NOTES AND CORRESPONDENCE

On Tidal Damping in Laplace's Global Ocean

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ABSTRACT

Laplace's tidal equations are augmented by dissipation in a bottom boundary layer that is intermediate in character between those of Ekman and Stokes. Laplace's tidal equation for a global ocean remains second-order and self-adjoint, but the operator and eigenvalues are complex with imaginary parts that are $O(\mathsf{E}^{1/2})$, where $\mathsf{E} = \nu/2\omega h^2$ (ν is the vertical component of the kinematic eddy viscosity, ω the rotational speed of the Earth, and h the depth of the global ocean). The imaginary part of the eigenvalue is expressed as a quadratic integral of the corresponding Hough function. The Q for a free oscillation is expressed as the ratio of two quadratic integrals that represent the mean energy and dissipation rates. Approximate calculations for the semidiurnal tides (with azimuthal wave number 2) are given.

1. Introduction

Tidal friction, beginning with G. I. Taylor's (1919) calculation for the Irish Sea, has traditionally been estimated by invoking either Chezy's formula

$$\tau = C\rho|\mathbf{q}|\mathbf{q} \tag{1.1}$$

for the shear stress induced by a flow of velocity $\mathbf{q}$ (just outside of the boundary layer), where C is an empirical friction coefficient, or Boussinesq's formula

$$\tau = \rho\nu\frac{\partial\mathbf{q}}{\partial n}, \tag{1.2}$$

where n is the normal to the boundary and $\rho\nu$ is a mixing coefficient (or eddy viscosity) that is determined either from a phenomenological model or through comparison with observation. The corresponding dissipation rate per unit area is given by the temporal average $\langle\tau\cdot\mathbf{q}\rangle$.

I consider here a somewhat more elaborate formulation, which allows for the dynamical structure of the boundary layer,[1] by incorporating the approximation (1.2) in Laplace's tidal equations (LTE) and calculating the complex eigenvalues of these equations for a global ocean of uniform depth. The uncertainties already implicit in (1.2), together with the fact that dissipation in the real oceans is concentrated in their shallow margins, render it unlikely that this model will provide any significant improvement, vis-à-vis Taylor's procedure, in the practical calculation of tidal friction; however, it does provide a rational incorporation of dissipation in a model that is of fundamental importance for the ocean (Lamb, 1932,[2] §§222, 223; Longuet-Higgins, 1968; Miles, 1974).

The dynamical parameters for the present model are

$$\beta = \frac{4\omega^2a^2}{gh}, \quad \lambda = \frac{\sigma}{2\omega}, \quad \mathsf{E} = \frac{\nu}{2\omega h^2} \ll 1, \tag{1.3a,b,c}$$

where β is Lamb's tidal parameter, a the radius of the Earth, h the depth of the ocean, ω the rotational speed of the Earth, σ the angular frequency of the tidal motion, and E the Ekman number, which is assumed to be small. The primary eigenparameter for LTE (or, more precisely, Laplace's tidal equation *singular,* which is obtained from LTE through the assumption of cyclic dependence of the pressure and velocity on both longitude and time) is β, rather than λ, and the numerical determination of the complex eigenvalues does not lead directly to the complex λ for the free oscillations. The real part of the latter eigenvalue may be determined through numerical interpolation (Longuet-Higgins, 1968) or through a variational approximation, although this is likely to be practical only for the dominant mode for prescribed azimuthal wave number, while the imaginary part or, equivalently, the inverse Q, may be determined through an ancillary calculation of the mean dissipation rate and mean energy. I give representative calculations for the semidiurnal tides in sections 5 and 6.

It is worth emphasizing that the adjective *tidal* in LTE, may be interpreted here in Lamb's (§168) sense "as descriptive of [all] waves in which the motion of the fluid is mainly horizontal, and therefore . . . sen-

[1] This boundary layer is locally of the type first described (for $\lambda \gg 1$) by Thorade (1931) and Marchuk and Kagan (1984, §5.3), and is intermediate in character between those of Stokes (1851) and Ekman (1905).

[2] The 1932 edition is implicit in all subsequent references to Lamb.

sibly the same for all particles in a vertical line." LTE also are of fundamental importance, but the present description of dissipation is inadequate, for the atmosphere, in which the vertical structure outside of the boundary layer is nonhydrostatic and boundary-layer damping may be dominated by molecular diffusion (of both heat and momentum), infrared cooling and ion drag at high altitudes (Chapman and Lindzen, 1970).

2. Equations of motion

Small perturbations about hydrostatic equilibrium in a shallow, incompressible ocean of uniform depth h on a spherical planet of radius a are governed by the equation of motion (Lamb, §214)

$$\partial_t(u, v) + (2\omega \cos\theta)(-v, u) = -(g/a)(\partial_\theta, \operatorname{cosec}\theta\partial_\phi)(\zeta - \bar{\zeta}) + \mathbf{F} \quad (2.1)$$

and the continuity equation

$$(a \sin\theta)^{-1}[\partial_\theta(u \sin\theta) + \partial_\phi v] + \partial_r w = 0, \quad (2.2)$$

where r, θ, ϕ are spherical polar coordinates (θ and ϕ are the co-latitude and longitude), w, u, v are the corresponding velocity components, ζ is the displacement of the free surface, $\bar{\zeta} \equiv -\Omega/g$ is the equilibrium displacement associated with the external potential Ω, ω is the angular velocity of the Earth, and $\mathbf{F}$ is the specific frictional force. The corresponding boundary conditions, on the assumption that friction is negligible at the free surface, are

$$u = v = w = 0 \ (r = a - h), \quad w = \partial_t\zeta \ (r = a). \quad (2.3\text{a,b})$$

Integrating (2.2) between $r = a - h$ and $r = a$ and invoking (2.3) for w, we obtain

$$(a \sin\theta)^{-1} \int_{a-h}^{a} [\partial_\theta(u \sin\theta) + \partial_\phi v]dr + \partial_t\zeta = 0. \quad (2.4)$$

Equations (2.1) and (2.4) differ from the traditional form of LTE (*ibid*) only in incorporating $\mathbf{F}$ and in allowing for the variation of u and v with r in consequence of friction.

We now adopt the approximation (1.2), which implies

$$\mathbf{F} \doteq \nu\partial_r^2(u, v), \quad (2.5)$$

assume that the motion is cyclic in t and ϕ, introduce dimensionless complex amplitudes according to

$$(u, v) = (gh/2\omega a) \operatorname{Re}\{(-iU, V)e^{i(\sigma t + s\phi)}\}, \quad (2.6\text{a})$$

$$(\zeta, \bar{\zeta}) = h \operatorname{Re}\{(Z, \bar{Z})e^{i(\sigma t + s\phi)}\}, \quad (2.6\text{b})$$

where Re signifies *the real part of*, introduce

$$\mu \equiv \cos\theta, \quad \mu_* \equiv \sin\theta, \quad z \equiv 1 + (r - a)/h, \quad (2.7\text{a,b,c})$$

and transform (2.1), (2.4) and (2.3a) to

$$(\lambda + i\mathsf{E}\partial_z^2)(U, V) - \mu(V, U) = (\mu_*\partial_\mu, -s/\mu_*)(Z - \bar{Z}), \quad (2.8)$$

$$\int_0^1 [\partial_\mu(\mu_* U) + (s/\mu_*)V]dz + \beta\lambda Z = 0, \quad (2.9)$$

where β, λ and E are defined by (1.3), and

$$U = V = 0 \ (z = 0). \quad (2.10)$$

3. Boundary-layer reduction

Introducing the adjoint operators

$$\mathcal{D}_\pm \equiv \mu_*\partial_\mu \pm (s/\mu_*), \quad \mathcal{D}_\pm^* \equiv \partial_\mu\mu_* \pm (s/\mu_*), \quad (3.1\text{a,b})$$

where, here and subsequently, the alternative signs are vertically ordered, we transform (2.8) and (2.9) to

$$(\lambda \mp \mu + i\mathsf{E}\partial_z^2)U_\pm = \mathcal{D}_\mp(Z - \bar{Z}), \quad U_\pm \equiv U \pm V, \quad (3.2\text{a,b})$$

$$\tfrac{1}{2}\mathcal{D}_\mp^* \int_0^1 U_+ dz + \tfrac{1}{2}\mathcal{D}_\pm^* \int_0^1 U_- dz + \beta\lambda Z = 0. \quad (3.3)$$

The solution of (3.2), subject to (2.10) and the requirement that U and V attain their inviscid values outside of the boundary layer,[3] is given by

$$U_\pm = (1 - e^{-\gamma_\mp z})(\lambda \mp \mu)^{-1}\mathcal{D}_\mp(Z - \bar{Z}), \quad (3.4)$$

where

$$\mathsf{E}^{1/2}\gamma_\pm = [i(\lambda \pm \mu)]^{1/2} = |\lambda \pm \mu|^{1/2}e_\pm,$$

$$e_\pm = e^{\frac{1}{4}i\pi \operatorname{sgn}(\lambda \pm \mu)}. \quad (3.5\text{a,b})$$

Substituting (3.4) into (3.3) and neglecting $\exp(-\gamma_\pm)$ [in keeping with the boundary-layer approximation implicit in (3.4)], we obtain

$$\{\mathcal{L} + i\mathsf{E}^{1/2}\mathcal{M} + \beta\}(Z - \bar{Z}) = -\beta\bar{Z}, \quad (3.6)$$

where

$$\mathcal{L} \equiv \frac{1}{2}\lambda^{-1} \sum_\pm \mathcal{D}_\mp^*(\lambda \pm \mu)^{-1}\mathcal{D}_\pm \quad (3.7\text{a})$$

$$= \partial_\mu\left(\frac{\mu_*^2}{\lambda^2 - \mu^2}\right)\partial_\mu - \frac{s(\lambda^2 + \mu^2)}{\lambda(\lambda^2 - \mu^2)^2} - \frac{s^2}{\mu_*^2(\lambda^2 - \mu^2)}, \quad (3.7\text{b})$$

$$\mathcal{M} \equiv \frac{1}{2}\lambda^{-1} \sum_\pm \mathcal{D}_\mp^*|\lambda \pm \mu|^{-3/2}e_\pm\mathcal{D}_\pm, \quad (3.8)$$

and $\sum_\pm$ implies the sum of the two expressions with alternative, vertically ordered signs.

[3] If the boundary layer were not assumed to be thin (2.3) would be supplemented by $\nu u_z = \nu v_z = 0$ at $r = a$, $\cosh[\gamma_\pm(1 - z)]/\cosh\gamma_\pm$ would appear in place of $\exp(-\gamma_\pm z)$ in (3.4), and (3.8) would be multiplied by $\tanh\gamma_\pm$.

4. Hough-function expansions

Laplace's tidal equation (*singular*), which is obtained by setting $\mathsf{E} = 0$ in (3.6), admits a complete set of orthogonal eigenfunctions, the Hough functions H_n, which are determined by

$$(\mathcal{L} + \beta_n)H_n = 0, \quad (4.1)$$

$$\int_{-1}^{1} H_m H_n d\mu = \delta_{mn}, \quad (4.2)$$

together with the requirement that H_n be regular in $-1 < \mu < 1$; see Lamb (§222, 223), Flattery (1967) and Longuet-Higgins (1968). Substituting the expansions

$$Z = C_n H_n, \quad \bar{Z} = \bar{C}_n H_n \quad (4.3a,b)$$

(where, here and subsequently, repeated indices are implicitly summed over the complete set of the H_n except as noted) into (3.6), multiplying the result through by H_m, and invoking (4.2), we obtain

$$[\delta_{mn}(\beta_n - \beta) + i\epsilon_{mn}](C_n - \bar{C}_n) = \beta\bar{C}_m, \quad (4.4)$$

where

$$\epsilon_{mn} = -\mathsf{E}^{1/2} \int_{-1}^{1} H_m \mathcal{M} H_n d\mu \quad (4.5a)$$

$$= \frac{1}{2}\mathsf{E}^{1/2} \sum_{\pm} \int_{-1}^{1} \frac{(\mathcal{D}_\pm H_m)(\mathcal{D}_\pm H_n)e_\pm d\mu}{\lambda|\lambda \pm \mu|^{3/2}}, \quad (4.5b)$$

and (4.5b) follows from (4.5a) through integration by parts. Self-attraction could be incorporated at this stage by replacing $\bar{C}_n$ in (4.4) by (Lamb, §222)

$$\bar{C}_n' = \bar{C}_n + \left(\frac{3}{2n+1}\right)\left(\frac{\rho}{\rho_0}\right)C_n, \quad (4.6)$$

where ρ is the fluid density and ρ_0 is the mean density of the Earth, including the ocean.

The solution of (4.4) in the limit $\mathsf{E} \downarrow 0$, obtained by discarding the off-diagonal terms, is given by

$$C_n = \frac{\beta_n \bar{C}_n}{\beta_n - \beta + i\epsilon_n}, \quad n \text{ not summed}, \quad (4.7)$$

in which $O(\mathsf{E})$ terms are neglected in both numerator and denominator and

$$\epsilon_n \equiv \epsilon_{nn} = \frac{\mathsf{E}^{1/2}}{2\lambda} \sum_{\pm} \int_{-1}^{1} \frac{(\mathcal{D}_\pm H_n)^2}{|\lambda \pm \mu|^{3/2}} e_\pm d\mu. \quad (4.8)$$

It follows that the complex eigenvalues of (3.6) are given by $\beta_n + i\epsilon_n + O(\mathsf{E})$, where β_n are the eigenvalues of (4.1).

5. Variational formulation

The self-adjoint differential equation (3.6) may be derived from the variational principle

$$\delta J = 0, \quad (5.1)$$

where

$$J = \frac{1}{2}\left[\frac{1}{2\lambda}\sum_{\pm}\left(\frac{1}{\lambda \pm \mu} + \frac{i\mathsf{E}^{1/2}e_\pm}{|\lambda \pm \mu|^{3/2}}\right)(\mathcal{D}_\pm\Phi)^2 - \beta\Phi^2\right] - \beta[\Phi\bar{Z}], \quad (5.2)$$

$$\Phi \equiv Z - \bar{Z}, \quad (5.3)$$

and

$$[f(\mu)] \equiv \int_{-1}^{1} f(\mu)d\mu. \quad (5.4)$$

We emphasize that J is not the Lagrangian (in the sense of Hamilton's principle) of the mechanical system (see §6).

Expanding Φ in the orthogonal functions defined by (4.1) and (4.2) and requiring J to be stationary with respect to variations of each of the expansion coefficients, we recover the formulation of section 4.

Now suppose that Φ_1 is a trial function that is bounded in $-1 \leqslant \mu \leqslant 1$ and for which $\mathcal{D}_\pm\Phi = O(\lambda \pm \mu)$ as $\mu \to \mp\lambda$. Substituting $\Phi = C\Phi_1$ into (5.2), solving $dJ/dC = 0$ for C, invoking (5.3), and dropping the subscript from Φ_1, we obtain[4]

$$Z = \bar{Z} + \frac{\beta[\Phi\bar{Z}]\Phi}{(\beta_1 + i\epsilon - \beta)[\Phi^2]}, \quad (5.5)$$

where

$$\beta_1 = \frac{\sum_{\pm}[(\lambda \pm \mu)^{-1}(\mathcal{D}_\pm\Phi)^2]}{2\lambda[\Phi^2]}, \quad (5.6)$$

$$\frac{\epsilon}{\mathsf{E}^{1/2}} = \frac{\sum_{\pm}[|\lambda \pm \mu|^{-3/2}e_\pm(\mathcal{D}_\pm\Phi)^2]}{2\lambda[\Phi^2]}. \quad (5.7)$$

We remark that $\beta_1 + i\epsilon$, as given by (5.6) and (5.7), is stationary with respect to first-order variations of Φ about the true solution to the homogeneous Laplace tidal equation [$\mathsf{E} = \bar{Z} = 0$ in (3.6)], that (5.6) reduces to the exact eigenvalue β_1 for $\Phi = H_1$, and that (5.7) reduces to ϵ_1 (4.8) for $\Phi = H_1$. If Φ is a reasonable approximation to H_1, we expect (5.6) to be an excellent approximation to the lowest eigenvalue and (5.5) to be a good approximation to the exact solution for $\beta \lessdot \beta_1$. Special precautions are necessary in the construction of Φ if β approaches or exceeds the next eigenvalue β_2, as in any Rayleigh-Ritz approximation.

We illustrate the accuracy of the variational approximation by considering the semidiurnal forcing described by (Lamb, §221)

$$\bar{Z} = A\mu_*^2, \quad s = 2. \quad (5.8a,b)$$

Lamb takes $\lambda = 1$ (a good/rough approximation for solar/lunar forcing) and shows that Φ may be expanded

[4] This result also may be obtained by substituting $Z - \bar{Z} = C\Phi_1$ into (3.6), multiplying through by Φ_1, and integrating by parts from $\mu = -1$ to $+1$.

in powers of μ_*^2, beginning with μ_*^4. It may be shown (cf. Lamb, §222) that the corresponding expansion for $\lambda \neq 1$ begins with μ_*^2 and is given by (within an arbitrary multiplier)

$$\Phi = (1 - \mu^2)(\mu_0^2 - \mu^2), \quad \mu_0^2 = \lambda(2\lambda - 1), \tag{5.9a,b}$$

where μ_0^2 is determined by the requirement that

$$\mathscr{D}_\pm\Phi = \pm 2\mu_*(1 \mp \mu)(2\lambda - 1 \mp 2\mu)(\lambda \pm \mu) \tag{5.10}$$

vanish (so that U and V are regular) at $\mu = \mp\lambda$. Substituting the trial function (5.9a) into (5.5) and (5.6), we obtain

$$\frac{[\Phi\bar{Z}]}{[\Phi^2]} = \frac{3A(7\mu_0^2 - 1)}{21\mu_0^4 - 6\mu_0^2 + 1}, \tag{5.11}$$

$$\beta_1 = \frac{6}{\lambda}\left(\frac{84\lambda^3 - 56\lambda^2 + \lambda + 1}{21\mu_0^4 - 6\mu_0^2 + 1}\right). \tag{5.12}$$

The approximation (5.9a) has simple zeros at $\mu = \pm\mu_0$ for $\lambda > ½$ and corresponds to $n - s = 0$ ($\nu = 1$) and $\lambda < 0$ in Longuet-Higgins' (1968) notation [Longuet-Higgins poses $\exp(-i\sigma t)$ versus $\exp(i\sigma t)$ in the present formulation]. The approximation (5.12) is within 1% of Longuet-Higgins' (*ibid,* Table 3) numerical result for this mode for $\lambda \geqslant 0.8$; on the other hand, it yields $\beta_1 < 0$, and therefore does not correspond to a free oscillation, for $\lambda \leqslant 0.6$ (although Laplace's tidal equation does admit negative eigenvalues).

Combining (5.8), (5.9), (5.11) and (5.12) in (5.5), setting $\lambda = 1$ and $\mathsf{E} = 0$, and dividing through by $\bar{Z}$, we obtain the amplification factor

$$\frac{Z}{\bar{Z}} = 1 + \frac{1.125\beta(1 - \mu^2)}{11.25 - \beta}\,(\lambda = 1). \tag{5.13}$$

The approximation to β_1, 11.25, differs from Longuet-Higgins's value of 11.22 by 0.3%. The approximation (5.13) is compared, for $\mu = 0$ (the equator), with Lamb's result in Table 1. The deterioration of (5.13) for $\beta \geqslant \beta_1$ (see last sentence in penultimate paragraph above) is a consequence of the second resonance at $\beta = \beta_2 \doteqdot 33.0$.

Substituting (5.9) and (5.10) into (5.7) and changing the sign of μ in the integral of $|\lambda - \mu|^{1/2}\ldots$, we obtain

TABLE 1. Comparison of diurnal tidal amplification at equator: (5.13) vs Lamb (§221)

	β				
	1	5	10	20	40
Eq. (5.13)	1.110	1.90	10.0	−1.57	−0.57
Lamb	1.110	1.92	11.3	−1.82	−7.43

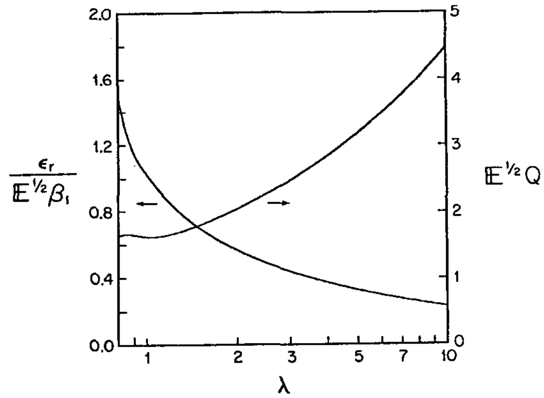

FIG. 1. The parameters $\epsilon_r/\mathsf{E}^{1/2}\beta_1$ and $\mathsf{E}^{1/2}Q$, as determined from (5.14) and (6.11) using the trial function (5.9).

$$\frac{\epsilon}{\mathsf{E}^{1/2}} = \frac{315\int_{-1}^{1} |\lambda + \mu|^{1/2}(1 - \mu^2)(1 - \mu)^2 \times (2\lambda - 1 - 2\mu)^2[1 + i\,\mathrm{sgn}(\lambda + \mu)]d\mu}{2^{5/2}\lambda(21\mu_0^4 - 6\mu_0^2 + 1)}, \tag{5.14}$$

the real part of which is plotted in Fig. 1. (Note that the real and imaginary parts of ϵ are equal for $\lambda > 1$. The imaginary part is of limited interest in the present context.) Setting $\lambda = 1$, we obtain $\epsilon = 11.52(1 + i)\mathsf{E}^{1/2}$.

6. Energy and dissipation

The mean kinetic and potential energies of the fluid motion are given by (after letting $dr = hdz$ and $r = a$ in the elements of volume and area)

$$\langle K\rangle = \frac{1}{2}\rho a^2 h \int_0^{2\pi} d\phi \int_{-1}^{1} d\mu \int_0^1 \langle u^2 + v^2\rangle dz,$$

$$\langle P\rangle = \frac{1}{2}\rho g a^2 \int_0^{2\pi} d\phi \int_{-1}^{1} \langle\zeta^2\rangle d\mu, \tag{6.1a,b}$$

where $\langle\ \rangle$ signifies an average over one period. We provisionally neglect dissipation, by virtue of which u_t, v and ζ are either in or 180° out of phase with $\bar{\zeta}$. Substituting (2.6) into (6.1) and choosing the initial time such that $\bar{Z}$, and therefore U, V and Z, are real, we obtain

$$\langle K\rangle = \frac{1}{2}E_0[U^2 + V^2], \quad \langle P\rangle = \frac{1}{2}\beta E_0[Z^2], \tag{6.2a,b}$$

where

$$E_0 \equiv (\pi\rho a^2 h)(gh/2\omega a)^2 \tag{6.3}$$

is a reference energy and [ ] is defined by (5.4). The inviscid approximations to U and V are determined by [cf. (3.4)]

$$U \pm V = (\lambda \mp \mu)^{-1}\mathscr{D}_\mp(Z - \bar{Z}). \tag{6.4}$$

We pose the (dimensionless) average Lagrangian of the motion in the form

$$L \equiv \langle K - P + \bar{P} \rangle / E_0, \quad (6.5)$$

where $\langle \bar{P} \rangle$, which may be regarded as a prescribed constant in the invocation of Hamilton's principle, is given by (6.1b) and (6.2b) with ζ and Z replaced by $\bar{\zeta}$ and $\bar{Z}$ therein. Combining (6.2)–(6.4) in (6.5), we obtain

$$L = \frac{1}{2}\left[\frac{1}{2}\sum_{\pm}\left(\frac{\mathcal{D}_{\pm}\Phi}{\lambda \pm \mu}\right)^2 - \beta\Phi^2\right] - \beta[\Phi\bar{Z}], \quad (6.6)$$

where $\Phi \equiv Z - \bar{Z}$, as in section 5. It is now evident, as anticipated, that J (5.2) with $\mathsf{E} = 0$ therein differs from the Lagrangian L and that the mean kinetic energy (in contrast to the mean potential energy) involves coupling among the eigenfunctions of section 4 and is not equal to the mean potential energy (in contrast to the equality that holds for surface waves in a nonrotating fluid).

The mean dissipation rate associated with the boundary layer is given by

$$\langle D \rangle = \rho\nu a^2 h \int_0^{2\pi} d\phi \int_{-1}^{1} d\mu \int_0^1 \langle (\partial_r u)^2 + (\partial_r v)^2 \rangle dz. \quad (6.7)$$

Substituting (2.6) into (6.7) and invoking (6.3) and (3.4), we obtain

$$\langle D \rangle = 2\omega E_0 \mathsf{E}\left[\int_0^1 \{|\partial_z U|^2 + |\partial_z V|^2\} dz\right] \quad (6.8a)$$

$$= \omega E_0 (\mathsf{E}/2)^{1/2} [\sum_{\pm} |\lambda \pm \mu|^{-3/2} |\mathcal{D}_{\pm}\Phi|^2] \quad (6.8b)$$

$$= \sigma E_0 \epsilon_r [\Phi^2], \quad (6.8c)$$

where ϵ_r is the real part of ϵ (5.7).

The Q of a lightly damped free oscillation is given by

$$Q = \sigma\langle K + P \rangle / \langle D \rangle. \quad (6.9)$$

Approximating $\langle K \rangle$ and $\langle P \rangle$ by their inviscid values (6.2), invoking (6.4), setting $Z = \Phi$ and $\bar{Z} = 0$ (free oscillations), invoking (6.8c) for $\langle D \rangle$, and defining Φ to be real, we obtain

$$Q = \frac{\frac{1}{2}\sum_{\pm} [(\lambda \pm \mu)^{-2} (\mathcal{D}_{\pm}\Phi)^2] + \beta[\Phi^2]}{2\epsilon_r[\Phi^2]}. \quad (6.10)$$

Remarking that, from (5.5), β must approximate β_1 (differing from it only by ϵ_i) for the assumed free oscillation and invoking (5.6), we reduce (6.10) to

$$Q = \frac{\sum_{\pm} [(\lambda \pm \mu)^{-2} (2\lambda \pm \mu)(\mathcal{D}_{\pm}\Phi)^2]}{4\epsilon_r \lambda [\Phi^2]}. \quad (6.11)$$

The approximation obtained by substituting (5.9) into (6.11) is plotted in Fig. 1.

An alternative representation of Q, obtained by invoking $\langle D \rangle = 2\sigma_i \langle K + P \rangle$ [since $K + P$ decays like $\exp(-2\sigma_i t)$] for the mean dissipation rate and calculating $\sigma_i = 2\omega\lambda_i$ from the resonance condition $\beta = \beta_1 + i\epsilon$ in (5.5), is

$$Q = \frac{1}{2}(\lambda_r/\lambda_i) = -\frac{1}{2}\epsilon_r^{-1}(\lambda d\beta_1/d\lambda), \quad (6.12)$$

where the subscripts r and i identify real and imaginary parts, and λ is regarded as real on the right-hand side of (6.12). Substituting β_1 from (5.6) into (6.12) and invoking the variational principle that β_1 is independent of first-order variations of Φ (including those associated with variations of λ), we recover (6.11). The approximations (6.11) and (6.12) may differ if β_1 in (6.12) is determined from (5.6) using the same trial function Φ as is used in (6.11), and we then should expect (6.11) to provide the better approximation. For example, for $\lambda = 1$ we obtain $Q = 1.61\mathsf{E}^{-1/2}$ by substituting (5.9) and (5.10) and the corresponding approximation to ϵ_r (see last sentence in §5) into (6.11) or $Q = 1.49\mathsf{E}^{-1/2}$ by substituting (5.12) into (6.12).

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