"Pycnobathic" Currents over the Upper Continental Slope*

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ABSTRACT

The dynamic interaction of a sloping seafloor with along-isobath density variation is calculated for cases involving a sharp pycnocline and a surface-to-bottom front. Pycnocline depth is supposed to vary in the alongshore direction only, over a sloping plane seafloor; the bottom trace of the surface-to-bottom front is supposed to cut across isobaths. The calculations are diagnostic and make use of linearized equations with a linear bottom friction term.

The results illustrate the manner in which cross-isobath baroclinic flow is converted at the slope into a barotrophic flow field, with velocities nearly parallel to the isobaths. The "forward" portion of the slope (that which lies in the direction of Kelvin wave propagation) only is affected, if there is only one inflow or outflow region. However, for equal and opposite inflow and outflow, a closed circulation pattern arises connecting these two regions, accompanied by a secondary closed circulation cell forward of both the inflow and outflow legs. The pressure distribution at the coast is a strongly filtered and phase-shifted version of the off-shore steric-height variation.

In the example of a surface-to-bottom front, the divergence of the intensifying baroclinic flow over increasing depth is again drawn from the forward sector, or discharged into that sector when the baroclinic flow is convergent, i.e., passes from deep into shallow water.

1. Introduction

A previously neglected aspect of ocean circulation has recently begun to attract attention: this is the interaction of a baroclinic current with a sloping ocean floor. Hansen (1959) has already drawn attention to the potential importance of departures from Ekman's "law of parallel solenoids", i.e., of the variation of density along isobaths. Welander (1959) discussed the problem further, and pointed out that along-isobath variations of density introduce an extra pressure term into the vorticity tendency equation. The importance of this effect was later demonstrated by "diagnostic" numerical calculations of large-scale ocean circulation of Sarkisyan and Ivanov (1971) and Holland and Hirschman (1972); see also the review of Holland (1977). A recent paper by Rattray (1982) contains a particularly clear application of the diagnostic-modeling approach to the circulation of the Southern Ocean, designed to minimize inaccuracies due to the imperfectly known distribution of bottom density.

In simplest terms, the interaction in question between bottom slope and density field arises when the baroclinic transport (defined as by Fofonoff, 1962, but without postulating bottom density constant) has a component along the depth gradient. The baroclinic transport is then divergent, so that a barotropic transport field has to arise to restore mass balance (Shaw and Csanady, 1983). In the vorticity tendency equation the divergence becomes translated into vortex stretching terms, due respectively to the along-isobath gradients of density and pressure.

While the basic physical cause of the interaction between bottom slope and density field is thus clear enough, very little is known about its effects, qualitative and quantitative. The diagnostic numerical models of large-scale circulation offer some tantalizing clues, but they are complex and they also exhibit some puzzling discrepancies with observation (Sarmiento and Bryan, 1982). It would be clearly desirable to examine the interaction between bottom slope and density field with the aid of analytical models so simple that the physical consequences become quite transparent. This is the objective of the present paper.

The terminology of the phenomenon in question has not so far evolved sufficiently to avoid the need for very lengthy, clumsy or unfelicitous expressions. "Bottom slope–density field interaction" is not likely to stick any more than "baroclinic current–ocean floor interaction" or "baroclinity-topography interaction". Some numerical modelers have referred to the phenomenon in question as "JEBAR" (Joint Effect of Baroclinity and Relief). A better alternative may be to call the (barotropic) currents induced by the interaction "pycnobathic currents". This term was suggested by a colleague as a replacement for "pycnongraphic currents", which was my first tentative proposal.

The most likely places to find pycnobathic currents are steep continental slopes, which may be expected
to be intersected by an uneven oceanic pycnocline. Huthnance (1984) gives a general discussion of this possibility and suggests that some observed upper-slope currents (e.g., off Scotland) are pycnobic. He then goes on to consider the more difficult question, how along-isobath density variations can coexist with the currents they induce. The aim of the calculations below is more modest, being confined to exhibiting the effects of certain simple prescribed pycnocline shapes, but in specific detail, not in terms of generalitys.

The prototype example to be discussed is a pycnocline possessing an alongshore gradient that is perpendicular to the gradient of the seafloor. This idealizes a case thought to exist on the west coast of North America, where the steric level of the ocean varies in the alongshore direction (Reid and Mantyla, 1976; see Fig. 1 here). As Montgomery (1969) and Sturges (1974) before them, Reid and Mantyla suppose that the same alongshore sea-level variation occurs right at the coast. This amounts to assuming that the main pycnocline, the alongshore setup of which is responsible for the level variations in question, runs into the continental slope more or less along straight generators. (And also that the pressure remains constant across the continental shelf.) This is not necessarily the case, and certainly not the case off the Carolina coast, for example, where the Sargasso Sea pycnocline rises sharply near the coast. However, as Hickey and Pola (1983) have pointed out, the west-coast alongshore gradients implied by Reid and Mantyla’s results are not inconsistent with the observed circulation on the west-coast continental shelf, which should respond to such gradients. The alongshore pressure gradients necessary to drive the southwestward flow along the east-coast continental shelf north of Cape Hatteras have also been attributed to deepwater effects (Csany, 1978; Beardsley and Winant, 1979), most probably an alongshore pycnocline setup such as previously deduced from hydrographic observations by Sturges (1974).

Supposing that these inferences are correct, and that an uneven pycnocline indeed runs into the continental slope along these coasts, and ignoring the question of how such a density field may be maintained in equilibrium, the calculations below are designed to show what steady-state circulation arises in consequence over the continental slope, and how exactly coastal sea levels are affected. The main dissipative effect, which ultimately counteracts the circulation-inducing influence of the along-isobath density variation, is taken to be bottom friction. This is thought to be realistic for weak flow in contact with the bottom.

A second example treated is a surface-to-bottom front, the bottom trace of which cuts across isobaths. Such fronts are observed in a number of shelf-seas, the one over the east-coast continental shelf stretching out over the upper slope. Being induced ultimately by runoff from land (Kao, 1981; Csany, 1984a), such fronts must cut across isobaths somewhere. The distribution of the associated pycnobic currents should prove to be illuminating. The two rather different examples, together with earlier results on currents induced by density gradients associated with a freshwater plume (Csany, 1984b) should, between them, throw some light on the physics of pycnobic currents.

2. Basic idealizations

The theoretical approach builds upon a series of earlier contributions on problems of frictionally con-

![Fig. 1. Alongshore variation of coastal sea level on the west coast of North America, deduced from dynamic heights in deep water just offshore. From Reid and Mantyla (1976).](image)
trolled steady circulation over a sloping continental shelf, summarized in a recent text (Csanyi, 1982). A method of calculating density-driven circulation outlined elsewhere (Csanyi, 1979, referred to as C1) will be applied. The problems to be treated are made tractable by the neglect of momentum advection, of the variation of Coriolis parameter with latitude, and of the cross-isobath component of the bottom stress. The bottom topography of a shelf or slope will be idealized by an infinite sloping plane, the density distribution by two layers of constant density. These simplifications bring with them some limitations on the validity of the theory.

As discussed in greater detail in earlier studies of this kind, the neglect of the cross-isobath component of bottom stress is easily justified, because this component competes with the geostrophic balance of the principal along-isobath flow. The cross-isobath component of bottom velocity, \( u_b \), need only be postulated moderately weaker than the along-isobath component \( v_b \):

\[
\frac{u_b}{v_b} < 1. \tag{1}
\]

Linearized bottom-stress components are taken to be \( (\tau_{ub}, \nu_{vb}) \), with \( \tau \) a resistance coefficient, equal to a drag coefficient times average fluctuating velocity magnitude.

The variation of Coriolis parameter with latitude \( \beta \) introduces a term into the important vorticity tendency equation which, except for an east–west coast, is of order \( \beta v H \) with \( H \) water depth. Over a steep continental slope one expects the vortex stretching term associated with cross-isobath flow to be much more important, i.e.:

\[
\left| \frac{\beta v H}{f \nu_s} \right| \ll 1, \tag{2}
\]

where \( s = dH/dx \) is bottom slope.

Provided that \( u/v \) is of the same order as \( u_b/v_b \) this can be satisfied only if

\[
\left| \frac{\beta H}{sf} \right| \ll \left| \frac{u_b}{v_b} \right| < 1. \tag{3}
\]

Typically, \( u_b/v_b \) might be of order 0.1, \( sf = 10^{-6} \) s\(^{-1}\) over a steep midlatitude continental slope, \( \beta = 10^{-11} \) m\(^{-1}\) s\(^{-1}\). Equation (3) can then be satisfied for \( H \) up to about 1 km, i.e., over the upper slope, but not in much deeper water.\(^1\) Over a midlatitude continental shelf, \( sf \) is an order of magnitude smaller, but the depth is only of order 100 m, so that (3) is again satisfied. Over a low-latitude shelf or slope \( \beta H/sf \) is \( O(0.1) \), but earth rotation is generally no longer a strong enough effect to ensure that the cross-isobath bottom stress is unimportant. Equation (2) may then still be valid, except of course very close to the equator, but without the simplification of neglecting cross-isobath bottom stress the circulation problem becomes somewhat more complex. Low-latitude shelf and slope problems are not further considered here.

Observed currents over the upper continental slope, not considering western boundary currents, have along-isobath velocities of order 0.1 m s\(^{-1}\). Along-isobath surface elevation gradients are \( O(10^{-7}) \), implying geostrophic cross-isobath velocities \( O(0.01 \text{ m s}^{-1}) \). Apart from satisfying the above discussed constraints, these velocities are also small enough for the Rossby numbers to be negligible:

\[
\frac{v}{fL_y} \ll 1, \quad \frac{u}{fL_x} \ll 1, \tag{4}
\]

where \( L_x, L_y \) are across- and along-isobath length scales, typically \( 10^4 \) and \( 10^6 \) m. The along-isobath length scale is imposed by forcing, while, with the present idealizations, \( L_x \) arises from the solution of the problem, and its order of magnitude has to be verified \emph{a posteriori} in each case investigated. Note especially that in the examples calculated below any scales imposed by shelf-slope geography are neglected (principally the width of the shelf region of moderate bottom gradient). The inclusion of such a scale in the analysis might significantly modify the results.

Another major idealization adopted in the calculations below, more drastic than any so far discussed, is the supposition that a certain known density distribution is consistent with the calculated flow fields. This so-called diagnostic approach is justified by an appeal to observational evidence, such as discussed earlier, but is of course ultimately unsatisfactory: the fluid might conveniently solve its own equations for the advection and diffusion of heat and salt, but our understanding of the key controls remains incomplete until these equations are also made part of the model. The results obtained below with the aid of the diagnostic approach should be viewed in this light. As already pointed out, recent work by Huthnance (1984) addresses some limitations of the diagnostic approach.

3. Governing equations

Steady, low Rossby-number flow over an upper continental slope will be taken to be subject to the equations of motion:

\[
-f u = -g \frac{\partial \tau_r}{\partial x} - g \int_0^x \frac{\partial \sigma_g}{\partial x} dz + \frac{1}{\rho_0} \frac{\partial \tau_x}{\partial z}, \tag{5}
\]

\[
f u = -g \frac{\partial \tau_r}{\partial y} - g \int_0^y \frac{\partial \sigma_g}{\partial y} dz + \frac{1}{\rho_0} \frac{\partial \tau_y}{\partial z}
\]

\(^1\) Unless, of course, the flow is trapped near the bottom, to depths of order 1 km; cf. the case discussed by Stommel and Arons (1972).
Here $\rho_0$ is reference density, $\epsilon_p = (\rho - \rho_0)/\rho_0$ the proportionate density excess, $\zeta$ is surface elevation and $(\tau_x, \tau_y)$ are the components of Reynolds shear stress in horizontal planes. Only the shear stress arising from flow over the bottom will be considered, and bottom stress will be parameterized as discussed here and in C1:

$$\frac{T_{\phi b}}{\rho_0} = \frac{rg}{f} \left( \frac{\partial \epsilon_p}{\partial x} + \int_{-H}^{0} \frac{\partial \epsilon_p}{\partial x} \, dz \right),$$  \hspace{1cm} (6)

where $r$ is a bottom resistance coefficient with dimension of velocity. The cross-isobath component of the bottom stress is ignored because it is smaller, and is also less important as it competes in the first Eq. (5) with the geostrophic balance of the principal along-isobath flow.

Depth integration of the equations of motion now results in the following expressions for the components of transport (depth-integrated velocity):

$$U = -\frac{gH}{f} \left[ \frac{\partial \zeta}{\partial y} + \int_{-H}^{0} (1 + z/H) \frac{\partial \epsilon_p}{\partial y} \, dz \right] - \frac{rg}{f} \left( \frac{\partial \epsilon_p}{\partial x} + \int_{-H}^{0} \frac{\partial \epsilon_p}{\partial x} \, dz \right),$$

$$V = \frac{gH}{f} \left[ \frac{\partial \zeta}{\partial x} + \int_{-H}^{0} (1 + z/H) \frac{\partial \epsilon_p}{\partial x} \, dz \right],$$  \hspace{1cm} (7)

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

where the equation of continuity has also been written. Continuity is satisfied by the introduction of a transport streamfunction $\psi$ such that

$$U = -\frac{\partial \psi}{\partial y}, \hspace{1cm} V = \frac{\partial \psi}{\partial x}. \hspace{1cm} (8)$$

The topography of the upper continental slope will also be idealized: as usual in simple models of shelf circulation, the coast will be supposed straight, the depth a function of the cross-isobath coordinate alone;

$$H = H(x). \hspace{1cm} (9)$$

Upon substituting the first two Eqs. (7) into the third an equation is now obtained for the pressure (surface elevation) field:

$$\frac{r}{f} \frac{\partial^2 \zeta}{\partial x^2} + \frac{dH}{dx} \frac{\partial \zeta}{\partial y}$$

$$= - \frac{dH}{dx} \int_{-H}^{0} \frac{\partial \epsilon_p}{\partial y} \, dz - \frac{r}{f} \frac{\partial \epsilon_p}{\partial x} \int_{-H}^{0} \frac{\partial \epsilon_p}{\partial x} \, dz. \hspace{1cm} (10)$$

Following the recipe of C1, the surface elevation will be regarded as the sum of a steric elevation $\zeta_1$ and a residual field $\zeta_2$, the latter arising from the along-isobath gradient of bottom density, as will become apparent. The steric elevation is defined as

$$\zeta_1 = -\int_{-H}^{0} \epsilon_p \, dz - \int_x^{\infty} \frac{dH}{dx} \, dx,$$  \hspace{1cm} (11)

where $\epsilon_p$ is the value of $\epsilon_p$ at $z = -H$, and it has been supposed that $\epsilon_b$ vanishes at large $x$.

Substituting $\zeta = \zeta_1 + \zeta_2$ into Eq. (10), with $\zeta_1$ as defined above, one arrives at the following equation for the residual elevation field:

$$\frac{r}{f} \frac{\partial^2 \zeta_2}{\partial x^2} + \frac{dH}{dx} \frac{\partial \zeta_2}{\partial y} = \frac{dH}{dx} \int_{-H}^{0} \frac{\partial \epsilon_b}{\partial y} \, dx. \hspace{1cm} (12)$$

As foreshadowed, the forcing term in this equation contains the along-isobath density gradient, supposed known from observation. Note, however, that the density gradient appears in an integrated form (in the cross-isobath direction) so that the magnitude of the forcing term is less subject to observational error than in Eq. (10). In this respect the method used here is similar to Rattray's (1982) approach.

The solution of these equations may now proceed as follows. A simple bottom topography and a density distribution are chosen to represent observed conditions over some continental shelf or slope. The steric elevation $\zeta_1$ may be directly determined from these, but only the source term distribution for $\zeta_2$ (rhs of Eq. (12)). Upon integration of Eq. (12) the total pressure field is known, and transports or the transport stream function field may be calculated from Eqs. (7) and (8).

The boundary conditions to be applied will be the same as in similar shelf circulation problems. In the absence of wind stress, the boundary condition at the coast is, from Eq. (7), with $H \to 0$ as $x \to 0$:

$$\frac{\partial \zeta}{\partial x} = 0, \hspace{1cm} x = 0. \hspace{1cm} (13)$$

At large distances from the coast the bottom density excess $\epsilon_b$ is supposed to vanish. Any effects of varying $\epsilon_b$, exhibited by solutions of Eq. (12), may then legitimately be supposed trapped at the coast so that

$$\zeta_2 = 0, \hspace{1cm} x \to \infty. \hspace{1cm} (14)$$

By the definition in Eq. (11), the cross-shore slope of $\zeta_1$ vanishes at the coast, with $H$:

$$\frac{\partial \zeta_1}{\partial x} = -\int_{-H}^{0} \frac{\partial \epsilon_b}{\partial x} \, dx = 0, \hspace{1cm} x = 0, \hspace{1cm} (15)$$

so that the same boundary condition applies at the coast to the $\zeta_2$ field:

$$\frac{\partial \zeta_2}{\partial x} = 0, \hspace{1cm} x = 0. \hspace{1cm} (16)$$

Far from the coast $\zeta_1$ is steric elevation relative to the bottom and is arbitrary, determined by the prescribed density field. Any features of interest of the
problem discussed here are contained in the $\xi_2$-field, and in the manner it combines with the steric elevation.

4. Pycnocline running into a slope

As a simple model of pycnocline–bottom slope interaction suppose that pycnocline depth $h$ is a function of the along-isobath coordinate $y$ only, while the total water depth $H$ increases with $x$:

$$
h = h(y),
H = sx
$$

so that $s$ is bottom slope. The pycnocline will then intersect the bottom along the curve

$$
X = \frac{h(y)}{s}
$$

(see Fig. 2 for schematic illustration). No attempt will be made to represent a shelf of lesser bottom gradient shoreward of the steeper slope, and the coastal boundary conditions will be applied at $x = 0$, where the depth vanishes given the shoreward extrapolation of the slope $s$.

Below the pycnocline the density will be taken equal to the reference density $\rho_0$ so that $\epsilon_p$ is zero. Above the pycnocline $\epsilon_p = \epsilon$ is a (negative) constant. Across the bottom–pycnocline intersection $\epsilon_p$ thus jumps from 0 to $\epsilon$, idealizing a narrow zone of high along-bottom density gradient. The bottom density distribution in this idealization is thus

$$
\epsilon_b = \epsilon \mathcal{H}(X - x),
$$

where $\mathcal{H}(\_\_)$ is the Heaviside unit function (see, e.g., Schwartz, 1966).

The distribution of density within the water column is, in similar notation:

$$
\epsilon_p = \epsilon \mathcal{H}(z + h), \quad x > X
\epsilon_p = \epsilon, \quad x \leq X
$$

Substituting (19) and (20) into Eq. (11), one finds the steric elevation field:

$$
\xi_1 = -seX = -eh(y),
$$

which is simply the deep-water steric elevation continued to the coast and is independent of the cross-shore coordinate.

The source term on the right of Eq. (12) becomes, with (19):

$$
s^2 \int_x^\infty \frac{\partial \epsilon_b}{\partial y} \, dx = s\epsilon \frac{dh}{dy} \mathcal{H}(X - x).
$$

The source intensity is thus constant at given $y$, from $x = 0$ to $X$, beyond which it vanishes.

Upon integrating Eq. (12) from $x = 0$ to infinity one finds, taking account of the boundary conditions (14) and (16):

$$
\frac{\partial}{\partial y} \int_0^\infty \xi_2 \, dx = \epsilon \frac{dh}{dy}
$$

or, noting Eq. (18) and integrating with respect to $y$:

$$
\int_0^\infty \xi_2 \, dx = \frac{\epsilon}{2s} h^2(y) + \text{constant}.
$$

The integration constant may be determined if conditions at some “backward” (sufficiently high $y$) section are known. For example, if $dh/dy$ vanishes for all $y > 0$, so does $\xi_2$ on account of the parabolic nature of Eq. (12). The constant in (24) is then $-\epsilon/2s$, and

$$
M = \int_0^\infty \xi_2 \, dx = \frac{\epsilon}{2s} [h^2(y) - h^2(0)],
$$

where $M$ is the total volume or “mound” of fluid piled up against the coast (or missing from there, in case of negative $\xi_2$). The volume of the mound varies as $-h^2$ (noting that $\epsilon$ is negative). Where $h$ increases in the time-like negative $y$ direction, a negative pressure anomaly and a negative mound develop; in the opposite case a positive anomaly and mound.

Certain implications of the result for the pattern of transport streamlines are revealed by calculating the value of the streamfunction along a line parallel to the coast, far offshore. From Eq. (7), with neither $\xi_1$ nor $\epsilon_p$ depending on $x$, one has

$$
V = \frac{\epsilon_{sx} \, \xi_2}{\xi_2}.
$$

so that, upon partial integration and use of Eq. (25):
\[ \psi_\infty = \int_0^\infty g \frac{\partial \xi_2}{\partial x} dx = -g \frac{x}{f} \int_0^\infty \xi_2 dx \]

\[ = -\epsilon \frac{g}{2f} [h^2(y) - h^2(0)] = -g \frac{x}{f} M, \quad (27) \]

the coast being chosen the \( \psi = 0 \) streamline. One may wonder how this result is reconciled with the first of Eqs. (7). At such distances from the coast that bottom stress is negligible (this is necessarily well beyond the pycnocline-bottom intersection) cross-isobath transport is entirely due to baroclinic flow in the top layer:

\[ U = -\frac{g h}{f} \frac{\partial \xi_1}{\partial y} = \frac{gh}{f} \frac{dh}{dy}, \quad x \to \infty. \quad (28) \]

This vanishes at \( y \to 0 \), where \( h = h(0) \) is constant. The \( y = 0 \) line is therefore also part of the \( \psi = 0 \) streamline. Putting \(-\partial \psi/\partial y \) for \( U \) in (28) and integrating, one recovers Eq. (27).

Physically, the baroclinic flow, toward or away from the coast, where the pycnocline slopes alongshore, is converted into barotropic flow along isobaths in the domain \( y < 0 \). Inflow turns toward negative \( y \), while outflow is drawn from the same region.

For greater detail, the solution of Eq. (12) has to be found, with the source term given by Eq. (22). The equation may be written as:

\[ \kappa \frac{\partial^2 \xi_2}{\partial x^2} + \frac{\partial \xi_2}{\partial y} \frac{\partial y}{\partial x} = \epsilon \frac{dh}{dy} \mathcal{F}(X - x), \quad (29) \]

where \( \kappa = \tau/\sigma_0 \).

This is of the form of the equation of heat conduction with an internal heat generation term, and its solution may be readily written, noting that according to (16) the coast is a reflecting boundary:

\[ \xi_2 = \frac{\epsilon}{2} \int_X \left[ \text{erf} \left( \frac{X(y') - x}{2\kappa(y' - y)^{1/2}} \right) \right. \]

\[ + \left. \text{erf} \left( \frac{X(y') + x}{2\kappa(y' - y)^{1/2}} \right) \right] \frac{dh}{dy} dy' \quad (30) \]

supposing again a flat pycnocline for \( y > 0 \). If the pycnocline reverts to a flat plane \( h = h_1 \) constant forward of some \( y = -Y \) (i.e., at \( y < -Y \)), the source region is limited to \( -Y < y < 0 \). Sufficiently far forward in the time-like negative \( y \)-direction then only the total backward source strength \( M \) [Eq. (25)] enters the result, the elevation field being given by

\[ \xi_2 = \frac{M}{\pi \kappa(-y)^{1/2}} \exp \left( \frac{x^2}{4\kappa Y} \right), \quad y \to -Y. \quad (31) \]

Note that this asymptotic result is especially sensitive to the infinite-slope (in the \( x \)-direction) assumption. Analytical results are available also for more complex models, however.

Once \( \xi_2 \) is determined, the streamfunction \( \psi \) is calculated from:

\[ \psi = \int_0^x \left( V dx' - \frac{g}{f} \left[ x \xi_2 - \int_0^x \xi_2(x') dx' \right] \right). \quad (32) \]

5. Calculated examples

Two concrete examples will be discussed further in detail. Example 1 is a pycnocline with a limited section \( (-Y < y < 0) \) of constant alongshore slope:

\[ h = h_0 = \text{constant}, \quad y > 0 \]

\[ = h_0 + my, \quad -Y < y < 0 \]

\[ = h_1, \quad y < -Y \]

At \( y > 0 \) and \( y < -Y \), the depth is constant \( h = h_0, h_1 \), so that

\[ m_0 = \frac{h_1 - h_0}{Y}. \quad (34) \]

In Example 2 the pycnocline is "bowl-like," the depths far backward and forward being equal, \( h = h_0 \). In the middle of the bowl there is a constant depth section \( cY \) long:

\[ h = h_0, \quad y > 0 \]

\[ = h_0 - my, \quad -Y < y < 0 \]

\[ = h_1, \quad -(1 + c)Y < y < -(1 + c)Y \]

\[ = h_0 + (h_1 - h_0)(2 + c) + my, \quad -(2 + c)Y < y < -(1 + c)Y \]

\[ = h_0, \quad y < -(2 + c)Y \]

where again \( m = (h_1 - h_0)/Y \).

In the second example, forward of \( y = -(2 + c)Y \) (in the negative \( y \)-direction) the effects of the two oppositely inclined pycnocline sections nearly cancel: according to Eq. (25), \( M = 0 \) here, and the asymptotic distribution (31) vanishes. However, in some forward neighborhood, depending on how far the two inclined sections are separated, positive and negative loops of \( \xi_2 \) remain: close to the coast \( \xi_2 \) is positive, because the effect of the closer positive \( \xi_2 \) sources overcome those of the more distant negative \( \xi_2 \) sources. Farther from the coast, however, only the tail of the broader negative \( \xi_2 \) "plume" is present. A weak cyclonic gyre may thus be expected here.

The calculations are simplified by using the following scales:

- cross-shore distance \( L_x = (\kappa Y)^{1/2} \)
- alongshore distance \( L_y = Y \)
- depths \( h \) and \( H \) \( \Delta h = h_0 - h_1 \)
- surface elevation \( \xi_1, \xi_2 \) \( -\epsilon \Delta h \)
streamfunction $\psi = -\frac{g\epsilon}{f L_x} \Delta h$

bottom slope $s = \frac{\Delta h}{L_x}$ (36)

The variables $x$ and $X$ thus come to mean $x/L_x$, $X/L_x$, the elevations $\zeta_1(\epsilon\Delta h)$, etc. The scaled equations are

$$
\begin{align*}
\zeta_2 &= h \\
x &= \frac{h}{s} \\
\zeta_2 &= \frac{1}{2} \int_y \left[ \text{erf}\left( \frac{X(y') - x}{2(y' - y)^{1/2}} \right) \\
&\quad + \text{erf}\left( \frac{X(y') + x}{2(y' - y)^{1/2}} \right) \frac{dh}{dy'} \right] dy' \\
&+ \int_y \zeta_2(x') dx'
\end{align*}
$$

(37)

Figures 3–6 illustrate the elevation (pressure as hydraulic head) and streamfunction fields of the two examples. “Typical” parameters used in the calculations are listed in Table 1, along with the various scales.

In Example 1 the pycnocline shoals in the forward (negative) direction, meaning that the baroclinic flow offshore is shoreward. The surface elevation contours and streamlines turn generally parallel to the coast over the slope. However, right at the coast the elevation gradient is much less than far offshore, and the low elevation of 2.0 (20 cm, corresponding to a pycnocline 200 m deep, using the typical parameters of Table 1) is only reached at a long distance forward, not discernible in Fig. 3. Corresponding to the diverging elevation field, the streamlines also have a small offshore inclination, the maximum alongshore transport occurring at a distance from the coast that increases slowly in the forward direction. The pattern over the slope is indeed very much as discussed earlier (Csanady, 1981) for the case of a sloping shelf, over which a “mound” is maintained by any form of

![Fig. 3. Surface elevation (pressure as hydraulic head) field induced by shoaling pycnocline, such as illustrated in Fig. 2. The shoaling section extends from $y = 0$ to $-1$. The coordinate system is intended to simulate the west coast of North America, looking out to sea along positive $x$, with a northward drop of steric height offshore between $0 < y < 1$, and a northward extending far field over the continental slope. The coastal elevation gradient is much less than the gradient of the steric height in deep water, which is seen undisturbed at $x > 5$ (i.e., in water deeper than 500 m, using typical data of Table 1).](image1)

![Fig. 4. Transport streamlines belonging to Fig. 3. The streamline corresponding to the minimum value ($-2.5$) of the streamfunction is difficult to trace after turning shoreward. Streamlines are dense near the sloping pycnocline–bottom intersection, but the maximum transport shifts gradually offshore in the forward direction, under the influence of bottom friction.](image2)

**Table 1.** Typical parameters used in calculations and scales of various variables.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Typical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>density perturbation, $-\epsilon$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>bottom slope, $s$</td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>pycnocline depth, $h_0$ or $h_1$</td>
<td>300 m–200 m</td>
</tr>
<tr>
<td>Coriolis parameter, $f$</td>
<td>$10^{-4}$ s$^{-1}$</td>
</tr>
<tr>
<td>acceleration of gravity, $g$</td>
<td>$10$ m s$^{-2}$</td>
</tr>
<tr>
<td>resistance coefficient, $r$</td>
<td>$2 \times 10^{-4}$ m s$^{-1}$</td>
</tr>
<tr>
<td>pycnocline length scale, $Y$</td>
<td>$5 \times 10^6$ m</td>
</tr>
<tr>
<td>pycnocline slope, $m$</td>
<td>$2 \times 10^{-4}$</td>
</tr>
<tr>
<td>pycnocline bowl, $c$</td>
<td>2</td>
</tr>
<tr>
<td>equivalent conductivity, $\kappa = \tau/\rho_s$</td>
<td>200 m</td>
</tr>
<tr>
<td>crossshore length scale, $L_x = V_x Y$</td>
<td>$10^4$ m</td>
</tr>
<tr>
<td>alongshore length scale, $L_y$</td>
<td>$5 \times 10^5$ m</td>
</tr>
<tr>
<td>elevation scale, $-m Y$</td>
<td>0.1 m</td>
</tr>
<tr>
<td>depth scale, $\Delta h$</td>
<td>100 m</td>
</tr>
<tr>
<td>transport streamfunction scale, $\frac{g\rho_s L_x}{f} (-\epsilon)m Y$</td>
<td>$10^6$ m$^3$ s$^{-1}$</td>
</tr>
</tbody>
</table>
of forcing, wind stress, coastal freshening or, as shown by this example, baroclinic shoreward transport.

The second example is even more striking: the elevation increase at the coast is much less than offshore and it is displaced in the forward direction by some 2 or 3 alongshore scales, i.e., by a distance of the order of a thousand kilometers. The streamline pattern consists principally of equal baroclinic inflow and outflow, connected by pycnoclastic currents along the slope in between. However, there is also a residual cyclonic circulation cell in the forward direction, as could be foreseen from the analytical results, the intensity of which is about 40% of the total inflow or outflow, as measured by the peak total transport.

6. Shelf–sea front, cutting across isobaths

As a final example, consider a surface-to-bottom shelf–sea front, cutting across the isobaths as illustrated in Fig. 7. The seafloor is again supposed to slope linearly with distance from shore, so that the anchor depth of the front, \( H_f(y) \), traces out the curve \( x = X(y) \):

\[
\begin{align*}
H &= sx \\
X &= \frac{H_f(y)}{s}
\end{align*}
\]  

The density excess is again a constant (negative) \( \epsilon \) above and shoreward of the front:

\[
\epsilon_f(x) = \begin{cases} 
\epsilon, & x < X(y) \\
\epsilon \mathcal{H}(z + h), & x > X(y)
\end{cases}
\]

where the depth of the front \( h(x, y) \) varies in some arbitrary manner in the cross-front direction:

\[
h(x, y) = H_f(y) \phi \left[ \frac{x - X(y)}{l(y)} \right]
\]

![Fig. 5. Elevation field of a bowl-like pycnocline, with shoaling sections between \(-1 < y < 0\) and \(-4 < y < -3\). At the coast, only a fraction of the elevation change is felt, phase-shifted by about 2 length units in the forward (negative \( y \)) direction. There is, however, a considerable tail of this reduced elevation change reaching far forward.](image1)

![Fig. 6. Streamlines belonging to the previous figure. Pycnoclastic currents connect the inflow to the outflow, but a separate cyclonic cell is induced over the forward portion of the slope, centered near \( x = 4 \) (or at about 400 m depth, using typical quantities). The maximum transport in the induced cell is about 1 streamfunction unit, or 40% of the inflow–outflow circulation.](image2)

![Fig. 7. Surface-to-bottom front cutting across isobaths: (a) plan, showing bottom trace of front \( X(y) \); (b) section A-A looking backward.](image3)
with \( k(y) \) being the stretch of the front, i.e.
\[
\begin{align*}
\phi(0) &= 1 \\
\phi(l) &= 0
\end{align*}
\]  
(41)

The bottom density varies again according to Eq. (19), so that the \( \xi_2 \) equation is Eq. (29), with \( dH/dy \) replacing \( dh/dy \). The main difference lies in the \( \xi_1 \) field, which is now
\[
\xi_1 = \begin{cases} 
-\epsilon H_f(y), & x < X \\
-\epsilon h(x, y), & x > X.
\end{cases}
\]  
(42)

Note that this vanishes at \( x > X + l \), unlike the case of the pycnocline with horizontal generators. Thus the alongshore transport is, with the second of Eqs. (7):
\[
V = \begin{cases} 
gsx \frac{\partial \xi_2}{\partial x}, & x < X \\
gsx \frac{\partial \xi_2}{\partial x} - \frac{\epsilon gh}{f} \frac{\partial h}{\partial x}, & x > X
\end{cases}
\]  
(43)

The additional term above the front is the baroclinic transport \( V_c \), the cross-shore integrated value of which is
\[
\psi_{c,0} = \int_0^\infty V dx = \frac{\epsilon g H_f^2}{2f}.
\]  
(44)

The streamfunction is, upon integration of (43):
\[
\psi = \frac{gs}{f} x_0 \xi_2 - \frac{gs}{f} \int_0^x \xi_2 dx' 
+ \frac{\epsilon g}{2f} (H_f^2 - h^2), \quad x > X.
\]  
(45)

The cross-shore integrated value of the barotropic transport (first two terms in Eq. (45), as \( x \to \infty \)) is, as before in Eq. (25), putting \( H_f(y) \) for \( h(y) \):
\[
\psi_{b,0} = -\frac{\epsilon g}{2f} [H_f^2(y) - H_f^2(0)]
\]  
(46)

with \( H_f(0) \) the anchor depth over the backward section, which is supposed constant for \( y > 0 \). Consequently, the total is
\[
\psi_0 = \psi_{c,0} + \psi_{b,0} = \frac{\epsilon g H_f^2(0)}{2f}
\]  
(47)

which is constant, equal to the total baroclinic transport over the backward sector.

Suppose now that the anchor depth changes in the sense indicated in Fig. 7, increasing toward negative \( y \). The baroclinic flow is then everywhere toward negative \( y \), increasing in magnitude with the square of anchor depth, Eq. (44). The increase is supplied by barotropic transport toward positive \( y \), in the total amount given by Eq. (46). The general shape of the streamlines is indicated in Fig. 8. In detail, the barotropic flow streamlines are much as in the examples calculated before. For example, if the anchor depth is

\[
H_f = \begin{cases} 
H_0, & y > 0 \\
H_0 - \frac{H_1 - H_0}{y} y, & -Y < y < 0 \\
H_1, & y < -Y.
\end{cases}
\]  
(48)

the streamlines of the barotropic flow are the same as in Fig. 4, with the flow direction reversed. However, the baroclinic flow just offshore of the anchor depth of the front absorbs the streamlines turning offshore, instead of allowing them to escape to infinity. The baroclinic flow becomes more intense over greater depth, as already pointed out.

7. Discussion

Although the above examples are highly idealized, the following general conclusions regarding the physics of pycnocathatic currents may safely be drawn. The divergence of baroclinic currents over bottom relief acts as a source–sink distribution for barotropic flow. The sources and sinks induce primarily along-isobath currents over a steep slope, affecting those portions of the slope in the “forward” direction, i.e., in the direction of topographic or Kelvin wave propagation. Bottom friction causes weak cross-isobath flow and a slow spreading of the streamlines in the forward direction.

The principal weakness of the theory is that it ignores the question, how the prescribed density field is maintained. Is it reasonable to suppose that he calculated barotropic flow can coexist with the prescribed density field? The answer perhaps is that it is in some cases, but not in others. The results of the last example, illustrated in Fig. 8, show that the intensifying baroclinic current draws its supply mostly from the lighter fluid on the shore side of the front. A small amount of fluid drawn from the heavier fluid
(dotted streamline) may perhaps be rationalized as bottom layer transport. On the other hand, were the front to cut across isobaths in the opposite sense, as illustrated schematically in Fig. 9, the theory would predict barotropic flow in the heavy fluid, fed by the divergence of the baroclinic transport of light fluid. This is clearly inconsistent with the maintenance of the hypothesized density field, unless a strong over-riding flow toward positive y is present. If such a density field were actually observed, one would have to infer just such alongshore flow: as Huthnance (1984) points out, the divergence of the baroclinic flow is a real physical effect, which somehow has to be accommodated.

Similar remarks apply to the example of a sloping pycnocline running into a sloping seafloor: if the bottom density variations are indeed as supposed, the pattern of induced barotropic currents follows, embedded as it perhaps may be in flow generated otherwise, e.g., by wind stress curl. On the other hand, regarding the application of these results to a specific case such as the west coast of North America, it must be noted that the bottom density variations have not actually been observed to be what they are generally thought to be. In view of the surprisingly large pycnoclinic transports indicated by such simple calculations as have been made above, which essentially agree with the results of more elaborate models, it would clearly be of great importance to map bottom density over at least the upper continental slope in various parts of the world.

A final point is that inferences on coastal sea levels, drawn from steric heights in deep water offshore, must be treated with great caution. The pressure field shown in Fig. 5 illustrates forcefully the “insulating” effect of a sloping seafloor, deduced from general considerations elsewhere (Shaw and Csanady, 1983). The offshore pressure field undoubtedly influences currents over the upper slope, and perhaps the outer shelf not too far from the pycnocline-bottom intersection, but it is transmitted to the coast only in a heavily filtered and phase-shifted form.

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