Steady Wind- and Wave-Induced Currents in the Open Ocean

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(Manuscript received 20 May 1982, in final form 21 October 1982)

ABSTRACT

Steady wind-drift currents in a deep viscous, rotating ocean are studied theoretically. The analysis is based on the Lagrangian description of motion.

A mean wind-stress at the surface yields the traditional Ekman current. In addition, the wind-stress is assumed to contain a fluctuating part which transfers energy to the surface waves and compensates for loss due to viscous dissipation. The induced drift due to such waves is investigated. The wave-drift depends on the eddy viscosity as well as the earth's rotation.

We assume a fully developed sea, and take the eddy viscosity to be proportional to the friction velocity times a characteristic depth. Hence the total current (Ekman current plus wave-induced current) can be expressed as functions of the wind speed. The results show that the magnitude of the total surface current lies between 3.1 and 3.4% of the wind speed at 10 m height for winds between 5 and 30 m s^{-1}. The deflection angle away from the wind direction varies from 23 to 30° in this range of wind speeds.

1. Introduction

It has been known for some time that the earth's rotation influences the mass transport (the Stokes drift) associated with surface waves in the open sea. Ursell (1950) in fact proved that for swell in an *inviscid* rotating ocean, the induced horizontal mean motion would be purely inertial. Hence, there is no net mass transport in this case. Similar conclusions were reached by Hasselmann (1970) and Pollard (1970).

The effect of viscosity alters this picture considerably. When there is no wind, i.e., in the case of swell, viscous dissipation must inevitably lead to attenuation of the wave field, as well as attenuation of the induced secondary flow. Taking these effects into account, it has been demonstrated by the author that this leads to a net mass transport associated with swell in a rotating ocean (Weber, 1983).

The presence of wind may result in energy transfer to the waves, which compensates the energy loss due to dissipation. Hence, the waves may remain permanent in the sense that their amplitude will not decay in time. An attempt to analyze the induced mean flow in an infinitely deep ocean due to such waves was done by Madsen (1978). His analysis, however, is incomplete since he applies Longuet-Higgins' (1953) stress-free condition at the free surface for this problem. Accordingly, the surface mass transport velocity obtained by Madsen has the deficiency that it tends towards infinity as the Ekman depth becomes much larger than the Stokes depth, as one would expect from Longuet-Higgins' analysis.

Analogous to Madsen (1978), the present analysis is based on the Lagrangian formulation of motion. The solutions are written as series expansions after a small parameter ϵ (Pierson, 1962), where ϵ essentially is proportional to the amplitude of the surface wave. Since the wave problem to $O(\epsilon)$ is identical to what one obtains by using an Eulerian formulation, the complete solution to $O(\epsilon)$ may be obtained directly from Lamb (1932). The application of a nonzero stress condition at the free surface in order to sustain the waves against frictional decay is shown to have a profound influence upon the spatially averaged second-order solution.

2. Mathematical formulation

We consider a homogeneous, incompressible viscous fluid rotating about the vertical axis with a constant angular velocity $f/2$, where f is the Coriolis parameter. The depth of the fluid is infinite, and the horizontal extent is unlimited. When undisturbed, the surface is horizontal. A Cartesian co-ordinate system is chosen such that the x-y-axes are situated at the undisturbed surface and the z-axis is positive upwards.

We describe the motion by using a Lagrangian formulation. Let a fluid particle (a, b, c) initially have coordinates (x_0, y_0, z_0). Its position (x, y, z) at later times will then be a function of a, b, c and time t. Velocity components and accelerations are given by (x_t, y_t, z_t) and (x_{tt}, y_{tt}, z_{tt}), respectively, where subscripts denote partial differentiation. By including rotation, the equations for conservation of momentum and mass can be written (Lamb, 1932; Pierson, 1962) as

$$x_{tt} - fy_t = -\frac{1}{\rho_0}\frac{\partial(p, y, z)/\partial(a, b, c)}{\partial(x_0, y_0, z_0)/(\partial(a, b, c)} + \nu\nabla^2 x_t, \tag{2.1}$$

$$y_{tt} + fx_t = -\frac{1}{\rho_0}\frac{\partial(x, p, z)/\partial(a, b, c)}{\partial(x_0, y_0, z_0)/\partial(a, b, c)} + \nu\nabla^2 y_t, \tag{2.2}$$

$$z_{tt} + g = -\frac{1}{\rho_0}\frac{\partial(x, y, p)/\partial(a, b, c)}{\partial(x_0, y_0, z_0)/\partial(a, b, c)} + \nu\nabla^2 z_t, \tag{2.3}$$

$$\frac{\partial(x, y, z)}{\partial(a, b, c)} = \frac{\partial(x_0, y_0, z_0)}{\partial(a, b, c)}, \tag{2.4}$$

where p is the pressure, ρ_0 the constant density, ν the (constant) coefficient of kinematic viscosity and g the acceleration due to gravity. The operator connected with the pressure terms and the continuity equation is the Jacobian, defined by

$$\frac{\partial(\ )}{\partial(a, b, c)} = \det\begin{vmatrix} \frac{\partial}{\partial a} & \frac{\partial}{\partial a} & \frac{\partial}{\partial a} \\ \frac{\partial}{\partial b} & \frac{\partial}{\partial b} & \frac{\partial}{\partial b} \\ \frac{\partial}{\partial c} & \frac{\partial}{\partial c} & \frac{\partial}{\partial c} \end{vmatrix}. \tag{2.5}$$

The Laplacian operator ∇^2 becomes rather complicated when expressed in Lagrangian form, involving the Jacobian of a Jacobian. The reader is referred to Pierson (1962) for the explicit expression.

The equations (2.1)–(2.4) will be solved by considering small perturbations from a basic state, i.e.,

$$\left.\begin{aligned} x &= a \\ y &= b \\ z &= c \\ p &= p_0 - \rho_0 gc \end{aligned}\right\}. \tag{2.6}$$

The free surface is given by $c = 0$ for all times. This is a linear kinematic boundary condition as opposed to the equivalent nonlinear Eulerian version. This shows some of the advantages by using a Lagrangian description for problems involving a freely moving boundary. Alternatively, in Eulerian form, one will have to use some sort of curvilinear coordinates along the free surface (Longuet-Higgins, 1953).

According to the adopted approach (Pierson, 1962), we write the solutions

$$\left.\begin{aligned} x &= a + \epsilon x^{(1)} + \epsilon^2 x^{(2)} + \cdots \\ y &= b + \epsilon y^{(1)} + \epsilon^2 y^{(2)} + \cdots \\ z &= c + \epsilon z^{(1)} + \epsilon^2 z^{(2)} + \cdots \\ p &= p_0 - \rho_0 gc + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \cdots \end{aligned}\right\}. \tag{2.7}$$

The expansion parameter ϵ is essentially proportional to the amplitude of the initial surface wave, as will be described in more detail later.

The free surface is given by $z = \zeta$ in Eulerian form. By the aid of (2.7), the appropriate Lagrangian description of the surface form becomes

$$\zeta = \epsilon z^{(1)} + \epsilon^2 z^{(2)} + \cdots, \quad c = 0. \tag{2.8}$$

Since we work in an infinitely deep ocean, all perturbation quantities are assumed to vanish when $c \to -\infty$.

We introduce vertical and horizontal external stresses $P^{(zz)}$ and $P^{(xz)}$ at the *sloping* free surface. For reasons which become clear in the next paragraph, we neglect differentiation with respect to b in the perturbations and put $y^{(1)} = 0$. Using the expansions (2.7) the dynamic boundary conditions in the vertical and horizontal directions at the free surface can be written, respectively, as

$$\begin{aligned} P^{(zz)} = -p_0 &+ \epsilon[-p^{(1)} + 2\mu z_{tc}^{(1)}] + \epsilon^2[-p^{(2)} \\ &+ \mu(2z_{tc}^{(2)} + 2x_a^{(1)}z_{tc}^{(1)} - 2x_c^{(1)}z_{ta}^{(1)} \\ &- x_{tc}^{(1)}z_a^{(1)} - z_{ta}^{(1)}z_a^{(1)})] + O(\epsilon^3), \quad c = 0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} P^{(xz)} = \epsilon\mu[x_{tc}^{(1)} &+ z_{ta}^{(1)}] + \epsilon^2[p^{(1)}z_a^{(1)} + \mu(x_{tc}^{(2)} \\ &+ z_{ta}^{(2)} + x_a^{(1)}x_{tc}^{(1)} - x_{ta}^{(1)}x_c^{(1)} + z_{ta}^{(1)}z_c^{(1)} \\ &- z_a^{(1)}z_{tc}^{(1)} - 2x_{ta}^{(1)}z_a^{(1)})] + O(\epsilon^3), \quad c = 0, \end{aligned} \tag{2.10}$$

where $\mu = \rho_0\nu$[see also Chang (1969)].

3. The permanent primary wave

The linearized version of (2.1)–(2.4), or the solution to $O(\epsilon)$, yields the primary wave. We are looking at ocean waves in the presence of wind, i.e., relatively short, high frequency gravity waves. This means that the problem can be considerably simplified. Typically, such waves will have wavelengths λ of $\sim$50 m and frequencies σ of $\sim 1\ s^{-1}$. With a value of the inertial frequency $f \sim 10^{-4}\ s^{-1}$, the Rossby radius of deformation for such waves is of the order 10^5 m. Hence, we can safely neglect the effect of rotation on the solution to $O(\epsilon)$.

Letting the wave propagate along the x-axis ($\partial/\partial b = 0$ in the perturbations) and taking $y^{(1)} = 0$ since we neglect rotation, the first-order equations then reduce to

$$\left.\begin{aligned} x_{tt}^{(1)} + gz_a^{(1)} &= -\frac{1}{\rho_0}p_a^{(1)} + \nu\nabla_L^2 x_t^{(1)} \\ z_{tt}^{(1)} + gz_c^{(1)} &= -\frac{1}{\rho_0}p_c^{(1)} + \nu\nabla_L^2 z_t^{(1)} \\ x_a^{(1)} + z_c^{(1)} &= 0 \end{aligned}\right\}, \tag{3.1}$$

where $\nabla_L^2 = \partial^2/\partial a^2 + \partial^2/\partial c^2$. We have here assumed that $\partial(x_0, y_0, z_0)/\partial(a, b, c) = 1 + O(\epsilon^2)$ which will be verified *a posteriori*.

Formally, (3.1) is identical to what one obtains from a linearized Eulerian description. Therefore, the solution of (3.1) can be obtained directly from Lamb (1932, p. 625). Following his procedure, we separate the solutions into an irrotational part $\phi^{(1)}$ and a rotational part $\psi^{(1)}$, such that

$$\left.\begin{aligned} x_t^{(1)} &= -\phi_a^{(1)} - \psi_c^{(1)} \\ z_t^{(1)} &= -\phi_c^{(1)} + \psi_a^{(1)} \end{aligned}\right\}. \quad (3.2)$$

Hence, from (3.1)

$$\nabla_L^2\phi^{(1)} = 0, \quad (3.3)$$

$$\psi_t^{(1)} = \nu\nabla_L^2\psi^{(1)}, \quad (3.4)$$

$$p^{(1)} = \rho_0\phi_t^{(1)} - \rho_0 g z^{(1)}. \quad (3.5)$$

Eqs. (3.3)–(3.4) have solutions of the form

$$\left.\begin{aligned} \phi^{(1)} &= Ae^{kc}e^{ika+nt} \\ \psi^{(1)} &= Be^{mc}e^{ika+nt} \end{aligned}\right\}, \quad (3.6)$$

where

$$m^2 = k^2 + n/\nu. \quad (3.7)$$

The surface elevation to lowest order then becomes

$$\zeta = \epsilon z^{(1)}(c = 0) = -\epsilon kn^{-1}(A - iB)e^{ika+nt}. \quad (3.8)$$

In the present study, we assume that the work by the wind stress at the surface compensates exactly the energy loss in the fluid due to viscous dissipation. Hence. the wave amplitude will not attenuate in time. This means that n must be purely imaginary, or

$$n = -i\sigma, \quad (3.9)$$

where σ is a real and positive frequency. The minus sign means that we let the waves propagate in the positive x-direction.

Following Lamb (1932), it is easy to show that (3.7) has an approximate solution

$$\left.\begin{aligned} m_r &= (\sigma/2\nu)^{1/2} \equiv \gamma \\ m_i &= -(\sigma/2\nu)^{1/2} \equiv -\gamma \end{aligned}\right\}, \quad (3.10)$$

where subscripts r and i denote real and imaginary parts, respectively. The solution above rests on the assumption

$$\frac{k}{\gamma} \ll 1. \quad (3.11)$$

This condition is very well fulfilled for the kind of waves we are looking at, even with a turbulent eddy viscosity in (3.10).

Let us assume that we have a *vanishing* horizontal stress to $O(\epsilon)$ at the surface, i.e.,

$$P^{(xz)} = 0, \quad c = 0,$$

or from (2.10)

$$x_{tc}^{(1)} + z_{ta}^{(1)} = 0, \quad c = 0. \quad (3.12)$$

Eqs. (3.2) and (3.6) then yield approximately

$$\begin{aligned} B &= 2\nu k^2\sigma^{-1}A, \\ \sigma^2 &= gk. \end{aligned} \quad (3.13)$$

Take a permanent surface wave which propagates along the positive x-axis with amplitude ζ_0, i.e.,

$$\zeta = \zeta_0 \sin(ka - \sigma t). \quad (3.14)$$

For this wave to stay permanent, we note from (2.9) that the vertical wind-stress component at the surface to $O(\epsilon)$ must vary as

$$P^{(zz)} = -p_0 - 4\mu k\sigma\zeta_0\cos(ka - \sigma t). \quad (3.15)$$

This means maximum numerical vertical stress at the rear and minimum stress at the front of the wave.

The alternative stress-distribution considered by Lamb (1932) of vanishing normal stress and a suitably distributed horizontal stress along the surface, will be discussed later.

We normalize the solutions to $O(\epsilon)$ by taking $A = 1$ in (3.13). By the aid of (3.8), (3.13) and (3.14) this means that

$$\epsilon = \zeta_0\sigma k^{-1}, \quad (3.16)$$

as stated in Section 2. The solutions to $O(\epsilon)$ can then be written as

$$\left.\begin{aligned} x^{(1)} &= \frac{k}{\sigma}e^{kc}\cos(ka - \sigma t) \\ &\quad + \frac{k^2}{\gamma\sigma}e^{\gamma c}[\sin(ka - \gamma c - \sigma t) \\ &\qquad - \cos(ka - \gamma c - \sigma t)] \\ z^{(1)} &= \frac{k}{\sigma}e^{kc}\sin(ka - \sigma t) \\ &\quad - \frac{k^3}{\gamma^2\sigma}e^{\gamma c}\cos(ka - \gamma c - \sigma t) \\ p^{(1)} &= \frac{\rho_0 g k^3}{\gamma^2\sigma}e^{\gamma c}\cos(ka - \gamma c - \sigma t) \end{aligned}\right\}, \quad (3.17)$$

where we have utilized the fact that $k \ll \gamma$. We note from this that $\partial(x_0, y_0, z_0)/\partial(a, b, c) = 1 + O(\epsilon^2)$ as previously assumed.

4. The steady secondary mean flow

For the equations to $O(\epsilon^2)$ we again refer to Pierson (1962). Averaging the second order equations over one wavelength, and including rotation, we obtain for the steady horizontal mean flow:

$$\begin{aligned} \nu\bar{x}_{tcc}^{(2)} + f\bar{y}_t^{(2)} &= -\frac{1}{\rho_0}\overline{p_a^{(1)}x_a^{(1)}} - \frac{1}{\rho_0}\overline{p_c^{(1)}z_a^{(1)}} \\ &\quad + \nu[2\overline{x_a^{(1)}x_{taa}^{(1)}} + 2\overline{z_c^{(1)}x_{tcc}^{(1)}} + 2\overline{z_a^{(1)}x_{tac}^{(1)}} \\ &\quad + 2\overline{x_c^{(1)}x_{tac}^{(1)}} + \overline{x_{ta}^{(1)}\nabla_L^2 x^{(1)}} + \overline{x_{tc}^{(1)}\nabla_L^2 z^{(1)}}], \end{aligned} \quad (4.1)$$

$$\nu\bar{y}_{tcc}^{(2)} - f\bar{x}_t^{(2)} = 0, \quad (4.2)$$

where the overbar denotes spatial average. We have here assumed that there is no mean horizontal pressure gradient to $O(\epsilon^2)$.

Let us discuss the physical interpretation of the Lagrangian mean motion. Since $x^{(1)}$, $z^{(1)}$ are periodic in the horizontal [and $y^{(1)} = 0$] we obtain from (2.7) that the *mean* displacement of a particle to this order can be written

$$\left.\begin{aligned}\bar{x} &= a + \epsilon^2\bar{x}^{(2)}\\ \bar{y} &= b + \epsilon^2\bar{y}^{(2)}\\ \bar{z} &= c + \epsilon^2\bar{z}^{(2)}\end{aligned}\right\} . \quad (4.3)$$

Take now a thin infinitely long fluid tube initially laying parallel to the direction of averaging, and with its center of mass at the position (a, b, c). The mean motion defined above can be related to the displacement of this center of mass (Andrews and McIntyre, 1978). Hence it is a direct measure of the net mass transport associated with the waves.

For convenience, we define horizontal mean flow components u, v as

$$u = \epsilon^2\bar{x}_t^{(2)}, \quad v = \epsilon^2\bar{y}_t^{(2)}. \quad (4.4)$$

Accordingly, (u, v) represent the horizontal velocity of a particle associated with the previously defined center of mass. Furthermore, as usual in problems involving rotation, we introduce a complex velocity W by

$$W = u + iv. \quad (4.5)$$

By inserting for $x^{(1)}$, $z^{(1)}$ and $p^{(1)}$ from the (3.17) into (4.1) and (4.2), and using the definitions above, the equation for the mean motion reduces to

$$\nu W_{cc} - ifW = \nu\zeta_0{}^2\sigma k^3[4e^{2kc} - 4\gamma k^{-1}e^{\gamma c} \times (\cos\gamma c - \sin\gamma c)]. \quad (4.6)$$

Here again we have utilized the fact that $\gamma \gg k$. This means that terms proportional to $\exp(\gamma c)$, $(k/\gamma)\exp(\gamma c)$ etc. have been neglected inside the parenthesis on the rhs of (4.6). This can be done, because when integrating, one finds that these terms only introduce small corrections to the mean velocity or the mean velocity gradient.

An equation similar to (4.6) was derived by Madsen (1978). The last term in the parenthesis was lacking, however, due to an incomplete treatment of the first order problem. In what follows, we shall demonstrate that this surface boundary layer term is very important, and cannot be neglected.

We assume a constant mean horizontal wind stress τ in the x-direction at the surface. By averaging the boundary condition (2.10) we obtain, after some algebra,

$$W_c = -2\zeta_0{}^2\sigma k^2 + u_*^2\nu^{-1}, \quad c = 0. \quad (4.7)$$

Here $u_* = (\tau/\rho_0)^{1/2}$ is the friction velocity. We note that this boundary condition, with $u_* = 0$, is different from that obtained by Longuet-Higgins (1953, 1960) and Ünlüata and Mei (1970) for steady wave drift in the absence of rotation, and by Weber (1983) for the attenuated problem including rotation. The cases referred to consider a different problem, however, in which both the horizontal and vertical stresses at the surface were assumed to vanish. This leads to $W_c = 0$ at $c = 0$ for the mass transport velocity.

At infinity we must require

$$W \to 0, \quad c \to -\infty. \quad (4.8)$$

5. The non-rotating case

To simplify the discussion, we shall briefly look at the non-rotating case ($f = 0$, $v = 0$, $u = W$). Furthermore, let us assume that there is no mean wind stress at the surface, i.e., $u_* = 0$. If $\nu \neq 0$, (4.6) may be integrated to yield

$$u_c = \zeta_0{}^2\sigma k^2(2e^{2kc} - 4e^{\gamma c}\cos\gamma c) + K, \quad (5.1)$$

where K is a constant of integration. From the boundary condition (4.7) with $u_* = 0$, we obtain $K = 0$. Hence, from (5.1)

$$u = \zeta_0{}^2\sigma k[e^{2kc} - 2k\gamma^{-1}e^{\gamma c}(\cos\gamma c + \sin\gamma c)], \quad (5.2)$$

which fulfills the condition $u \to 0$, $c \to -\infty$. Below a thin boundary layer of thickness γ^{-1} this is just Stokes' classic solution (Stokes, 1847). To get an idea of the scale involved, we put $\sigma \sim 1\ \mathrm{s}^{-1}$ and take an eddy viscosity of order $10^2\ \mathrm{cm}^2\ \mathrm{s}^{-1}$. From (3.10) we then obtain $\gamma^{-1} \sim 10$ cm.

Since $k\gamma^{-1} \ll 1$, the last term in (5.2) introduces only a small correction to Stokes solution. However, the derivative of this term is of order *unity*, and hence it plays a crucial role in connection with the boundary condition at the surface.

Let us now return to the second way of sustaining a wave train against friction, as discussed by Lamb (1932). The vertical component of the dynamic shear stress now vanishes at the surface, while the horizontal component varies periodically, i.e., to $O(\epsilon)$:

$$\left.\begin{aligned}P^{(zz)} &= -p_0\\ P^{(xz)} &= 4\mu k\sigma\zeta_0\sin(ka - \sigma t)\end{aligned}\right\} . \quad (5.3)$$

In the computations of the induced mean motion to $O(\epsilon^2)$ the assumptions above will result in a *plus* sign in the last term on the right-hand side of (4.6). Furthermore, the boundary condition (4.7) will be replaced by

$$W_c = 4\zeta_0{}^2\sigma k^2 + u_*^2\nu^{-1}, \quad c = 0, \quad (5.4)$$

which is the same boundary condition as used by Madsen (1978). For the non-rotating case with $u_* = 0$, the constant of integration K in (5.1) now becomes $K = -2\zeta_0{}^2\sigma k^2$ from (5.4). Accordingly, by repeated integration of (5.1), we obtain that the mass transport velocity increases with depth below the Stokes layer, i.e., there is no way of getting a bounded solution when the depth tends to infinity. Therefore, to have a physically acceptable solution in the case of no rotation, we must assume that the wind-stress distribution at the surface to $O(\epsilon)$ must be of the form (3.12), (3.15).

6. The rotating ocean

We now return to the full problem including rotation. Three vertical length scales appear in the problem. By definition they are

$$\left.\begin{array}{ll} L = 1/2k & \text{(Stokes depth)} \\ l = \gamma^{-1} = (2\nu/\sigma)^{1/2} & \text{(Vorticity layer depth)} \\ D = (2\nu/f)^{1/2} & \text{(Ekman depth)} \end{array}\right\} . \quad (6.1)$$

It is natural to separate the current $W^{(E)}$ induced by the mean stress at the surface (the Ekman current; Ekman 1905) from the wave-induced current $W^{(w)}$, so we write

$$W = W^{(E)} + W^{(w)}. \quad (6.2)$$

From (4.6) and the boundary conditions (4.7) and (4.8), together with the fact that $l^2/D^2 = f/\sigma \ll 1$, we find

$$W^{(E)} = \frac{2u_*^2}{(1+i)fD} e^{(1+i)c/D}, \quad (6.3)$$

$$W^{(w)} = \frac{\zeta_0^2\sigma k}{(1+i)}\left(\frac{D}{L}\right)\left(1 - \frac{1}{1-2iL^2/D^2}\right)e^{(1+i)c/D} + \frac{\zeta_0^2\sigma k}{1-2iL^2/D^2}e^{2kc} - 2\zeta_0^2\sigma k\left(\frac{k}{\gamma}\right)e^{\gamma c} \times (\cos\gamma c + \sin\gamma c). \quad (6.4)$$

We note the important result that the wave-induced flow $W^{(w)}$ is influenced by the earth's rotation. Generally it is *not* directed along the direction of wave propagation.

The result (6.4) differs essentially from that of Madsen (1978). In particular, this becomes obvious in the asymptotic limit of large Ekman depth, i.e., $D \gg L$. Then

$$W^{(w)} = \zeta_0^2\sigma k\left[e^{2kc} - \frac{2i}{1+i}\left(\frac{L}{D}\right)e^{(1+i)c/D}\right] + O\left(\frac{k}{\gamma}\right).$$

In this case, the wave-induced flow is only a small modification of the classic Stokes drift. The reason is, of course, that now the depth over which the Stokes flow varies (the Stokes depth) is so small that the Ekman veering is negligible. This is contrary to Madsen's result of an infinite surface mass transport velocity for this limit.

For the other limit, $L/D \to \infty$, or $\nu \to 0$ when k is finite, we obtain $W^{(w)} \to 0$, which confirms Ursell's (1950) result of no net mass transport in an inviscid rotating ocean (see also Madsen, 1978; Weber, 1983).

In the general case, the Stokes depth and the Ekman depth will both be functions of the wind speed U_{10} at 10 m height, say. For a well-developed sea the dominant wavelength λ for surface waves is connected with the wind at 19.5 m above sea level by

$$\lambda = (2.803 \times 10^{-3} U_{19.5}^2 \text{ s}^2 \text{ cm}^{-2}) \text{ cm} \quad (6.5)$$

(Jacobs, 1978; Delnore, 1980). By assuming a logarithmic velocity profile in the air above the sea surface, we find

$$U_{19.5} = U_{10}[1 + (c_{10}{}^{1/2}/\kappa)\ln(19.5/10)], \quad (6.6)$$

where $\kappa = 0.4$ (Pierson, 1964). The drag coefficient c_{10} depends on the wind speed as well as the vertical stability of the air. For the present purpose we take $c_{10} = 1.8 \times 10^{-3}$ when $U_{10} < 15$ m s^{-1} and $c_{10} = 2.7 \times 10^{-3}$ when $U_{10} > 20$ m s^{-1}, with a linear interpolation in between. This choice has been shown to yield excellent agreement between computed wind surge and observed surface elevation in the North Sea (Timmerman, 1977). Hence the Stokes depth is determined by the wind at 10 m height through (6.5) and (6.6).

For the friction velocity at the surface, we obtain

$$u_* = \left(\frac{\rho_a c_{10}}{\rho_0}\right)^{1/2} U_{10}, \quad (6.7)$$

where ρ_0 and ρ_a are the densities of sea water and air, respectively.

The most difficult problem in this connection is to relate the eddy viscosity to the wind speed. By analogy with shear generated turbulence in pipes etc., the eddy viscosity should be proportional to the friction velocity. Furthermore, it should increase in the shear zone as we move away from the boundary. In the present paper, we have neglected this vertical variation. On average, the eddy size should be proportional to a typical overall dimension H of the system. Arguing along these lines, Long (1977) uses the laboratory measurements by Robertson (1959), and obtains

$$\nu = 0.12u_*H. \quad (6.8)$$

For the present problem, H is a typical vertical scale of motion, which we take to be the arithmetic mean of the Ekman depth and the Stokes depth, i.e., $H = (D + L)/2$. By utilizing (6.1), (6.6) and (6.7), this equation determines the eddy viscosity and hence the Ekman depth as function of the wind speed at 10 m height.

The present theory is assumed to be valid for fully developed waves. We take the steepness of such waves to be 0.055, as often done. Hence the wave amplitude ζ_0 is determined as function of wind speed through (6.5). Furthermore, we have chosen a typical value of 1.26×10^{-4} s^{-1} for the Coriolis parameter in the

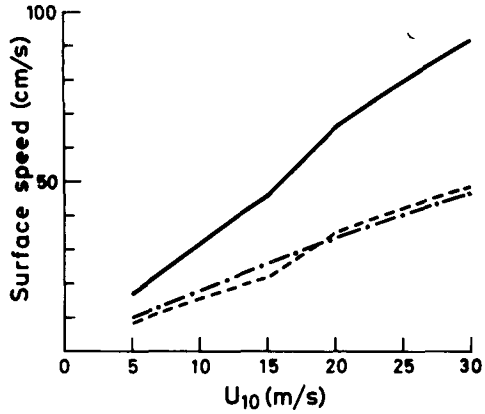

FIG. 1. Speed of total surface current versus wind speed at 10 m height (solid line). Dashed line indicates contribution from Ekman flow [(6.3)] and dot-dash line the contribution from wave-induced flow [(6.4)].

computations. As before, the wind-stress and wave propagation are along the x-axis.

The surface current is obtained from (6.2) by setting $c = 0$. In Fig. 1, we have displayed the speed of the steady surface current as function of wind speed at 10 m. The two broken curves are the Ekman current (6.3) and the wave-induced flow (6.4), respectively. We note the interesting result that these contribute nearly equally to the total current at the surface. The relative increase in the Ekman part between 15 and 20 m s^{-1} is due to the increase in the drag coefficient in this region. In the displayed domain the total current has a value which lies between 3.1 and 3.4% of the wind speed.

The deflection of the steady surface current to the right of the wind direction (on the Northern Hemisphere) is displayed in Fig. 2, where the deflection angle is plotted as function of the wind speed. The broken curves are again contributions from the Ekman current and the wave-induced current, respectively. Since we here have assumed constant eddy viscosity, we obtain the familiar 45°-deflection of the Ekman current for a deep ocean. The moderate deflection to the right of the wave-induced current (4–14°) is a novel result, however. The graph shows that the resulting total surface current is deflected from ~23 to 30° for winds between 5 and 30 m s^{-1}.

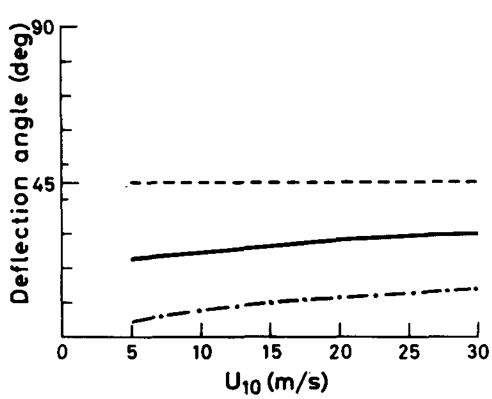

FIG. 2. Deflection angle of total surface current versus wind speed at 10 m height (solid line). Dashed line indicates contribution from Ekman flow [(6.3)] and dot-dash line the contribution from wave-induced flow [(6.4)].

In Fig. 3, we have displayed the variation with depth of the total drift current for $U_{10} = 10$ m s^{-1} (solid line). We note the expected veering to the right with depth. The broken line is the Ekman part (6.3) of the total current. The wave-induced current vector is obtained immediately from the graph by an orientated straight line from the broken to the solid curve at a specific depth. We note that the magnitude of the wave-induced drift decreases rather quickly with depth. At five meters the current speed is ~10% of the surface value. For the present typical wind speed, the eddy viscosity from (6.8) becomes 172 cm^2 s^{-1}, and the dominant wavelength for a fully developed sea (6.5) is 32 m. Accordingly, $D \approx 16.5$ m and $L \approx 2.6$ m from (6.1). This means that the Ekman current will dominate at depths larger than L, which also is obvious from the graph in Fig. 3.

7. Summary and discussion

We have investigated theoretically steady mean drift currents in a homogeneous, deep, viscous, rotating ocean. The analysis is based on the Lagrangian description of motion. The currents result from wind blowing at the surface. A constant mean wind stress yields the traditional steady Ekman solution. In addition, we assume that the vertical wind stress varies over a typical wavelength for a fully developed sea. The variation is such that there is an energy transfer to the waves which compensates for loss due to viscous dissipation. This results in non-decaying small amplitude surface waves. Assuming the sea to be a narrow band process, we consider the induced mass transport due to a single dominant wave component.

In the Lagrangian formulation, the solutions are written as series expansions after a small parameter ϵ. This parameter is essentially proportional to the amplitude of the surface wave. The wave-induced mean flow (the mass transport) is found by averaging the solution to $O(\epsilon^2)$ over one wavelength. The analysis assumes that the parameter $k\gamma^{-1} \equiv k(2\nu\sigma^{-1})^{1/2}$ is much smaller than one, and that the frequency σ of the waves is much larger than the inertial frequency.

The results show that the wave-induced steady mean flow is influenced by the earth's rotation. This was also pointed out by Madsen (1978), although his results were not correct. In general one cannot just use Stokes' (1847) result to assess the relative importance of wave-induced motion in a rotating ocean as done by Bye (1967) and Kenyon (1969). However, as shown here, if the Ekman depth is much larger

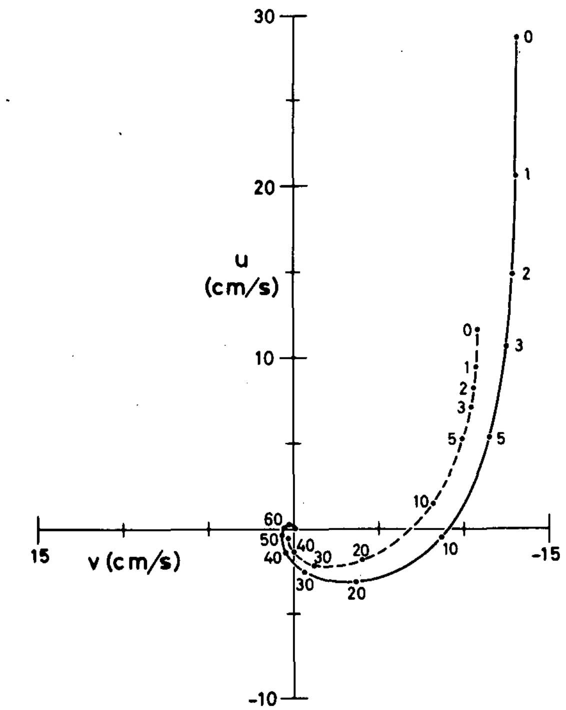

FIG. 3. Hodograph showing the depth dependence of the total drift current [(6.2)] for U_{10} = 10 m s^{-1} (solid line). The numbers on the graph denote depth in meters. The broken line is the contribution from the Ekman current (6.3).

than the Stokes depth, the total current can be approximately written as the sum of Ekman's and Stokes' classic solutions.

The present analysis shows that the wave-induced current and the Ekman current at the surface have approximately the same magnitude. The total current amounts to ~3% of the wind speed at 10 m height, and the deflection to the right of the wind is from 23 to 30° for winds between 5 and 30 m s^{-1}. However, in a real ocean, the eddy viscosity will increase with depth in the top layer. This causes a smaller deflection of the surface Ekman current to the right of the wind direction (Madsen, 1977; Weber, 1981).

It is a commonly employed rule of thumb that oil slicks are advected with a velocity which is ~3% of the wind speed, and in a direction which is approximately 15° to the right of the wind (on the Northern Hemisphere). If one takes the effect of a variable eddy viscosity into account, it is possible that an analysis like the present one will yield results for the surface current which are even closer to this empirical rule. When applied to a real situation, one must, of course, also remember that the wind changes in direction as well as in strength. Also, for a constant wind, the current will approach the steady limit obtained here through damped inertial oscillations.

Finally it should be stressed that the present theory does not involve any assumptions concerning the generation and growth of wind waves, as described by Miles (1957), Phillips (1957), Longuet-Higgins (1969) and others. The present equilibrium theory rests on the assumption that the growth has stopped, and that there is a balance between loss and gain of energy for the particular wave in question. Although this clearly is an idealized situation, it still may prove to form a basis for a more realistic theory of wind-drift currents in the open ocean.

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