Boundary Currents, Free Currents and Dissipation in the Low-Frequency Scattering of Shelf Waves

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ABSTRACT

The low-frequency scattering of barotropic shelf waves is considered in the limit of small but nonzero dissipation. For a rectilinear shelf it is shown that an intense oscillatory boundary layer forms on the incident side of any obstacle. This layer destroys incident shelf wave energy and turns volume flux to pass smoothly round the obstacle. By confining attention to low frequencies the results of Wilkin and Chapman for scattering at a discontinuity in shelf width are extended to more general changes in width and more general shelf profiles. A spreading along-shelf free shear layer is present downstream of a widening shelf. For a narrowing shelf this layer carries fluid from the dissipation layer and for a headland the layer represents an intense free midshelf current. It is also shown that a change in shelf width can scatter energy from an oscillatory geostrophic current into shelf waves. This corresponds in a flow with a free surface to scattering of Kelvin wave energy into shelf waves.

1. Introduction

Wilkin and Chapman (1987, called WC herein) present solutions for the scattering of barotropic continental shelf waves of arbitrary subinertial frequency at discontinuities in width of an exponential shelf in a channel. An examination of their results shows that energy flux and mode amplitudes of the scattered field are almost independent of frequency for frequencies up to 0.1 f, where f is the Coriolis parameter. Thus, it is to be expected that a low frequency analysis should reproduce many of their results. For the particular case of a discontinuously widening shelf with an exponential profile Middleton and Wright (1988) show that this is indeed the case, obtaining extremely close agreement with the results of WC in comparable cases. Wilkin and Chapman conclude by noting the necessity of generalizing their results to more realistic geometries and to considering the effects of narrowing shelves. It is the purpose of the present note to show that both these effects can be dealt with directly in the low-frequency limit.

Barotropic long waves are unidirectional: energy and phase propagate along isobaths with shallow water to their right in the Northern Hemisphere. There is no mechanism in the long wave equation for energy to be reflected. When a barrier is placed across the isobaths the continuous arrival of energy forces a singularity. Johnson (1985) gives an example of such a case for topographic waves along a step abutting a wall. In the present case of a rectilinear shelf in the low-frequency limit a singular boundary layer forms on the incident side of any obstruction. This is a region of arbitrarily large velocity gradients and it is shown that even vanishingly small viscosity, so small as to be negligible elsewhere in the flow, is sufficient to dissipate the incoming energy.

Section 2 gives the equations of motion for a fluid of vanishingly small but nonzero viscosity, deriving the requisite matching conditions for the long-wave equation and discussing the structure of the intense currents possible at low frequencies. Section 3 gives specific results for the case considered by WC. Previous analyses have concentrated on the effects of incidence shelf waves with zero net instantaneous volume flux. Section 4 shows that an oscillatory geostrophic flow, when forced to cross bottom contours at a change in shelf width, scatters energy into shelf waves. The results and their relevance to observations are discussed briefly in section 5 where it is also shown that free midshelf currents can occur behind headlands jutting across shelves.

2. Governing equations

The topographic wave equation can be written

$$\partial_t + \zeta \nabla \cdot (H^{-1} \nabla \Psi) + \zeta \cdot [\nabla \Psi \times \nabla (f/H)] = 0,$$

(2.1)

where H(x'/l, y'/l) is the local depth, \zeta a unit vertical vector and \Psi the depth-averaged streamfunction. This is the form given in Rhines (1969) with an extra term modeling a linear bottom friction. The friction coefficient is \zeta > 0 and is taken to be vanishingly small in
the subsequent analysis. For finite depth flows at low frequencies \( \hat{v}^2 \) is proportional to the Ekman number. Consider an infinite channel with side walls \( C_1 \) at \( y' = l (\infty < x' < \infty) \); and \( C_0 \) at \( y' = 0 (x' < 0) \), \( y' = Lx'/a \) \( (0 \leq x' \leq a) \) and \( y' = llx' (x' > a) \). For \( L > 0 \) this gives a channel with a continuous linear narrowing and for \( L < 0 \) a widening channel (Fig. 1). For simplicity take the topography to be rectilinear, i.e., \( H = H(y'/l) \), including the exponential topography considered by WC. More general topography is considered in Johnson (1989a).

Consider the scattering of an incoming shelf wave of frequency \( \omega f \) and introduce nondimensional variables \((x, y) = (x'/l, y'/l)\). Write \( \Psi = \text{Re} \times [\exp(-i\omega f)\psi(x, y)] \). Then

\[
(-i\omega + \hat{v})(\psi_{xx} + \psi_{yy}) + G(y)(i\omega \psi_y - \psi_x) = 0.
\]  

(2.2)

Here \( G(y) = (\log H)_y \) and for definiteness is taken to be strictly positive so long waves are incident from \( x < 0 \). If there is no net oscillatory flux along the channel, the boundary conditions on \( \psi \) at the rigid walls are

\[
\psi = 0 \quad \text{on} \quad C_0, C_1.
\]  

(2.3)

The case of nonzero flux is considered in section 4.

Following Johnson (1989b), introduce the slowly varying \( x \) scale, \( X = \omega x \), and consider the long-wave, low-frequency, weakly viscous limit \( \omega \rightarrow 0, \hat{v} \rightarrow 0 \) with \( X, y, a \) and \( \nu = \hat{v}/\omega \) fixed. The leading order equation corresponding to (2.2) becomes, dropping dashes,

\[
(1 + i\nu)\psi_{yy} - G(\psi_y + i\psi_x) = 0.
\]  

(2.4)

The changes in shelf width appear discontinuous on these length scales, occurring at \( X = 0 \) (Fig. 2). The boundary conditions on \( \psi \) are (2.3) and the specified incident field. The flow in \( X > 0 \) is determined entirely by the value of \( \psi \) on \( X = 0^+ \), \( (L < y < 1) \). It is shown below that the sole information required from \( X < 0 \) to determine \( \psi|_{X=0^+} \) is \( \psi|_{X=0^-} \). Note that for a given value of the incident streamfunction at \( X = 0^- \), \( \psi \) increases exponentially as \( X \rightarrow -\infty \). Thus for \( \nu > 0 \), it must be supposed that the incident field is generated at some (large, since \( \nu^{-1} \gg 1 \)) distance upstream and decays slow towards \( X = 0 \). On the scale of the shelf width such attenuation is negligible.

To obtain \( \psi(0^+, y) \) consider (2.2) on the scale \( x \) (i.e., \( \omega^{-1} X \)). Then the sole surviving term in the present limit gives

\[
\psi_x = 0.
\]  

(2.5)

The channel geometry on this scale is given by Fig. 1.

For the widening channel \( (L < 0) \) (Fig. 1a) integrating (2.5) from \( -\infty \) to \( \infty \) with respect to \( x \) in \( 0 < y \leq 1 \) gives
\[ 0 = [\psi]_{x=-\infty}^{x=0} = [\psi]_{x=0^+}^{0^-}, \quad (2.6) \]

i.e., \( \psi \) is continuous across \( X = 0 \). For \( L \leq y < 0 \), integrating (2.5) from \( C_0 \) to \( \infty \) along a line \( y = \text{constant} \) gives

\[ \psi|_{x=0^+} = \psi|_{x=\infty} = \psi|_{C_0} = 0. \quad (2.7) \]

Both (2.6) and (2.7) give at their junction \( \psi(0, 0) = 0 \) and so the solution in \( X > 0 \) is completely determined.

For the narrowing channel \( (L > 0) \) (Fig. 1b) the incident field generally conflicts with the condition that \( \psi \) vanishes on \( X = 0, (0 \leq y \leq L) \). To resolve this difficulty reconsider the scale \( X \). Then for \( L < y < 1 \), integrating (2.5) gives (2.6) and the continuity of \( \psi \) across \( X = 0 \). For \( 0 \leq y < L \), integrating (2.5) from \( -\infty \) to \( C_0 \) gives

\[ \psi|_{C_0} = \psi|_{x=-\infty} = \psi|_{x=0^-}. \quad (2.8) \]

Condition (2.8) conflicts with (2.3) in the region marked \( D \) in Fig. 1b. To consider this region rotate (2.2) and introduce tangential and normal coordinates \((s, n)\). Then introduce the stretched variable \( N = n/\omega \), giving the scale of viscously damped waves. Equation (2.8) with conditions (2.8) and (2.3) becomes

\[ (1 + iv)\psi_{NN} + ig(s)s_N = 0, \quad (2.9) \]

\[ \psi = 0 \quad (N = 0), \quad \psi = F(s) \quad (N = \infty), \quad (2.10) \]

where \( g(s) \) and \( F(s) \) are known functions of \( s \). This system has the solution

\[ \psi = \{1 - \exp[-(i + v)g(s)N/(1 + v^2)]\}F(s), \quad (2.11) \]

where \( g(s) \) is positive since \( G(y) \) is. The viscous boundary layer is passive, being driven by the external flow and dissipating the energy of the incident wave. The tangential component of velocity in the layer is

\[ -\omega^{-1} \frac{\partial \psi}{\partial N} = -\frac{1}{\omega} \left( \frac{i + v}{1 + v^2} \right) g(s) \]

\[ \times F(s) \exp[-(i + v)g(s)N/(1 + v^2)]. \quad (2.12) \]

It is large, of order \( \omega^{-1} \gg 1 \), decaying exponentially away from the boundary over a distance of order \( \omega(1 + v^2)^{-1/2} \) and fluctuating with wavelength of order \( \omega(1 + v^2) \) and zero mean. Figure 3 gives profiles of the velocity within the layer at quarter period intervals for \( v = 1 \) and \( v = 0.2 \), taking \( g(s) = 1 \) and \( F(s) = 1 \). For larger values of the slope \( g(s) \) the layer is thinner. The magnitude of the velocity scales directly on \( g(s) \) and \( F(s) \) by (2.12). For large \( v \) the layer is thin and the current decays rapidly. For small \( v \) the layer is thicker and fluctuations extend further from the wall. Nevertheless, the waves have no signature on the scale of the shelf width \( l \) provided \( v \gg \omega \). The phase of the velocity fluctuations propagates inwards corresponding to outward energy propagation by short waves. The

![Fig. 3. The profile across the boundary layer D of the tangential component of velocity. The velocity has been multiplied by Ω since it is of order \( \omega^{-1} \) in D. The spatial coordinate is N of order \( \omega l \), small compared to the shelf width. The profiles are shown at successive quarter period intervals for period \( 1 = 2\pi/\omega \). (a) \( v = 1 \). The current decays rapidly away from the wall. (b) \( v = 0.2 \). The current direction oscillates and its strength decays only slowly away from the wall. The phase of the disturbance propagates towards the wall corresponding to short waves carrying energy outwards.](image)

layer turns the incident flux from being parallel to the isobaths on the scale \( l \) to run along the edge \( y = Lx/a \) of the contraction before emerging into \( x > a \) from a source at \((a, L)\).

The details of the source region do not affect the downstream flow but are noted briefly for completeness. The flow carried by the layer \( D \) is turned to run again parallel to the isobaths in a small square region of dimension \( \omega l \times \omega l \) centered on \((a, L)\). In this region, labeled \( S_0 \) in Fig. 1b, (2.2) becomes

\[ (1 + iv)(\psi_{xx} + \psi_{yy}) - iG(L)\psi = 0, \quad (2.13) \]

where \( \xi = (x - a)/\omega, \eta = (y - L)/\omega \). Since the coefficient of \( \psi \) in (2.13) is constant, solutions of (2.13) are equivalent to viscously damped Rossby waves on a \( \beta \)-plane with \( \beta = G(L) \). It is sufficient to note that (2.13) can be matched to (2.11) and to the boundary condition \( \psi = 0 \) on \( C_0 \), i.e., on \( \eta = 0, \xi > 0 \).

The value of \( \psi \) at \( x = a, y > L \) is given by (2.5) and (2.6) as \( \psi(0^+, y) \) and the value of \( \psi(a, L) \) is given by (2.3) as zero. The jump in the streamfunction on the line \( x = 0 \) between its zero value at \( y = L \) and its value at \( y = L^+ \) represents a volume source of strength \( \psi(0^+, L^+) \). This is the flux carried by \( D \) emerging through \( S_0 \) into \( x > a \). Over distances of order \( l \) the emerging
flux forms a spreading boundary current flowing along
the wall \( y = L \). This follows by introducing the
stretched cross-stream coordinate \( \hat{\eta} = (y - L)/\omega^{1/2} \).
Then in the thin layer of dimensions \( l \times \omega^{1/2} \) labeled
\( \Sigma_1 \) in Fig. 1b, Eq. (2.2) becomes, with \( \hat{x} = x - a \)
\[
(1 + iv)\psi_{\eta\eta} - iG(L)\psi_x = 0. \tag{2.14}
\]
On this scale the region \( \Sigma_0 \) shrinks to the point \( \hat{x} = 0 \),
\( \hat{\eta} = 0 \) and so (2.14) is to be solved in the quarter plane
\( \hat{x} > 0, \hat{\eta} > 0 \) subject to the boundary conditions
\[
\psi = \psi(0^-, L) \quad (\hat{x} = 0, \hat{\eta} > 0) \tag{2.15}
\]
\[
\psi = 0, \quad (\hat{\eta} = 0, \hat{x} > 0). \tag{2.16}
\]
The jump in the \( \psi \) at \( (\hat{x}, \hat{\eta}) \) gives a source
for fluid carried by \( D \) and turned in \( \Sigma_0 \). The region \( \Sigma_1 \)
need not be analyzed separately as (2.14), (2.15) and
(2.16) are contained in the long-wave equation (2.4)
and its boundary conditions and so solutions on the
long-wave scale necessarily contain the required solutions
(2.14). The solution, however, is straightforward and compact, giving the qualitative structure of the flow at the contraction on scales of order the shelf-width.

Equations (2.14), (2.15), (2.16) are equivalent to a heat conduction problem with complex diffusivity \( (\nu - i)/G(L) \), having a similarity solution
\[
\psi(\hat{x}, \hat{\eta}) = \psi(0^-, L) \text{erf}(\gamma x), \tag{2.17}
\]
where erf is the complex error function, \( \gamma \) is the square root with positive real part of
\[
\gamma^2 = (\nu + i)/(1 + \nu^2),
\]
and \( \chi \) is the similarity variable,
\[
\chi = \frac{1}{2} \hat{\eta} G(L)/\hat{x}^{1/2}. \tag{2.18}
\]
Since \( \nu > 0, \gamma^2 \) lies in the first quadrant so \( |\arg\gamma| < \frac{1}{4}\pi \) and, as required by (2.15) and (2.16),
\[
\text{erf}(\gamma x) \rightarrow 1, \text{ as } x \rightarrow \infty \text{ (i.e., } \hat{x} = 0),
\]
\[
\text{erf}(\gamma x) = 0, \text{ on } x = 0 \text{ (i.e., } \hat{\eta} = 0). \tag{2.19}
\]
Streamlines are the parabolas \( \chi = \text{const} \). The tangential component of velocity in the layer is
\[
-\omega^{-1/2} \frac{\partial \psi}{\partial \hat{\eta}} = \psi(0^-, L)[G(L)/\pi\omega \hat{x}]^{1/2}
\times \exp \left[-\frac{1}{4} \frac{G(L) (\nu + i)}{1 + \nu^2} \frac{\hat{\eta}^2}{\hat{x}} \right]. \tag{2.18}
\]
Unlike the boundary layer on the incident face of the obstacle, which has constant thickness for constant \( g(s) \), this layer thickens downstream. Over distances of order \( l \) it has width of order \( [\omega(1 + \nu^2)/\nu]^{1/2}/l \), spreading to cover the shelf width by a downstream distance of order \( l/\omega \), the long-wave scale. Velocities in the layer are again large, of order \( \omega^{-1/2} \), but not as

large as in the narrower current \( l \) and so dissipation
is weaker in \( \Sigma_1 \), vanishing to leading order in \( \nu \) as \( \nu \rightarrow 0 \). Figure 4 gives profiles of the velocity within the
layer at quarter period intervals for \( \nu = 1 \) and \( \nu = 0.2 \), at \( x = G(L) \), taking \( \psi(0^-, L) = 1 \). For smaller \( \hat{x} \) the current is more intense and narrower while for larger \( \hat{x} \) it is weaker and wider. For large \( \nu \) the current is thin and for weak dissipation the current spreads rapidly. As in \( D \), the phase of the velocity fluctuations propagates inwards corresponding to outward energy propagation. That (2.17) is contained in solutions to (2.4) follows from noting that the similarity variable can be expressed entirely in terms of the long-wave variables
\[
\chi = \frac{1}{2} \frac{(y/\omega^{1/2}) [G(L)/\hat{x}]^{1/2}}{\nu G(L)/X^{1/2}}.
\]

On the long-wave scale regions \( \Sigma_0 \) and \( \Sigma_1 \) collapse onto the point \( (0, L) \) (Fig. 2b). The flux carried in \( \Sigma_1 \)
is given by a source whose strength is the difference on
\( X = 0^+ \) between \( \psi(0^+, L^+) \) and \( \psi(0^-, L^-) \), the latter vanishing by (2.3). The specification of \( \psi \) on \( X = 0^+ \) is thus complete, viz:

\[
\text{FIG. 4. The profile across the layer } \Sigma_1 \text{ of the tangential component of velocity. The velocity has been multiplied by } \omega^{1/2} \text{ since it is of order } \omega^{-1/2} \text{ in } \Sigma_1 \text{. The spatial coordinate is } \hat{x} \text{ of order } \omega^{1/2}, \text{ small compared to the shelf width. Profiles are shown at successive quarter period intervals, } \tau = 2\pi/\omega. \text{ (a) } \nu = 1. \text{ At any instant the current is almost unidirectional. (b) } \nu = 0.2. \text{ The current oscillates with distance from the wall and its strength decays less rapidly than the more viscous flow of (a). The phase propagates towards the wall corresponding to outward energy propagation. The profiles are flat at the wall. Profiles for the layer } \Sigma_1 \text{ of section 5 and Fig. 7a are thus given by continuing the profile to be even about the wall. The ordinate is then the distance from the centerline of the current.}
\[ \psi(0^+, y) = \psi(0^-, y), \quad L < y \leq 1 \quad \text{if} \quad L \geq 0 \]
\[ \psi(0^+, y) = \begin{cases} \psi(0^-, y), & 0 \leq y \leq 1 \\ 0, & L \leq y \leq 0 \end{cases} \quad \text{if} \quad L \leq 0. \tag{2.19} \]

Since all changes with along-shelf scales of order \( l \) appear abrupt on the long-wave scale, the rate of change \( L/a \) does not appear in (2.19).

Although the long-wave problem for the widening shelf requires no flux-carrying boundary layers, the above discussion for the narrowing shelf allows the form of the solution over scales of order \( l \) to be deduced. The streamfunction is continuous at the origin by (2.3), but in general \( \psi_x(0^-, 0^+) \) is nonzero and so the along-shelf transport is discontinuous. This discontinuity is smoothed out in the free shear layer \( S_1 \) of Fig. 1a, of dimension \( l \times \omega^{1/2} \) centered on \( y = 0, x > 0 \) and governed by (2.14). Within this layer the streamfunction is again a function of \( x \) alone, constant on parabolas. The layer separates parallel flow in \( y > 0 \) from quiescent fluid in \( y < 0 \), spreading to occupy the whole shelf width by downstream distances of order \( l/\omega \), i.e., \( X \) of order unity. A shadow region of length of order \( l/\omega \) is thus present behind the widening shelf. Just such a region is remarked on by WC and is visible in their Figs. 7 and 8. Velocities in the layer are of the same order as in the parallel flow in \( y > 0 \) but no larger since, unlike the case of a narrowing shelf, the present layer carries no flux.

The energy conservation relation corresponding to (2.1) can be written

\[ \left( \partial_t + 2\tilde{v} \right) E + \nabla \cdot F = 0, \tag{2.20} \]
where \( E = \frac{1}{2} H^{-1} |\nabla \psi|^2 \) is the local kinetic energy density. There is no unique expression for the energy flux \( F \) as the addition to \( F \) of any quantity whose divergence is zero leaves (2.20) unchanged. A number of forms for \( F \) are given in Johnson (1989c). A convenient choice for \( F \) in the present geometry is

\[ F = (\Psi/H)(\partial_t + \tilde{v})\nabla \Psi - \frac{1}{2} \nabla \tilde{z} \times \nabla (f/H), \tag{2.21} \]

since it vanishes on both \( C_0 \) and \( C_1 \). Integrating (2.20) over a region bounded by cross-channel planes (as indicated, for example, by \( A \) and \( B \) in Fig. 1a) and using (2.3) gives

\[ \left( \frac{d}{dt} + 2\tilde{v} \right) \int E dx dy = \left[ \int F \cdot d\hat{x} d\hat{y} \right]_A^B. \tag{2.22} \]

Energy crosses into the region at \( A \) and out at \( B \), decaying within the region on a time-scale of order \( \tilde{v}^{-1} \gg 1 \). Consider (2.22) for periodic low-frequency motion and planes fixed relative to the origin as \( \omega \to 0 \). For the case of a widening shelf \( (L > 0) \), (2.21) gives, to leading order in \( \omega \),

\[ \frac{1}{2} \int_0^L \Psi^2 (H^{-1})_x dy |_{x=0^+} = \frac{1}{2} \int_0^L \Psi^2 (H^{-1})_y |_{x=0^-}. \tag{2.23} \]

Energy is conserved across the change in width. This follows also from (2.19). For a narrowing shelf \( (L > 0) \), (2.19) and (2.22) show that there is a net flux,

\[ \frac{1}{2} \int_0^L \Psi^2 (H^{-1})_y dy |_{x=0^-}, \tag{2.24} \]

into the boundary layer on the sloping wall, where it is dissipated. As the integrand in (2.24) is non-negative, the greater the narrowing for a given incident field the greater the dissipation and loss of transmitted energy.

Although the analysis can be pursued for nonzero \( \nu \), both the present results and comparison with previous work are more straightforward in the further limit \( \nu \to 0 \). Solutions are to be regarded as the inviscid solution obtained in the limit of vanishing viscosity. The sole effect of the limit is to set \( \nu \to 0 \) in (2.4). This limit is discussed in greater detail in section 5.

3. Particular examples

For arbitrary positive \( G(y) \) the inviscid form of (2.4) can be solved by separation of variables. Hsieh and Buchwald (1985) point out that the cross-stream eigenfunctions satisfy a standard Sturm–Liouville problem and so form a complete orthogonal set. Any piecewise continuous function, and so in particular \( \psi(0^+, y) \), can be expanded in a convergent series of these functions and so the solution downstream of the change in width can be given as a series of shelf wave modes. The required normalization constants and orthogonality relations for general \( H \) are in Hsieh and Buchwald (1985). As a specific example consider the exponential profile

\[ H(y) = H_0 e^{2by}, \quad H_0, b \text{ const.} \tag{3.1} \]

Write \( \psi = \exp[b(y - 1)] \Phi \). Then incident modes are given by

\[ \Phi = \sin \pi y \exp \left[ i(m^2 \pi^2 + b^2)(X/2b) \right], \quad m = 1, 2, \ldots \tag{3.2} \]

and the scattered field in \( X > 0 \) is given by

\[ \Phi^s = \sum_{n=1}^\infty a_n \sin \left[ n \pi \left( \frac{y - L}{1 - L} \right) \right] \times \exp \left[ i \left( \frac{n \pi}{1 - L} \right)^2 + b^2 \right] \left( \frac{X}{2b} \right), \tag{3.3} \]
where, provided \((1 - L)m\) is not an integer,

\[
a_n = \begin{cases} 
\frac{2}{\pi} \frac{m(1 - L) \sin \left( \frac{n \pi L}{1 - L} \right)}{[n^2 - m^2(1 - L)^2]}, & \text{if } L < 0, \\
\frac{2}{\pi} \frac{n \sin (m \pi L)}{[n^2 - m^2(1 - L)^2]}, & \text{if } L > 0.
\end{cases}
\]

(3.4)

If \((1 - L)m = M\), an integer, then for \(L > 0\), mode \(M\) is the sole transmitted mode with amplitude unity, whereas for \(L < 0\), all modes are present, the coefficient \(a_M\) being changed to simply

\[
a_M = (-1)^{M-m} / (1 - L), \quad L < 0.
\]

For widening shelves \(L > 0\) and \(m = 1\), the expression for \(\Phi^*\) reduces to that given by Middleton and Wright (1988). They show that for \(b = 1\) the departure of the values of \(a_n\) given by (3.4) from values calculated by WC by solving (2.1) at \(\omega = 0.1\) is of order \(10^{-3}\). The present analysis has relaxed the assumption of an abrupt step and has extended the results to narrowing shelves, \(L > 0\). The observations by WC and Middleton and Wright (1988) that at low frequencies the amplitude of a scattered mode depends on its similarity to the incident field and is independent of \(b\), the topographic slope, are equally valid in the present more general context. The energy flux in mode \(n\) in the scattered field, normalized on the incident flux, is given by \((1 - L)a_n^2\). The total normalized flux is unity for \(L < 0\) and for \(L > 0\) is

\[
1 - L + \sin 2m\pi L / 2m\pi < 1, \quad (3.5)
\]

showing a flux \(L - \sin 2m\pi L / 2m\pi\) into the dissipative boundary layer. Figure 5 shows the normalized flux for the various modes and the total transmitted energy as a function of \(L\). This figure reproduces closely the results in Fig. 3 of WC for the same topography at \(\omega = 0.1\), where their \(L_2/L_1 = 1 - L\). Figure 5a shows that for a narrowing shelf the transmitted field for an incident first mode is almost entirely mode 1, whereas Fig. 5b shows that for an incident second mode the strength of the transmitted second mode decreases and that of the first mode increases as \(L\) increases from zero until for \(L > 0.5\) the transmitted field is once again dominated by the fundamental.

The presence of a source at \((0, L)\) for \(L > 0\) causes the terms in series (3.3) to decay only slowly with increasing wavenumber in this completely inviscid problem. Restoring the vanishingly small viscous effect represented by nonzero \(\nu\) introduces an imaginary part proportional to the square of the mode number to the otherwise real-channel wavenumber. Then for any \(\nu > 0\) the series for \(\Phi^*\) converges exponentially fast in \(X > 0\) as do the series for the velocity field and all its higher order derivatives.

FIG. 5. The distribution of energy flux amongst the transmitted modes at a change in shelf width normalized on the incidence flux. For a widening shelf \((L < 0)\) energy is conserved and the total transmitted and incident fluxes are equal. For a narrowing shelf \((L > 0)\) energy is dissipated in an intense oscillatory boundary current on the incident side of the narrowing. (a) Mode 1 incident. (b) Mode 2 incident.

4. Wave generation by an oscillatory geostrophic flow

The analysis of the previous sections concentrates on the case of no net instantaneous volume flux in the channel. If the net flux is nonzero the flow upstream can be separated into an incident shelf-wave field as treated in sections 2, and 3 and an oscillatory, parallel current. Then, even if the incident shelf-wave field is
absent, changes in shelf width force fluid to cross bottom contours, generating shelf waves which propagate into \( X > 0 \). In a flow with a free surface the parallel current corresponds to a Kelvin wave and the present results give a linear mechanism for the scattering of Kelvin wave energy into shelf waves. The boundary conditions when the instantaneous flux is nonzero are

\[
\psi = 0 \quad \text{(on } C_0), \quad \psi = \alpha \quad \text{(on } C_1) \tag{4.1}
\]

where \( \alpha \) is a nonzero complex constant (with dimensions of volume per unit time). The solution of the inviscid form of (2.4) in \( X < 0 \) then gives

\[
\psi = \alpha \int_0^\eta H(\eta) d\eta \int_0^\eta H(\eta) d\eta, \quad x < 0, \tag{4.2}
\]

and the solution in \( X > 0 \) can be written

\[
\psi = \alpha \int_L^\eta H(\eta) d\eta \int_L^\eta H(\eta) d\eta + \psi^w, \quad x > 0, \tag{4.3}
\]

where \( \psi^w \) consists of scattered shelf waves, satisfying (2.4), the homogeneous conditions (2.3), and determined at \( X = 0^+ \) by (2.13).

The energetics of the flow follow by considering the inviscid, time-dependent form of (2.4),

\[
\Psi_{yyT} - G(y)(\Psi_{yT} + \Psi_x) = 0, \tag{4.4}
\]

where \( T = \omega t \) is the slow time scale and the boundary conditions are

\[
\Psi = 0 \quad \text{(on } C_0), \quad \Psi = Q(T) \quad \text{(on } C_1), \tag{4.5}
\]

where the volume flux \( Q(T) \) is given by \( Q(T) = \Re \{ \alpha e^{iT} \} \) for periodic motion. In the absence of a wave-field the flow is parallel with uniform speed in \( X < 0 \),

\[
U_0(T) = Q(T)/A_0, \tag{4.6}
\]

where \( A_0 = l \int_0^L H(y) dy \) is the cross-sectional area of the shelf.

The associated vertically integrated local kinetic energy is \( \frac{1}{2} HU_0^2 \) and the instantaneous energy flux over a cross-shelf plane is \( \frac{1}{2} QU_0^2 \), with a time-average of zero. In \( X > 0 \) the deviation of the energy density from that associated with the uniform stream

\[
U_1(T) = Q(T)/A_1, \tag{4.7}
\]

for cross-sectional area \( A_1 \), in a volume \( V \) bounded by cross-shelf planes at \( B \) and \( B' \) is

\[
\frac{d}{dT} \int_V \left( \frac{1}{2} H^{-1} \Psi_y^2 - \frac{1}{2} HU_1^2 l^2 \right) dX dy = \frac{d}{dT} \int_V \left( \frac{1}{2} \Psi^w (H^{-1} \Psi_y^w + 2 U_1 l) \right) dX dy = \left[ \int_L^L \frac{1}{2} \left( f/H)_y (\Psi^w)^2 dy \right]_B^{B'}. \tag{4.8}
\]

The energy carried by the waves causes fluctuations about the energy of the current. For periodic motion (4.8) shows that the time-averaged wave energy flux over a cross-channel plane is independent of its station.

As a specific example, consider the exponential topography (3.1). Then the scattering of a current is given by

\[
\Phi = \begin{cases} 
\alpha \sinh b(y)/\sinh b(1 - L) + \alpha \Phi^w, & X > 0 \\
\alpha \sinh b(y - L)/\sinh b(1 - L) & X < 0
\end{cases}
\]

where \( \Phi^w \) is given by (3.3) with

\[
a_n = \begin{cases} 
2b(1 - L) \sin(n \pi L/1 - L)/ & L < 0 \\
[2^n \pi^2 + 2(1 - L)^2 \sinh b, & L > 0
\end{cases}
\]

The time-averaged energy flux carried over a cross-shelf plane by the transmitted waves is given by

\[
\int_L^L \frac{1}{2} (f/H)_y (\Psi^w)^2 dy = \frac{1}{2} (1 - L) b |\alpha|^2 (f/H_0) e^{-2b} \sum_{n=1}^\infty a_n^2. \tag{4.11}
\]

A convenient normalization that removes the parametric dependence on \( \alpha, H_0, \), and \( f \), is given by \( f \) times the time-averaged kinetic energy per unit length associated with the downstream uniform current, i.e.,

\[
\frac{1}{2} \int_L^L HU_1^2 dy = b |\alpha|^2 (f/H_0)/(e^{2b} - e^{2b L}). \tag{4.12}
\]

Then the energy flux associated with mode \( n \) normalized on the kinetic energy of the current is given by

\[
\frac{1}{2} (1 - L) (1 - \exp[-2b(1 - L)]) a_n^2.
\]

Figure 6 shows these fluxes as a function of \( L \) for \( b = 1 \). Figure 6a gives the distribution of flux for moderate \( L \), showing that for a narrowing shelf the wave field is far less energetic than the current squeezed to pass at high speed through the contraction, the energy associated with the wave field not exceeding 3% of that associated with the current. Figure 6b gives the distribution of flux for significant widenings of the shelf. The flux rapidly attains its maximum as \( -L \) increases, corresponding to progressively higher modes since low modes are almost constant across the shelf junction at large \( -L \). Even for this case where the continuing geostrophic flow is relatively weak, the total transmitted
wave flux is less than 15% of the energy associated with the geostrophic flow. The majority of the energy in the flow is associated with the oscillatory current or transmitted Kelvin wave in free-surface flows. For geometries where this current is absent all transmitted energy is carried by the shelf waves, as in the forcing by an oscillatory coastal volume source considered in Middleton (1988).

5. Discussion

The low-frequency scattering of barotropic shelf waves has been considered in the limit of small but nonzero dissipation. For the particularly simple case of a rectilinear shelf it has been shown that against the incident side of any obstacle there forms an intense oscillatory boundary layer destroying incident wave energy and turning smoothly any volume flux to pass round the obstacle. The structure of this layer has been discussed in detail for the case of a shelf narrowing over distances of order the shelf width \( l \). Its thickness is of order \( \omega l / \nu = \omega^2 / \tilde{\nu} \) for small \( \nu \), and viscous effects on the long-wave scale \( l/\omega \) (i.e., of \( X \)) are of order \( \nu \). Thus provided \( 1 \gg \nu \gg \omega \) (i.e., \( \omega \gg \tilde{\nu} \gg \omega^2 \)) the layer is of negligible thickness compared to \( l \) and dissipation is unimportant outside the layer. This is the dissipation range relevant for the examples in sections 3 and 4. For weaker dissipation where \( \omega \gg \nu \gg \omega^2 \) reflected short-wave energy survives over scales of order \( l/\omega \) before being dissipated but again has no effect on the long-wave scale. It is only for the weakest dissipation, \( \tilde{\nu} \ll \omega^2 \), where \( \omega \ll 1 \) that reflected waves survive over scales of order \( l/\omega \) and affect scattering on the length scale of the long waves. This latter range includes of course the purely inviscid flows with \( \tilde{\nu} = 0 \). The analysis points to intense currents and strong dissipation on the incident side of obstacles, confined to a layer of thickness \( \sigma^2 l / \tilde{\nu} \), where \( \sigma \) is the dimensional dominant frequency of long incoming waves and the dissipation \( \tilde{\nu} \) can be estimated from (2.1) as the inverse of the e-folding time for the destruction of vorticity.

It has also been shown that an oscillatory current forced to cross bottom contours scatters energy into shelf waves. In an unbounded flow with a free-surface, the current becomes a Kelvin wave and so this scattering gives a linear mechanism for Kelvin wave energy to transfer to shelf waves.

For clarity the analysis has been presented for piece-wise linear width changes. However, the argument and conclusions remain equally valid for arbitrary width changes. A simple example of a further geometry that can be considered is given by Fig. 7, consisting of an arbitrary headland of length \( IL \) and breadth \( al \). In the low frequency limit a dissipation layer \( D \) is present on the incident side of the headland. The volume flux from \( D \) is turned smoothly in a region \( S_0 \) governed by (2.13), emerging as a free current \( S_1 \) governed by (2.14) (Fig. 7a). In contrast to the free shear layer of Fig. 1a, \( S_1 \) carries a nonzero volume flux derived from \( D \). The boundary conditions on the layer are (2.15) and the impermeability condition

\[ \psi = 0 \quad (x = 0, \hat{\eta} < 0), \quad (5.1) \]

and so the similarity solution is

\[ \psi(x, \hat{\eta}) = \frac{1}{2} \psi(0^-, L)[1 + \text{erf}(\gamma x)], \quad (5.2) \]
with \( \gamma, x \) as before. The along-shelf component of velocity in the layer is

\[
-\omega^{-1/2} \frac{\partial \psi}{\partial \hat{\eta}} = \frac{1}{2} \psi(0^-, L) \left[ G(L) / \pi \omega x \right]^{1/2}
\]

\[
\times \exp \left[ -\frac{1}{4} G(L \left( 1 + \frac{\nu^2}{\omega^2} \right) \hat{\eta}^2 \right], \quad (5.3)
\]

being half as large as that of (2.18) but spreading in both \( \hat{\eta} > 0 \) and \( \hat{\eta} < 0 \) so containing the same flux. The profile is symmetric about \( \hat{\eta} = 0 \) with Fig. 4 again giving the cross-shelf structure. This intense spreading current, with velocities of order \( \omega^{-1/2} \gg 1 \), separates parallel flow in \( y > L \) from quiescent flow in \( y < L \), \( x > 0 \). This example and that of a narrowing shelf point to the possibility of intense midshelf or boundary currents in the shadow of headlands and along the edge of constrictions. In terms of the parameters defined above such currents would have width of order \( \sigma l / (\hat{\eta} f)^{1/2} \) for small \( \hat{\eta} \).

On the long-wave scale regions \( S_0 \) and \( S_1 \) of Fig. 7(a) collapse to \( (0, L) \) and the headland becomes a barrier of height \( L \) at the origin (Fig. 7b). The solution in \( x > 0 \) is given by (2.3) and (2.4) with the condition

\[
\psi(0^+, y) = \begin{cases} 
\psi(0^-, y), & L < y \leq 1 \\
0, & 0 \leq y < L.
\end{cases} \quad (5.4)
\]

For the exponential topography (3.1) and incident mode (3.2) the scattered field is given by (3.3) with

\[
a_n = \begin{cases} 
\sin[(n + m) \pi L] / (n + m) \pi, & n \neq m \\
-\sin[(n - m) \pi L] / (n - m) \pi, & n = m \\
\sin 2m \pi L / 2m \pi + 1 - L, & n = m.
\end{cases} \quad (5.5)
\]

Figure 8 shows the distribution of energy flux among the modes as a function of \( L \). The flux into the dissipation layer is the same as for the narrowing shelf treated in section 3.

The analysis has been confined to barotropic flows and appears directly relevant to observations on high-
latitude shelves as in Middleton, Foster and Foldvik (1987) who explain motions in the South Weddell Sea in terms of short barotropic waves. For midlatitude shelves, stratification may be sufficiently strong and rotation sufficiently weak that coastal-trapped waves are unidirectional at all wavelengths and frequencies (Chapman 1983). Incident wave energy could then be scattered into transmitted waves and the dissipation regions discussed here would be weak or absent.

Equation (2.1) follows directly from the conservation of barotropic potential vorticity on introducing the depth-integrated volume-flux streamfunction and noting that nonlinearity is negligible provided relative vorticity is small compared to $f$. In the bulk of the fluid relative vorticity has magnitude $U/l$ for a typical velocity $U$, and nonlinear terms are negligible provided the Rossby number $\text{Ro} = U/f l$ is small. In the dissipative boundary layer, however, the enhanced velocities and decreased length scales give a typical magnitude of $U/\omega^2 l$ for the relative vorticity there in the low-frequency limit. Nonlinearity is negligible in the layer only if $\text{Ro} \ll \omega^2$. With increasing wave amplitude nonlinear interactions first become important in this layer where they cause a time independent, rectified current of strength $\omega^{-2} \text{Ro}$ compared to the incident wave field or incident oscillatory current. For Rossby numbers of order $\omega^2$ or larger, yet still small compared to unity, the bulk of the flow is governed by linear wave dynamics while advection is important in the boundary layer and oscillatory motion is rectified there by strong nonlinearity. These dynamics will be considered in greater detail elsewhere.

The present results have a wider applicability than those examples given here and in Johnson (1989a) are used to obtain connection formulae for low frequency scattering at simultaneous changes in shelf width, depth and direction.

REFERENCES


