zeta]_{z=\zeta} - \int_{-d}^{\zeta} \epsilon dz. \quad (5)$$

In deriving (4) and (5) the bottom stress was not taken into account because the case in which we are particularly interested corresponds to $d \to \infty$. In equation (5) $\epsilon = 2\mu(e_{ij})^2$ is the rate of energy dissipation by molecular viscosity per unit volume, so that $\int_{-d}^{\zeta} \epsilon dz$ is the rate of energy dissipation per unit area.

Let us consider now those statistically homogeneous motions where all mean values, including those taken at $z = \zeta$, are independent of the horizontal coordinates. We shall distinguish the *Eulerian mean* values of variables from their *statistical average*. The first will be designated by capital letters corresponding to the variable chosen, and the latter by angle brackets about the random quantities. We will use the decomposition for Eulerian fields

$$\left.\begin{aligned} u_\alpha &= U_\alpha + u'_\alpha, \quad u_3 = w' \\ \zeta &= Z + \zeta' = \zeta', \quad \text{since} \quad Z = 0 \\ p &= P + p' \end{aligned}\right\}. \tag{6}$$

Then the momentum equation (4) can be written as

$$\partial_t\left\langle \int_{-d}^{\zeta} \rho U_\alpha dz \right\rangle + \partial_t\left\langle \int_{-d}^{\zeta} \rho u'_\alpha dz \right\rangle = \tau_{a\alpha}, \tag{7}$$

where

$$\tau_{a\alpha} = \langle (P\partial_\beta\zeta - \mu\partial_\beta u_\alpha \partial_\beta \zeta + \mu\partial_z u_\alpha)_{z=\zeta} \rangle \tag{8}$$

is the total flux of momentum from the atmosphere—the so-called wind stress. The energy equation (5) becomes

$$\partial_t\left\langle \frac{1}{2}\int_{-d}^{\zeta} \rho U_\alpha^2 dz \right\rangle + \partial_t\left\langle \int_{-d}^{\zeta} \rho U_\alpha u'_\alpha dz \right\rangle + \partial_t\left\langle \int_{-d}^{\zeta} \frac{1}{2}(\rho u'^2_\alpha + \rho w'^2) dz \right\rangle + \partial_t\left(\frac{1}{2}\rho g\langle \zeta^2 \rangle\right) = F_a - \left\langle \int_{-d}^{\zeta} \epsilon dz \right\rangle, \tag{9}$$

where

$$F_a = \langle [-p\partial_t\zeta + 2\mu(u_i e_{i3} - u_i e_{i\alpha}\partial_\alpha \zeta)]_{z=\zeta} \rangle \tag{10}$$

represents the total flux of energy from the atmosphere to the mean and fluctuating motions in the upper ocean. In a progressive wave train the rate of working by the atmospheric pressure distribution is $-\langle (pw)_\zeta \rangle = \langle cp\nabla\zeta \rangle$; so the part of the surface energy flux $-\langle p\partial_t\zeta \rangle_{z=\zeta}$, can be considered as an energy flux, to waves associated with the component of pressure variation in phase with the wave slope. It is also known (Phillips, 1977) that a fluctuating tangential stress applied at the free surface can transfer energy to the wave motion, and that is why we do not separate the rate of working by normal and tangential stresses at the surface. To proceed we need to evaluate the integrals on the left sides of (7) and (9) for the case where ζ is a random function with zero mean value. We adopt the following procedure as a rule for doing this

$$\left\langle \int_{-d}^{\zeta} a_i dz \right\rangle = \int_{-d}^{0} A_i dz + \left\langle \int_{0}^{\zeta} A_i dz \right\rangle + \left\langle \int_{0}^{\zeta} a'_i dz \right\rangle \approx \int_{-d}^{0} A_i dz + \langle a'_i(0)\zeta \rangle + (O\langle \zeta^2 \rangle). \tag{11}$$

Thus the momentum balance (7) reduces to

$$\partial_t \int_{-d}^{0} \rho U_\alpha dz + \partial_t M_\alpha = \tau_{a\alpha}, \tag{12}$$

where

$$M_\alpha = \rho\langle u'_\alpha(0)\zeta \rangle. \tag{13}$$

For cases where $u'_\alpha(0)$ is associated only with surface waves, M_α is the mean wave momentum per unit area. To calculate the first term in (12) we use the horizontal momentum equation (1). When there is no mean horizontal pressure gradient,

$$\partial_t(\rho U_\alpha) + \partial_z(\rho\langle u'_\alpha w' \rangle - \mu\partial_z U_\alpha) = 0, \tag{14}$$

so that

$$\partial_t \int_{-d}^{0} \rho U_\alpha dz = -(\rho\langle u'_\alpha w' \rangle - \mu\partial_z U_\alpha)_{z=0}. \tag{15}$$

Hence (12) takes the form

$$-(\rho\langle u'_\alpha w' \rangle - \mu\partial_z U_\alpha)_{z=0} + \partial_t M_\alpha = \tau_{a\alpha}. \tag{16}$$

Usually the part of the momentum flux $\tau_{a\alpha}$ from the atmosphere which is used for increasing the wave momentum is $\tau_{aw} = \partial_t M_\alpha$; so by neglecting the viscous stresses in (16) the boundary conditions for the Reynolds stresses in water can be well approximated as

$$-\rho\langle u'_\alpha w' \rangle_{z=0} = \tau_{a\alpha} - \tau_{w\alpha}. \tag{17}$$

There are still very few estimates of the ratio $\tau_{w\alpha}/\tau_{a\alpha}$ for growing wind waves in duration-limited situations (see the review of such data in Kitaigorodskii, 1973); but for fetch-limited cases the usual assumption is that $\tau_{w\alpha} \leq 0.2\tau_{a\alpha}$ (Hasselmann *et al.*, 1973). Using (11) the energy balance (9) can be written as

$$\partial_t E + \partial_t E_m + \partial_t\left(\frac{1}{2}\rho g\langle \zeta^2 \rangle\right) + \partial_t\left(\frac{1}{2}\rho\langle u'^2_i(0)\zeta \rangle\right) + U_\alpha(0)\partial_t M_\alpha = F_d - \left\langle \int_{0}^{\zeta} \epsilon dz \right\rangle - D, \tag{18}$$

where

$$E_m = \int_{-d}^{0} \frac{1}{2}\rho U_\alpha^2 dz \quad \text{and} \quad E = \int_{-d}^{0} \frac{1}{2}\rho\langle u'^2_i \rangle dz \tag{19}$$

are respectively the energy densities of the mean and fluctuating motions. $D = \int_{-d}^{0} \langle \epsilon \rangle dz$ characterizes the average viscous dissipation per unit area in the region below the *mean* water level (or, approximately, below the wave troughs). Calculating E_m from (14) and substituting it in (18) we can write the energy balance of the *fluctuating motion* in the form

$$\partial_t E + \frac{1}{2}\rho g \partial_t\langle\zeta^2\rangle = F_{aE} - D - \left\langle \int_0^\zeta \epsilon dz \right\rangle - \frac{1}{2}\rho\partial_t\langle u_i'^2(0)\zeta\rangle + \int_{-d}^0 \partial_z U_\alpha(\rho\langle u_\alpha' w'\rangle)dz - \int_{-d}^0 \mu(\partial_z U_\alpha)dz, \quad (20)$$

where

$$F_{aE} = F_a - [\tau_{a\alpha}U_\alpha(0) + M_\alpha\partial_t U_\alpha(0)] \quad (21)$$

represents the flux of energy from the atmosphere to the *fluctuating motion* in the water. Because the time scales $\tau_u^{-1} = \partial_t U_\alpha(0)/U_\alpha(0)$ and $\tau_m^{-1} = \partial_t M_\alpha/M_\alpha$ can be different (quite likely $\tau_u \gg \tau_m$), the role of the last term on the right side of (21) is probably negligible during active wave-generation conditions. However, because we do not know the exact form of F_a, this question is of minor importance here. We note that the term $M_\alpha\partial_t U_\alpha(0)$ appears in (21) in part due to our evaluation of the integral $\langle\int_0^\zeta U_\alpha u_\alpha' dz\rangle$ as $U_\alpha(0)M_\alpha$.

Equation (20) is of fundamental importance for a description of the evolution of the surface wave spectrum $\psi(\kappa)$, defined as

$$\rho g\langle\zeta^2\rangle = \int \psi(\kappa)d\kappa. \quad (22)$$

In the general case[1], when the velocity field consists of *turbulent* and *wave* motions, the balance of the surface wave energy can be given on the basis of equation (20) only if we are able to distinguish between the two types of motion and parameterize the possible dynamical interactions between them. In the next section we will attempt to make a clear distinction between the wave and turbulent components of the fluid dynamical fields in the presence of wind-generated surface gravity waves.

[1] It is instructive to notice that for *free surface gravity waves*, in a slightly viscous fluid, the *linear approximations* for the kinematic boundary condition

$$w(\zeta) \approx w(0) \approx \partial_t\zeta$$

and the velocity potential ϕ, for deep water,

$$\phi(x, z, t) = \kappa^{-1}\sigma a \exp[\kappa z] \sin(\kappa\cdot x - \sigma t)$$

give ($u_i = \nabla\phi$, $U_\alpha = 0$) the classical results, correct to second order in wave slope:

$$E = \frac{1}{2}\left\langle \int_{-d}^\zeta \rho u_i^2 dz \right\rangle \approx \frac{1}{2}\int_{-d}^0 \rho\langle u_i^2\rangle dz = \frac{1}{2}\rho g\langle\zeta^2\rangle,$$

and

$$\left\langle \int_{-d}^\zeta \epsilon dz \right\rangle \approx \int_{-d}^0 \epsilon_\phi dz = 4\rho\nu\kappa^2 g\langle\zeta^2\rangle.$$

Then equation (20) reduces to

$$\partial_t(\rho g\langle\zeta^2\rangle) = -4\nu\kappa^2\rho g\langle\zeta^2\rangle,$$

or in spectral form

$$\partial_t\psi(\kappa) = -4\nu\kappa^2\psi(\kappa).$$

3. The balance of turbulent kinetic energy in the upper ocean in the presence of random wind-generated waves

A rational procedure for dividing the random fluid dynamical fields into "turbulent" and "wave" components is absolutely necessary for both theoretical and experimental studies of the upper ocean under active wind generation conditions. This problem, up to the present time, has not been sufficiently discussed, and we know of only one paper (Benilov and Zaslavskii, 1974) where a similar question was raised regarding the turbulent atmospheric boundary layer above a wavy surface. However, there are very important differences between that case and the boundary layer in the ocean. In the atmospheric surface layer, turbulence is a dominant contributor to the energy of the fluctuating motions above the waves, whereas in the surface layer of the ocean the opposite picture occurs. Another difference is that the energy-containing wave components of the velocity field in the air are governed by the existence of a critical layer. However, there is, as a rule, no critical layer in the upper ocean, because the Eulerian surface drift is usually smaller than the phase velocity of the energy containing waves. Therefore, we can expect the mechanism of interaction between a turbulent shear flow and the wave-induced motion to be different for these two cases. Having this in mind, let us first examine a method of identifying turbulence based on phase-averaging. We define the *phase average* of a fluctuation, $f = F - \langle F\rangle$, of the random dynamical field $F(x, y, z, t)$, (the angle brackets now indicate an ensemble average), as the mean value of f over all points which satisfy the condition of constant phase $\theta = \kappa\cdot\mathbf{x} - \sigma t$ for a given Fourier component of the surface displacement $\zeta(\mathbf{x}, t)$. The wave component f^w of f is identified with the phase average of f, and the fluctuation of f relative to its phase-averaged value is called the turbulent component f^t. Hence we have the decomposition

$$F = \langle F\rangle + f^t + f^w,$$

where

$$f^w = \langle f\rangle_p, \quad f^t = f - f^w = F - \langle F\rangle - f^w, \quad (23)$$

and $\langle\ \rangle_p$ denotes the phase average. Note from the definition, $\langle f^t\rangle = {}_p0$. It is very important to understand that, from the point of view of the phase-averaging procedure, there is a vital difference between the turbulent boundary layer below a *random* sea surface (whose Fourier components have random amplitudes) and below a *deterministic* surface wave. In geophysical situations, because of the randomness of the surface displacement, the *phase average value must converge to the usual mean value by virtue of the very definition of phase average.* If we consider the motion in water, in the inviscid approximation, described by the equations

$$\partial_t v_i + \partial_k\left(v_i v_k + \frac{P}{\rho}\delta_{ik}\right) = -g\delta_{i3}, \qquad (24)$$

$$\partial_i v_i = 0, \qquad (25)$$

the substitution (23) gives us the well-known system of equations for the wave part of the fluctuating velocity field v_i^w and pressure p^w.

We will assume here, as in previous paragraphs, statistical homogeneity in the horizontal plane, so that

$$\langle v_i \rangle = [U(z), 0, 0], \qquad (26)$$

$$\langle P \rangle = P(z). \qquad (27)$$

In this case the equations for v_i^w can be written as

$$\partial_t v_i^w + v_3^w \delta_{i1}\partial_3 U + U(z)\partial_1 v_i^w + \partial_k(v_i^w v_k^w) + \partial_i \frac{p^w}{\rho} = -\partial_\kappa \langle v_i^t v_\kappa^t \rangle_p - g\delta_{i3}. \qquad (28)$$

Taking the difference between (24) and (28), using equation (23) multiplied by v_k^t, and introducing the notations

$$e_t = \frac{1}{2}\rho\langle v_i^t v_i^t \rangle_p \quad \text{and} \quad e_t' = \frac{1}{2}\rho\langle v_i^t v_i^t \rangle, \qquad (29)$$

we obtain an equation for the balance of turbulent energy

$$\partial_t e_t = -\rho\langle v_3^t v_1^t \rangle_p \partial_z U - \rho\langle v_i^t v_k^t \rangle_p \partial_k v_i^w - \partial_k[e_t v_k^w + \langle e_t' v_k^t \rangle + \langle p^t v_i^t \rangle_p]. \qquad (30)$$

Note that the turbulent viscous dissipation must be added in the general case of a viscous fluid. Possible wave–turbulence interactions are manifested here by the appearance of an additional production term, $\rho\langle v_i^t v_k^t \rangle_p \partial_k v_i^w$, and also a term describing the transport of turbulence by the wave-induced motion, $-\partial_k(e_t v_k^w)$. These terms are due to our use of phase-averaging to separate the wave-induced and turbulent parts of the fluctuating field. Note, however, that in the case of random surface waves, the phase average values must converge to the usual statistical average; so if $\zeta(\mathbf{x}, t)$ is random,

$$v_i^w \equiv \langle v_i \rangle_p \rightarrow \langle v_i \rangle = 0, \qquad (31)$$

and (30) will then be the usual equation for e_t, with v_i^t now denoting the total fluctuating part of the velocity field.

Therefore, in the case of random waves, we must use another approach to distinguish between wave and turbulent motions.

Let us consider a neutrally stratified fluid which occupies the space $-\infty < z < \zeta(\mathbf{x}, t)$. As before, we will consider ζ and the velocity field $\mathbf{u}(\mathbf{x}, z, t)$ to be statistically homogeneous in horizontal planes with the ensemble averages defined as:

$$\langle \zeta(\mathbf{x}, t) \rangle = 0,$$
$$\langle \mathbf{u} \rangle = \mathbf{U} = [U(z), 0, 0]. \qquad (32)$$

The fluctuating velocity field $\mathbf{u}' = \mathbf{u} - \mathbf{U}(z)$ will be decomposed as

$$\mathbf{u}' = \mathbf{u}^w + \mathbf{u}^t; \quad \mathbf{u}^w = \nabla\phi, \quad \mathbf{u}^t = \nabla \times \mathbf{A}. \qquad (33)$$

Now $\mathbf{u}^w$ denotes the irrotational part of $\mathbf{u}'$, derived from a scalar potential ϕ satisfying Laplace's equation $\nabla^2\phi = 0$; and $\mathbf{u}^t$ denotes the rotational part of $\mathbf{u}'$, satisfying $\mathbf{u}^t = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is a divergence-free vector potential, related to the vorticity by $\omega = -\nabla^2\mathbf{A}$. With the above definitions it follows from (32) and (33) that potential motions do not contribute to the *mean* motions. In analogy with (33) a similar decomposition can be made for $\zeta(\mathbf{x}, t)$. However, to first order, ζ is not strongly affected by the rotational motions associated with $\mathbf{u}^t$. Therefore we will always neglect the contribution of turbulence to the mean square surface displacement $\langle \zeta^2 \rangle$. The pressure field takes the form

$$p = P(z) + p_{0\phi} + p_t, \qquad (34)$$

where $P(z) = \langle p \rangle$ and the fluctuating part of the pressure $p' = p - P(z)$ is presented as a sum of two components: $p_{0\phi}$ which is defined as a solution of the linearized Bernoulli equation

$$\nabla\left[\partial_t\phi + gz + \frac{1}{\rho}(P(z) + p_{0\phi})\right] = 0, \qquad (35)$$

and $p_t = p - P(z) - p_{0\phi}$, which depends on the *turbulent* (rotational) velocity field and *nonlinearities* in the wave (potential) velocity field.

Then from the equations of motion

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = -g + \nu\nabla^2\mathbf{u} \qquad (36)$$

we can formally obtain the equations for the turbulent velocity fields

$$(\partial_t - \nu\nabla^2)\mathbf{u}^t + (\mathbf{u}^t\cdot\nabla)\mathbf{u}^t - \langle(\mathbf{u}^t\cdot\nabla)\mathbf{u}^t\rangle + (\mathbf{u}^t\cdot\nabla)\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{u}^t + Q = -\nabla\left[p_t/\rho + \frac{1}{2}(u^w)^2 - \frac{1}{2}\langle(u^w)^2\rangle\right], \qquad (37)$$

where

$$Q = (\mathbf{u}^t\cdot\nabla)\mathbf{u}^w - \langle(\mathbf{u}^t\cdot\nabla)\mathbf{u}^w\rangle + (\mathbf{U}\cdot\nabla)\mathbf{u}^w + (\mathbf{u}^w\cdot\nabla)\mathbf{u}^t - \langle(\mathbf{u}^w\cdot\nabla)\mathbf{u}^t\rangle + (\mathbf{u}^w\cdot\nabla)\mathbf{U}. \qquad (38)$$

An order of magnitude estimate of the different terms in (37, 38) is crucial for further analysis.

When analyzing motion in the upper ocean layer we can suppose that, even under conditions of active wave generation, a substantial part of the momentum flux $\tau_{a\alpha}$ in the atmospheric boundary layer (perhaps because of breaking waves) is transmitted to the mean current. This permits us to estimate a characteristic velocity u_* of turbulent motion in the water induced by the action of surface stress, as

$$u_* \approx \left(\frac{\rho_a}{\rho_w}\right)^{1/2} u_*^a, \qquad (39)$$

where $u_*^a = (\tau_{a\alpha}/\rho_a)^{1/2}$ is the friction velocity in the atmospheric boundary layer. The potential velocity $\mathbf{u}^w$ can be approximated by linear theory as

$$|\mathbf{u}^w| = u^w \approx a\sigma; \quad \sigma^2 = 2\pi \frac{g}{\Lambda}, \qquad (40)$$

where a is the characteristic wave amplitude $a \approx \langle \zeta^2 \rangle^{1/2}$, and σ is a characteristic wave frequency, which can be identified with the frequency σ_m corresponding to the wave spectrum maximum; Λ is then the wave length corresponding to σ_m. Despite the exponential decay of $u^w \approx \exp(-2\pi z/\Lambda)$, we can assume that inside a layer of thickness Λ, for wind speeds $U_a > 5$ m s^{-1}, the condition

$$\frac{u_*}{u^w} \ll 1 \qquad (41)$$

is satisfied. Moreover, the mean drift velocity $U(z)$ inside a layer of thickness Λ is also smaller than $\mathbf{u}^w$. *This means that terms which are quadratic in* $\mathbf{u}^w$ *will dominate in (37).* Contributions to p_t comprise *turbulence, nonlinearity in the wave motion,* and *turbulence-wave interactions,* and consequently cannot exceed $(u^w)^2$. More detailed analysis of equations (37) and (38) will be presented in the last section of this paper; but it is clear that, in the presence of waves, the $(u^w)^2$ term must be taken into account in the equations for $\mathbf{u}^t$.

Because the leading term in (37) is quadratic in u^w, the first term of the functional representation of $\mathbf{u}^t$ will be

$$\mathbf{u}^t = L[(u^w)^2 + \cdots], \qquad (42)$$

where the ellipsis indicates higher order terms (which are functions of the velocity field and boundary conditions). Since linear wave theory describes the energy-containing wave field accurately enough, the statistical distribution for u^w can be considered to be the same as for the surface displacement ζ. The latter, to good approximation, can be considered Gaussian. So the hypothesis which seems to be accurate to first order in $(u^w)^2$ is that the potential $\mathbf{u}^w$ and rotational u^t components of the velocity field are uncorrelated. It follows that if $\mathbf{u}^w$ is Gaussian, and $\mathbf{u}^t = L[(\mathbf{u}^w)^2]$, (which means that $\mathbf{u}^t$ is not Gaussian!) then the moments of the random fields $\mathbf{u}^w$ and $\mathbf{u}^t$ satisfy the following conditions:

$$\langle (u_i^w)^{2n-1}(u_j^t)^m \rangle = 0, \quad \langle (u_i^w)^{2n}(u_j^t)^m \rangle \neq 0,$$
$$\langle (u_i^t)^n (u_j^t)^m \rangle \neq 0; \quad n, m = 1, 2, \ldots. \qquad (43)$$

The odd-moments above are zero (by Gaussianity), while the even moments may be non-zero. From the equation for the mean velocity we obtain the Reynolds equation for the case under consideration (horizontally statistically homogeneous and stationary)

$$\langle u_3^t u_1^t \rangle - \nu \partial_z U = \text{constant} = u_*^2. \qquad (44)$$

The equation for the vertical component of the mean velocity is

$$\partial_z \frac{P}{\rho} + \partial_z \frac{1}{2} \langle (\mathbf{u}^w)^2 \rangle + \partial_z \langle (u_3^t)^2 \rangle = -g. \qquad (45)$$

Note that *the contributions to the Reynolds stress come only from the rotational components of the fluctuating motion,* whereas in (45) there is an additional term $\frac{1}{2}\langle (\mathbf{u}^w)^2 \rangle$ which is not present in flows over rigid surfaces.

We see that in the presence of surface waves the derivation of the equation of turbulent motion is not at all trivial. Using the method previously employed we can construct from (37) and (43) an equation for the turbulent energy

$$\partial_t e_t = -\langle u_3^t u_1^t \rangle \partial_z U - \partial_z \left[\langle e_t' u_3^t \rangle + \frac{1}{\rho} \langle p_t u_3^t \rangle + \frac{1}{2} \langle (\mathbf{u}^w)^2 u_3^t \rangle + 2\mu \langle u_i^t e_{i3}^t \rangle \right] - \epsilon_t, \qquad (46)$$

where

$$\epsilon_t = 2\mu \langle (e_{ij}^t)^2 \rangle. \qquad (47)$$

Eq. (46) differs from the standard form by the additional term $\partial_z \langle (\mathbf{u}^w)^2 u_3^t \rangle$, which represents the divergence of the turbulent transport of wave kinetic energy. The possible existence of this type of wave–turbulence interaction was first pointed out in the paper by Kitaigorodskii and Miropolskii (1968), and the simplest form of the turbulent energy balance containing the term $\langle (\mathbf{u}^w)^2 u_3^t) \rangle$ was subsequently derived by Benilov (1973). However, the physical mechanism behind this type of interaction has never been properly explained, and we address this question in the next section.

4. Physical mechanism of the wave-turbulence interaction in the upper ocean in the presence of a random wind-wave field

First of all, we note that in the phase-averaged version of the turbulent energy equation (30) the $\langle (u^w)^2 u_3^t \rangle$ term does not exist. It can also be seen immediately that if the wave induced velocity field $\mathbf{u}^w$ is due to a *monochromatic linear deterministic* wave, with surface displacement $\zeta = a \cos(\boldsymbol{\kappa} \cdot \mathbf{x} - \sigma t)$, then

$$(u^w)^2 = \frac{1}{2} (a\sigma)^2 \exp\left(+4\pi \frac{z}{\Lambda}\right), \qquad (48)$$

and $\langle (u^w)^2 u_3^t \rangle$ is again identically equal to zero (recall the $z < 0$ below the surface). Therefore, this term is important only in the presence of random surface waves.

To clarify the physical mechanism at work here let us consider a surface wave, with scalar wave number κ, initially in a non-turbulent fluid. The wave velocity u^w and pressure p^w associated with such a wave decay exponentially with depth. The kinetic energy of particle motions also decreases with depth as $\exp(+4\pi z/\Lambda)$. We can simplify things by considering the layer where the wave motions exist to have a characteristic thickness of order κ^{-1}, and assume that below this layer the fluid is practically stationary. Now if the fluid particles also participate in random turbulent motions (superimposed on the orbital wave motion) then the whole picture will change. The fluid elements *below* the wave layer (let us say at $z > \kappa^{-1}$) have turbulent velocities $\mathbf{u}^t$ and kinetic energy e_t. The fluid elements *inside* the wave layer participate in both turbulent and wave motions, so that their velocities and energies are *higher* than for those elements below the wave layer. Therefore vertical fluctuations will lead to an exchange of energy between the wave layer and the fluid below which, of course, results in a *downward flux of kinetic energy.* Because the energy taken from the surface layer must finally be dissipated, the divergence of this downward flux can be identified as a turbulent energy source (or sink) due to the transformation of wave energy into turbulence. Because at $z \gg \kappa^{-1}$ all motions must disappear, the amount of energy being transferred to the turbulence (and then dissipated to heat) will depend, in the horizontally homogeneous case, only on the surface value of the downward flux of wave energy. Such a flux, of course, simultaneously represents the dissipation of surface wave energy, and hence limits the continuing growth of the waves.

The picture described above is also valid in the case when there is a turbulent layer of thickness κ^{-1} with noticeable vertical gradients of $\mathbf{u}^w$ and wave kinetic energy. To describe the transformation of wave energy to turbulence in this layer, we must know the typical scales of the turbulent motions. Relative to the wavelength Λ and wave frequency σ, the whole spectrum of turbulent fluctuations with length scales L_t and time scales Ω_t^{-1} can be conveniently divided into four groups: "long" ($L_t > \Lambda$), "short" ($L_t < \Lambda$), "rapid" ($\Omega_t > \sigma$), and "slow" ($\Omega_t \leqslant \sigma$). Now it is clear that "long" fluctuations are the most effective for transporting wave energy downward, since they are able to cross the whole wave layer. However, both "short" and "long" fluctuations can act in the same way if there is a vertical gradient of $\mathbf{u}^w$ and $\frac{1}{2}(\mathbf{u}^w)^2$. It is probably more important to distinguish between rapid and slow fluctuations. Rapid fluctuations can interact with the periodic wave motion, since for them the latter appears as a slowly varying *shear* current. This type of wave–turbulence interaction is described effectively by the *phase-average* technique (the second term on the right side of Eq. (30) is exactly this wave shear production). However, if the typical turbulent velocities are of order u_*, then the rapid ($\Omega_t > \sigma$) fluctuations will have a maximum length scale $L_t \sim u_*/\sigma$. This is, of course, very small scale turbulence, [its integral length scale is probably less than the radius of the wave orbits, $a \exp(-\kappa z)$, everywhere!]. Such very short and rapid small scale turbulence will act as an effective eddy viscosity on the wave induced velocity field, but its role in the dissipation of wave energy is probably not very important (see Phillips, 1961). We also assume that its role in vertical mixing in the upper ocean is not crucial. The phase-averaging technique is a proper method to study such a range of scales but it is clear from the above that we should be more interested in long and slow turbulence. However, we cannot expect relatively slow and long turbulent fluctuations to have a strong *dynamical* effect for the case of a monochromatic wave since the turbulence will be "frozen" on the timescale of the orbital velocity. Hence the randomness of the waves must be taken into account, and the phase averaging technique fails. The turbulent energy budget, in the case of random surface waves, indicates that the primary effect of wave–turbulence interaction appears as the *vertical transport of wave kinetic energy* (but *not* momentum) *by the turbulence.* The fluctuations in $(\mathbf{u}^w)^2$ are due to the presence of a spectrum of surface waves, and so there is no restriction, in this case, to turbulent time scales much shorter than the wave period. In fact, we expect that the basic contribution to the turbulent transport of wave energy comes from turbulent motions having time scales on the order of the wave period (or the period of the wave groups) and length scales smaller than, or comparable to, the thickness of the "wave layer". The turbulence generated by wave-breaking in the thin surface layer can quite well satisfy such requirements.

It is interesting to note that surface-stress induced turbulence in the upper ocean probably also belongs to this category. Indeed, we have already pointed out that the characteristic velocity of turbulent motions in a wind-driven shear current is rather low ($u_* \sim u_*^a/30$, and with wind speeds of 10–15 meters per second we obtain a u_* of only a few centimeters per second). It is useful to compare the characteristic relaxation time τ_t and length scale L_t of such shear turbulence to the period T_m and wave length Λ_m of the energy containing components of the surface waves. The ratio of $\tau_t = L_t/u_*$ to $T_m = \Lambda_m/c_m$ (c_m is the phase speed of the wave) can be written as

$$\frac{\tau_t}{T_m} \approx \left(\frac{L_t}{\Lambda_m}\right)\left(\frac{c_m}{u_*^a}\right)\left(\frac{\rho_w}{\rho_a}\right)^{1/2}. \tag{49}$$

Since $c_m \approx U_a$ and $(u_*^a/U_a)^2 \approx 10^{-3}$, it follows that $\tau_t/T_m \sim 10^3 L_t/\Lambda_m$. Hence the characteristic relaxation time for this kind of shear-produced, three-dimensional turbulence, even with $L_t \ll \Lambda_m$, can be long compared to T_m. Such turbulence, from the point of view of an observer moving along the wave orbit, will appear frozen. Inasmuch as the speeds of the orbital motions in surface waves are, on the whole, greater than the drift current, such frozen turbulence, convected by the drift, will occupy a region of significantly lower frequency than the frequency of the dominant wind wave. This circumstance has apparently been confirmed by the recent measurements of Donelan (1978). His data also show that the Reynolds stress, in the region below the wave troughs, agrees rather well with estimates of $\rho_w u_*^2$ determined from the wind stress $\tau_{a\alpha}$. Thus the momentum flux in the surface layer turns out to be comparable to the momentum flux from the atmosphere, which lends support to the proposition of a weak correlation between the irrotational (wave) and rotational (turbulent) fluctuating velocity. It is quite possible, however, that such shear-induced turbulence will be correlated with fluctuations of $(\mathbf{u}^w)^2$ *in the presence of a spectrum of wind-generated waves,* and will therefore be responsible for the transformation of wave energy into turbulence inside the "wave" layer.

The fact that this correlation occurs only in a random field of surface waves makes a parameterization of the term $\langle(\mathbf{u}^w)^2 u_3^t\rangle$ in (47) even more straightforward than is the case for the third moments of the turbulent velocity field. Consider the expression (48) for $\frac{1}{2}(\mathbf{u}^w)^2$, in the case when we have a random ensemble of waves of different amplitudes a and wave numbers κ, so that

$$a = \langle a\rangle + a' \quad \text{and} \quad \kappa = \langle\kappa\rangle + \kappa'. \qquad (50)$$

Assuming for simplicity that $\kappa' = 0$ ($\langle\kappa\rangle = \kappa_m$ corresponds to the peak frequency of the wave spectrum via $\sigma_m = (g\kappa_m)^{1/2}$) we get

$$\frac{1}{2}\langle(\mathbf{u}^w)^2 u_3^t\rangle = \left(\frac{\sigma_m^2}{2}\right)\exp[+2\kappa_m z]\{\langle u_3^t a'^2\rangle + 2\langle a\rangle\langle a' u_3^t\rangle\}. \qquad (51)$$

Since a' has the same (Gaussian) distribution as $\zeta(\mathbf{x}, t)$ we obtain

$$\frac{1}{2}\langle u_3^t(\mathbf{u}^w)^2\rangle = \left(\frac{\sigma_m^2}{2}\right)\exp[+2\kappa_m z]\langle u_3^t(a')^2\rangle. \qquad (52)$$

One of the mechanisms, which can explain why u_3^t and a'^2 can be correlated is associated with the wave-breaking process. Indeed, the large values of a'^2 (for example $a'^2 > \langle a'^2\rangle$ must be correlated with the appearance of breaking events, which in turn increase the probability for the vertical turbulent velocity u_3^t ($z = 0$) being directed downwards. Therefore the largest absolute values of $(u_3^t a'^2)_{z=0}$ are likely to be negative, indicating that the transformation of wave energy into turbulence due to wave breaking can lead to a downward surface energy flux $\frac{1}{2}[u_3^t(\mathbf{u}^w)^2]_{z=0}$. We plan to discuss the parameterization of the correlation between u_3^t and a'^2 [or $(\mathbf{u}^w)^2$] in the future. However, we can already now establish from equation (47) constructive implications for the balance of wind waves and turbulence in the upper ocean. We will deal with these in the next section.

5. Some implications of the proposed balance of turbulent energy for the dynamics of wind-generated linear surface gravity waves

Now we are in a position to deal with equation (20) which governs the kinetic energy of the *fluctuating motions.* The hypothesis, that the wave and turbulent velocities are uncorrelated, permits us to separate the fluctuating energy into wave and turbulent parts, so that

$$E = E^w + E^t, \qquad (53)$$

where

$$\left.\begin{aligned} E^w &= \int_{-d}^{0}\frac{1}{2}\rho\langle(u_i^w)^2\rangle dz \approx \frac{1}{2}\rho g\langle\zeta^2\rangle \\ E^t &= \int_{-d}^{0}\frac{1}{2}\rho\langle(u_i^t)^2\rangle dz = \int_{-d}^{0} e_t dz \end{aligned}\right\}. \qquad (54)$$

Note we are assuming a linear relation between the wave velocity and surface displacement. Further, using the rules (43), arising from the assumption of Gaussianity for ζ and $\mathbf{u}^w$, we see that

$$\langle u_i'^2(0)\zeta\rangle = \langle[u_i^w(0)]^2\zeta\rangle + \langle[u_i^t(0)]^2\zeta\rangle = 0. \qquad (55)$$

Integrating (46) from to $z = -\infty$ to $z = 0$ and substituting the result in (20), we find for the final form of the energy balance for linear surface gravity waves in the presence of atmospheric forcing and wind-induced turbulence in the water

$$\begin{aligned} \rho g\partial_t\langle\zeta^2\rangle = F_{aE} &- (-\langle e_t' u_3^t\rangle)_{z=0} - (-\langle p_t u_3^t\rangle)_{z=0} \\ &- \left[-\frac{1}{2}\langle\rho(\mathbf{u}^w)^2 u_3^t\rangle\right]_{z=0} - \left\langle\int_0^\zeta \epsilon dz\right\rangle \\ &+ 2\mu\langle u_i^t e_{i3}^t\rangle|_{z=0} - 4\nu\kappa^2\rho g\langle\zeta^2\rangle. \end{aligned} \qquad (56)$$

The last term on the right hand side comes from expanding $\mu\partial_z\langle(u_3^w)^2\rangle_{z=0}$ to the second order in the wave slope.

Following the experiments reported by Stewart and Grant (1962), we consider that the effects of breaking are confined primarily to the region between the crest and trough [the term $\int_0^\zeta \epsilon dz$ thus plays an important role in (56)]. Since, in (21), F_{aE} represents the energy flux to the fluctuating motion due to atmospheric forcing, i.e., due to the pressure fluctuations to all wave components, it can be written as

$$F_{aE} = F_w + F_{br}, \qquad (57)$$

where F_{br} is the part of this flux going to the *breaking* waves, and F_w is the flux to the *growing* (or at least *non-breaking*) waves. As was suggested recently (Kitaigorodskii, 1983) the saturation regime in the wind wave spectrum $\psi(\kappa)$ associated with wave breaking is formed through the nonlinear energy cascade from larger to smaller wavelengths. Thus, the direct atmospheric input to the breaking waves can be negligible ($F_{br} \approx 0$) and as an approximate secondary balance in (56) we can consider that

$$(\langle e_t' u_3 \rangle)_{z=0} \approx \left\langle \int_0^\zeta \epsilon dz \right\rangle . \tag{58}$$

Dropping terms involving viscous transport of turbulent energy and using (58), we find the final form of equation (56)

$$\rho g \partial_t \langle \zeta^2 \rangle = F_w + \left[\frac{\rho}{2} \langle (\mathbf{u}^w)^2 u_3^t \rangle \right]_{z=0} - 4\nu\kappa^2 \rho g \langle \zeta^2 \rangle. \tag{59}$$

We have incorporated the term $\langle \rho_t u_3^t \rangle$ into $\langle (\mathbf{u}^w)^2 u_3^t \rangle$ because, as was mentioned earlier and also will be shown below, p_t cannot exceed $(\mathbf{u}^w)^2$. In the energy balance (59) wave-turbulence interactions are represented by the term $[\frac{1}{2}\rho \langle (\mathbf{u}^w)^2 u_3^t \rangle]_{z=0}$, which characterizes the downward flux of surface-wave energy due to turbulence inside the wave layer. The simplest parameterization of this term is

$$-\langle (\mathbf{u}^w)^2 u_3^t \rangle_{z=0} = \mathrm{R} u_* \langle [\mathbf{u}^w(0)]^2 \rangle, \tag{60}$$

where R is a correlation coefficient, assumed to be constant. In the case of strong wave-turbulence interaction $R = 1$ if, as was also assumed in (60),

$$[\langle (u_3^t)^2 \rangle^{1/2}]_{z=0} \approx u_* . \tag{61}$$

However, in the general case R must be a function of the nondimensional ratio $\langle \int_0^\zeta \epsilon dz \rangle / u_*^3$, which can be larger than one, because the value of $\langle \int_0^\zeta \epsilon dz \rangle$ represents a substantial part of the energy loss through the breaking per unit surface area. So it is possible that instead of u_* in (60) and (61) the proper velocity scale for $\langle (u_3^t)^2 \rangle$ will be $\langle \int_0^\zeta \epsilon dz \rangle^{1/3} \approx \epsilon_0^{1/3}$, where ϵ_0 is the nonlinear energy flux through the wavenumber spectrum $\psi(\kappa)$ (Kitaigorodskii, 1983). Because ϵ_0 is proportional to the wind speed to the third power, it will not change the form of the parameterization (60).

Using it and linear wave theory, we obtain an equation for the wind-wave energy spectrum $\psi(\kappa) = \rho g \langle \zeta^2 \rangle_\kappa$ in the form

$$\partial_t \psi(\kappa) = \Pi(\kappa) - R u_* \kappa \psi(\kappa) - 4\nu\kappa^2 \psi(\kappa) \tag{62}$$

$\Pi(\kappa)$ represents the forcing by turbulent atmospheric pressure fluctuations, and is normalized to

$$\int \Pi(\kappa) d\kappa = F_w(0). \tag{63}$$

It is clear from the spectral form of the wave-energy balance that the effect of wave-turbulence interactions can be accounted for by introducing an effective spectral eddy viscosity $\nu_e(\kappa)$, which, according to (62), is given by

$$\nu_e(\kappa) = R u_* \kappa^{-1} \gg \nu, \tag{64}$$

so that the *molecular dissipation* is negligible for all scales of practical importance. The critical length Λ_{cr} where direct viscous dissipation is of the same importance as wave-turbulence interactions is estimated to be

$$\Lambda_{cr} \approx \frac{2\pi\nu}{R u_*} . \tag{65}$$

For short waves, say less than 10 cm length, is it probably necessary to keep the last term in (62) only if $R < 10^{-2}$. Thus for most of the energy-containing waves the wave-energy balance, equation (62), reduces to

$$\partial_t \psi(\kappa) = \Pi(\kappa) - R u_* \kappa \psi(\kappa). \tag{66}$$

It is interesting to note that this provides a mechanism for a wind-dependent saturation regime in the energy-containing part of the wave spectrum without requiring nonlinear energy transfer from the spectral peak to higher wavenumbers or nonlinear transfer of wave action density in the opposite direction (Kitaigorodskii, 1981, 1982, 1983). To show this we start with the expansion (Phillips, 1977)

$$\Pi(\kappa) = \alpha(\kappa) + \beta(\kappa)\psi(\kappa). \tag{67}$$

Then (66) has the solution

$$\psi(\kappa, t) = \psi_{eq}(\kappa)\{1 - \exp[-(R u_* \kappa - \beta(\kappa))t]\}, \tag{68}$$

where the equilibrium spectrum $\psi_{eq}(\kappa) = \psi(\kappa, t \to \infty)$ is given by

$$\psi_{eq}(\kappa) = \frac{\alpha(\kappa)}{[R u_* \kappa - \beta(\kappa)]} . \tag{69}$$

The spectrum of turbulent atmospheric pressure fluctuations at the sea surface $E^P(\kappa, \sigma)$ can be expressed as (Phillips, 1977)

$$E^P(\kappa, \sigma) = E^P(\kappa)\delta(\sigma - \kappa \cdot \mathbf{U}_a), \tag{70}$$

where $\mathbf{U}_a$ is the mean wind speed, and $\langle p^2 \rangle = \int d^2\kappa d\sigma E^P(\kappa, \sigma)$. According to Kolmogorov–Obukhov theory, (Monin and Yaglom, 1975, Pt. 2), the spectrum $E^P(\kappa)$ scales as

$$E^P(\kappa) \propto \kappa^{-10/3} \tag{71}$$

so that

$$\alpha(\kappa) \approx \kappa E^P(\kappa) U_a / \rho g \propto \kappa^{-7/3}. \tag{72}$$

Hence the equilibrium spectrum $\psi_{eq}(\kappa)$ [supposing $R u_* \kappa > \beta(\kappa)$] goes as

$$\psi_{eq}(\kappa) \propto \kappa^{-10/3} \tag{73}$$

and the corresponding equilibrium frequency spectrum becomes

$$S_{eq}(\sigma) \approx \frac{\psi_{eq}(\kappa)\kappa}{\partial_\kappa \sigma} \propto \sigma^{-11/3}. \tag{74}$$

It is interesting to note, that this form of the spectrum is in exact agreement with the wave energy spectrum, based on constant wave action flux ϵ_N (Kitaigorodskii, 1983), when energy input from the wind must be concentrated at $\kappa = \infty$ and action flux ϵ_N due to nonlinear interactions is directed to $\kappa = 0$. Therefore the existence of the equilibrium form of wave spectra (74) can be due to two different mechanisms, and it remains to be seen which one is prevailing and how important the above type of stabilization will be in the statistical dynamics of nonlinear gravity waves.

6. Some implications of the proposed balance of turbulent energy for the structure of turbulence in the ocean surface layer

If we consider the region below the wave troughs where viscous stresses can be neglected in (45) and where the vertical transport of turbulent energy by vertical turbulent velocities and pressure fluctuations can be neglected compared with shear production and viscous dissipation, then equations (44 and 46) reduce, in the stationary case, to

$$\langle u_3^t u_1^t \rangle = \text{constant} = u_*^2, \qquad (75)$$

$$-\langle u_3^t u_1^t \rangle \partial_z U - \partial_z \frac{1}{2} \langle (\mathbf{u}^w)^2 u_3^t \rangle = \epsilon_t. \qquad (76)$$

In the absence of wave–turbulence interactions we can use the well-documented relationships in the constant flux surface layer; for example, the second moments have the form

$$\left.\begin{aligned} \sigma_1 &= \langle (u_1^t)^2 \rangle^{1/2} \equiv f_1(u_*, z) = A_1 u_* \\ \sigma_3 &\equiv \langle (u_3^t)^2 \rangle = f_3(u_*, z) = A_3 u_* \end{aligned}\right\}, \qquad (77)$$

where $A_1 \approx 2.3$, $A_3 \approx 0.9$ (Monin and Yaglom, 1974 Pt. 1). The term $\frac{1}{2}\langle (\mathbf{u}^w)^2 u_3^t \rangle$ can be parameterized as

$$\langle (\mathbf{u}^w)^2 u_3^t \rangle = \sigma_m^2 \exp[+2\kappa_m z] \langle u_3^t (a')^2 \rangle \approx R\sigma_3(z)\sigma_w^2 \exp[+2\kappa_m z], \qquad (78)$$

where

$$\sigma_w = \sigma_m \sigma_\zeta; \quad \sigma_\zeta^2 = \langle \zeta^2 \rangle \approx \langle a'^2 \rangle,$$

and we assume $R \approx$ constant. (79)

Therefore instead of (77), in the case when wave energy can be transformed into turbulence, we expect the more general relations

$$\left.\begin{aligned} \sigma_1 &= F_1(u_*, z, \sigma_w, \kappa_m) = u_* A_1'(\kappa_m z, \sigma_w/u_*) \\ \sigma_3 &= F_3(u_*, z, \sigma_w, \kappa_m) = u_* A_3'(\kappa_m z, \sigma_w/u_*) \end{aligned}\right\}. \qquad (80)$$

If we consider the layer close to the surface (asymptotically corresponding to $\kappa_m z \rightarrow 0$) then

$$\left.\begin{aligned} A_1' &\rightarrow A_{10}'\left(\frac{\sigma_w}{u_*}\right) \\ A_3' &\rightarrow A_{30}'\left(\frac{\sigma_w}{u_*}\right) \end{aligned}\right\}. \qquad (81)$$

Asymptotically the functions A_{10}', A_{30}' tend to

$$\left.\begin{aligned} A_{10}' &\rightarrow A_1 \\ A_{30}' &\rightarrow A_3 \end{aligned}\right\} \quad \text{for} \quad \frac{\sigma_w}{u_*} \rightarrow 0, \qquad (82)$$

$$\left.\begin{aligned} A_{10}' &\rightarrow B_1(\sigma_w/u_*) \\ A_{30}' &\rightarrow B_3(\sigma_w/u_*) \end{aligned}\right\} \quad \text{for} \quad \frac{\sigma_w}{u_*} \rightarrow \infty, \qquad (83)$$

so that the simplest linear interpolation formulae, such as

$$\left.\begin{aligned} A_{10}' &= A_1 + B_1(\sigma_w/u_*) \\ A_{30}' &= A_3 + B_3(\sigma_w/u_*) \end{aligned}\right\}, \qquad (84)$$

can be used as a first approximation to describe the variability of σ_1 and σ_3.

To find the variation with depth z of the turbulent energy we must make some approximations. If we assume that, close to the surface, the second term on the left side of (76), representing the divergence of the wave kinetic energy flux due to turbulence, is much larger than shear production, then (76) reduces to

$$\exp[+2\kappa_m z]\sigma_m^2 \kappa_m \langle u_3^t (a')^2 \rangle|_{z=0} \approx \epsilon_t. \qquad (85)$$

Using (78) and the estimate

$$\epsilon_t \approx \frac{\sigma_3^3}{L_t}, \qquad (86)$$

where L_t is the length scale of the energy containing eddies, we find

$$\frac{\sigma_3}{u_*} \approx (RL_t\kappa_m)^{1/2}\left(\frac{\sigma_w}{u_*}\right) \exp[+\kappa_m z]. \qquad (87)$$

We distinguish between two types of turbulence—one corresponding to large fluctuations, where

$$(L_t\kappa_m)^{1/2} \approx O(1), \qquad (88)$$

and the other corresponding to small scale fluctuations, $L_t\kappa_m \ll 1$. Turbulence with "large" scales $L_t \sim \Lambda_m = 2\pi/\kappa_m$, as was already mentioned in Section 4, is very efficient in extracting energy from waves, and in this case σ_3 scales as $\sigma_w \exp[+\kappa_m z]$, the rms wave velocity at depth z. Turbulence with scales much smaller than Λ_m can be defined as those having a vertical kinetic energy gradient on the same order as the vertical gradient of wave kinetic energy, so that

$$\frac{\sigma_3^2(0)}{L_t} \approx \frac{\sigma_w^2}{\Lambda_m}. \qquad (89)$$

In this case the turbulent diffusion of turbulent energy from the surface produces the same effect as the extraction of wave energy. Therefore, for $L_t/\Lambda_m \ll 1$, and close to the surface $\kappa_m z \ll 1$) we get from (87)

$$\frac{\sigma_3}{\sigma_3(0)} \approx \text{constant}, \qquad (90)$$

where

$$\sigma_3(0) \propto u_*. \qquad (91)$$

Of course, the description of the turbulent regime in the ocean surface layer is made much more difficult than in its counterpart, the atmospheric surface layer, by the existence of at least two additional scales, for example, the velocity scale σ_w and length scale κ^{-1}. However, the analysis of experimental data on the statistical characteristics of oceanic turbulence cannot be complete without attempts to discover the extent and nature of deviations from typical boundary layer turbulence. We will address this question in Part II, (Kitaigordskii *et al.*, 1983).

7. The linear potential velocity field and its interactions with turbulence

Consider turbulence with length and time scales comparable to the corresponding scales of surface gravity waves. If the turbulent intensity is much smaller than the rms wave velocity, then the waves will not be appreciably distorted by their interaction with the turbulence. Consequently, we expect that the quadratic term $(\mathbf{u}^w)^2$ in (37) can be adequately described by a linear model for the potential velocity field. An important point for our analysis is the inclusion in the turbulent pressure p_t of terms in $(\mathbf{u}^w)^2$. To investigate this question in more detail we begin with equation (37). Taking the divergence, we obtain an equation for the pressure.

$$\partial_j u_i^t \partial_i u_j^t - \langle \partial_j u_i^t \partial_i u_j^t \rangle + \partial_j U_i \partial_i u_j^t + \nabla \cdot Q = R$$
$$= -\nabla^2\left[p_t/\rho + \tfrac{1}{2}(\mathbf{u}^w)^2 - \tfrac{1}{2}\langle(\mathbf{u}^w)^2\rangle\right] = -\nabla^2 \frac{p''}{\rho}. \quad (92)$$

We will now use a Green's function technique to solve (92). If we write

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'), \quad G(\mathbf{x}^s, \mathbf{x}') = 0, \quad (93)$$

where $\mathbf{x}^s = (\mathbf{x}_\alpha, \zeta)$, then we may integrate over a region to obtain

$$-\iiint G(\mathbf{x}, \mathbf{x}')R(\mathbf{x})d\mathbf{x} - \frac{p''(\mathbf{x}')}{\rho} = -\iint \left[\frac{P''(\mathbf{x}^s)}{\rho}\right]\partial_i G d\sigma_i. \quad (94)$$

The integral on the right consists of integrals over the surface, over vertical sheets defining the sides of the region, and over the bottom. We presume that the fluctuating motion dies out at great depth, and that the motion may be considered periodic in the horizontal, with some very large wavelength, so that the lateral integrals will cancel. Hence, we can write

$$\frac{p''(\mathbf{x}')}{\rho} = \iint_{\text{surf}} \partial_i G[p''(\mathbf{x}^s)/\rho]d\sigma_i - \iiint G(\mathbf{x}, \mathbf{x}')R(\mathbf{x})d\mathbf{x} \quad (95)$$

over the free surface.

If we take free waves (without forcing), then the pressure at the free surface (ignoring surface tension) is constant. Thus

$$p(\mathbf{x}^s) = P(\mathbf{x}^s) + p_{0\phi}(\mathbf{x}^s) + p_t(\mathbf{x}^s) = \text{constant} = P_{\text{atm}}. \quad (96)$$

Hence, we can write

$$p_t(\mathbf{x}^s) = P_{\text{atm}} - P(\mathbf{x}^s) + \rho\partial_t\phi(\mathbf{x}^s). \quad (97)$$

From equation (45) we have

$$\frac{P(\mathbf{x}^s)}{\rho} = -g\zeta - \tfrac{1}{2}\langle(\mathbf{u}^w)^2\rangle - \langle(u_3^t)^2\rangle \quad (98)$$

so that

$$p_t(\mathbf{x}^s) = P_{\text{atm}} + \frac{\rho}{2}\langle(\mathbf{u}^w)^2\rangle + \rho\langle(u_3^t)^2\rangle + \rho g\zeta + \rho\partial_t\phi(\mathbf{x}^s). \quad (99)$$

We may now substitute this in (95) to obtain

$$\frac{p''(\mathbf{x}')}{\rho} = \iint_{\text{surf}} \partial_i G\left\{\tfrac{1}{2}(\mathbf{u}^w)^2 - \tfrac{1}{2}\langle(\mathbf{u}^w)^2\rangle + g\zeta + \partial_t\phi + \frac{P_{\text{atm}}}{\rho} + \tfrac{1}{2}\langle(\mathbf{u}^w)^2\rangle + \langle(u_3^t)^2\rangle\right\}d\sigma_i - \iiint G(\mathbf{x}, \mathbf{x}')R(\mathbf{x})d\mathbf{x}. \quad (100)$$

The constant terms, of course, will not contribute to any correlations with $\mathbf{u}^t$; in addition, the terms in ζ and $\partial_t\phi$ will not contribute (at least to first order) by the assumption embodied in Eq. (43). In addition, in (44), the only contributions from the volume integral will be the usual rapid and return-to-isotropy terms for the turbulent velocity field. With that exception, the only contribution to $\langle p''(\mathbf{x}')\partial_j u_i^t(\mathbf{x}')\rangle/\rho$ will come from the term $\tfrac{1}{2}(\mathbf{u}^w)^2 - \tfrac{1}{2}\langle(\mathbf{u}^w)^2\rangle$ in the surface integral.

To make this rigorous, we should construct the Green's function, and examine its order in ζ. First, we have the exact result

$$d\sigma_i = dx_1 dx_2(-\partial_1\zeta, -\partial_2\zeta, 1) \quad (101)$$

so that

$$\partial_i G d\sigma_i = dx_1 dx_2(\partial_3 G - \partial_1 G\partial_1\zeta - \partial_2 G\partial_2\zeta). \quad (102)$$

Now, the Green's function can be expanded in a series in the surface slope:

$$G = G_0 + G_1 + \cdots, \quad (103)$$

where

$$\nabla^2 G_0 = \delta(\mathbf{x} - \mathbf{x}'), \quad G_0(\mathbf{x}^a, \mathbf{x}') = 0, \quad (104)$$

where $\mathbf{x}^a = (x_1, x_2, 0)$ is the average location of the free surface. Then

$$\nabla^2 G_1 = 0, \quad (105)$$

and we can expand

$$G(\mathbf{x}^\zeta, \mathbf{x}') = G_0(\mathbf{x}^\zeta, \mathbf{x}') + G_1(\mathbf{x}^\zeta, \mathbf{x}') + \cdots$$
$$= G_0(\mathbf{x}^a, \mathbf{x}') + \partial_3 G_0(\mathbf{x}^a, \mathbf{x}')\zeta + G_1(\mathbf{x}^a, \mathbf{x}') + \cdots. \quad (106)$$

Keeping only first order terms,

$$G_1(\mathbf{x}^\alpha, \mathbf{x}') = -\zeta\partial_3 G_0(\mathbf{x}^\alpha, \mathbf{x}'). \quad (107)$$

The solution for G_0 is, of course, the classical one

$$G_0(\mathbf{x}, \mathbf{x}') = \frac{(1/r' - 1/r'')}{4\pi}, \quad r' = |\mathbf{x}' - \mathbf{x}|,$$
$$r'' = |\mathbf{x}'' - \mathbf{x}|, \quad \mathbf{x}'' = (x'_1, x'_2, -x'_3) \quad (108)$$

which gives

$$\partial_3 G_0(\mathbf{x}^a, \mathbf{x}') = \frac{x'_3}{2\pi r'^3} \quad (109)$$

(recall that $x'_3 < 0$ below the surface). For the fluctuating part, we may obtain by the same technique

$$G_1(\mathbf{x}, \mathbf{x}') = -\iint \zeta\partial_3 G_0(\mathbf{x}^a, \mathbf{x}')\partial_3 G_0(\mathbf{x}^a, \mathbf{x})dx_1 dx_2. \quad (110)$$

It is clear that by the same technique, we may obtain the solution for the Green's function to any order.

Finally, all quantities may be expanded about the mean water level. It is evident that a considerable number of other second order terms will be obtained in this way. We may eliminate many of these. First, let us set $P_{\text{atm}} = 0$ for simplicity. The only term of zero order in wave slope, in the first integral in (100), is $\langle (u^t)^2 \rangle$; we could in principle have a second order term from 1) the product of this with the second order term arising from the Green's function; 2) the product of the vertical gradient of this with the first order term arising from the Green's function; 3), the product of the vertical curvature of this with the zero-order term arising from the Green's function. However, these will all be smaller by the ratio of $\langle (u^t)^2 \rangle / \langle (\mathbf{u}^w)^2 \rangle$ than corresponding terms, and hence may be neglected.

We will obtain a non-negligible term from the product of the first-order Green's function and the first order term [in the first integral in (100)]: $(g\zeta + \partial_t\phi)\partial_3 G_1$ as well as from the zero-order Green's function and the vertical gradient of the first order term: $\partial_3\partial_t\phi\zeta\partial_3 G_0$. Both these terms are demonstrably of order if $(u_3^w)^2$. Hence, we may conclude that, in addition to the parts that contribute to the usual rapid and return-to-isotropy terms, the part of the pressure p'' that will be correlated with $\mathbf{u}^t$ is a linear functional of $(\mathbf{u}^w)^2$.

We may take as a paradigm for this

$$\langle \mathbf{u}^t(\mathbf{x}')p''(\mathbf{x}')/\rho \rangle = \iint_{\text{surf}} \tfrac{1}{2}\langle \mathbf{u}^t(\mathbf{x}')(\mathbf{u}^w)^2(\mathbf{x}^a)\rangle \partial_3 G_0 dx_1 dx_2, \quad (111)$$

where the additional terms will be of the same form. Note that $\partial_3 G_0 < 0$. We may crudely parameterize this as

$$\frac{1}{\rho}\langle \mathbf{u}^t(\mathbf{x}')p''(\mathbf{x}')\rangle \approx -\frac{1}{2}\langle \mathbf{u}^t(\mathbf{x}')(\mathbf{u}^w)^2(\mathbf{x}')\rangle \quad (112)$$

and this will lead to the additional term in (46).

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REFERENCES

Benilov, A. Y., 1973: On the generation of turbulence in the ocean by surface waves. *Izv. Acad. Sci. USSR Atmos. Ocean. Phys.*, **9**, 293–303.

——, and M. M. Zaslavskiy, 1974: Determination of wave and turbulent components of random hydrodynamic fields in the marine atmospheric surface layer. *Izv. Acad. Sci. USSR Atmos. Ocean. Phys.*, **10**, 628–635.

Donelan, M. A., 1978: Whitecaps and momentum transfer. *Turbulent Fluxes Through the Sea Surface, Wave Dynamics and Prediction.* A. Favre and K. Hasselmann, Eds., Plenum Press, 273–286.

Hasselmann, K. and collaborators, 1973: Measurements of wind wave growth and swell decay during the joint North Sea wave project (JONSWAP), *Dtsch. Hydrogr. Z.*, **A12**, 95–103.

Kitaigorodskii, S. A., 1973: *The Physics of Air-Sea Interaction,* 237 pp. (Translated from Russian by Israel Program for Scientific Translations), [NTIS TT-72-50062].

——, 1981: The statistical characteristics of wind-generated short gravity waves. *Spaceborne Synthetic Aperture Radar for Oceanography,* R. Beal, P. Deleonibus and I. Kats, Eds., The Johns Hopkins University Press, 32–140.

——, 1982: The equilibrium ranges in wave spectra-physical arguments and experimental evidence for and against their existence. *Proc. of NICRM Symp. Wave Dynamics and Radio Probing of the Ocean Surface,* Miami Beach, (in press).

——, 1983: On the theory of the equilibrium range in the spectrum of wind-generated gravity waves. *J. Phys. Oceanogr.*

——, and Y. A. Miropolskii, 1968: Turbulent energy dissipation in the ocean surface layer. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmos. Okeano,* **4**, pp. 647–659.

——, M. A. Donelan, J. L. Lumley and E. A. Terray, 1983: Wave-turbulence interactions in the upper ocean. Part II: *J. Phys. Oceanogr.*, **13**, 1988–1999.

Monin, A. S., and A. M. Yaglom, Statistical characteristics of wave and turbulent components of the random velocity field in the marine surface layer. 1975: *Statistical Fluid Mechanics, Mechanics of Turbulence, Pts. I and II.* The MIT Press, 1260 pp.

Phillips, O. M., 1961: A note on the turbulence generated by gravity waves. *J. Geophys. Res.*, **66**, 2889–2893.

——, 1977: *Dynamics of the Upper Ocean,* Second ed., Cambridge University Press, 261 pp.

Stewart, R. W., and H. L. Grant, 1962: Determination of the rate of dissipation of turbulent energy near the sea surface in the presence of waves. *J. Geophys. Res.*, **67**, 3177–3180.