The Role of Variable Coriolis Parameter in the Propagation of Inertia-Gravity Waves During the Process of Geostrophic Adjustment

J. E. GEISLER

Rosenstiel Institute of Marine and Atmospheric Science, University of Miami, Coral Gables, Fla. 33124

AND R. E. DICKINSON

National Center for Atmospheric Research,[1] Boulder, Colo. 80302

(Manuscript received 28 February 1972, in revised form 13 April 1972)

ABSTRACT

This analysis treats the transient inertia-gravity wave response of a shallow fluid to an impulsive addition of momentum. The Coriolis parameter varies with latitude, but Rossby waves are not considered. The square of the Coriolis term is approximated by a constant term plus a term linear in the northward coordinate. In this approximation, monochromatic waves, which reach a turning point at the latitude where the wave frequency equals the local Coriolis frequency, are given by Airy functions. A contour integral solution to the initial value problem is expressed as a Fourier integral over wave frequency with an Airy function argument and is evaluated approximately using the stationary phase technique. The solution at a given latitude is first dominated by waves from the source and then waves reflected from turning points poleward of the source. The results are applied to give a qualitative description of the wake of a hurricane moving over a stratified ocean.

1. Introduction

The problem of geostrophic adjustment was originally formulated by Rossby (1938) and Cahn (1945) to interpret the occurrence of geostrophically balanced motions in the ocean and atmosphere. Rossby introduced the fundamental length scale of the problem defined as the ratio of long wave speed to Coriolis parameter and referred to as the *radius of deformation*. In more recent years, interest has turned to the exploration of the question of the partition of energy between the inertia-gravity waves and the balanced current system (e.g., Veronis, 1956; Washington, 1964). The present status of adjustment theory has recently been reviewed by Blumen (1972). Further theoretical examination of the properties of transient inertia-gravity waves in the ocean now seems appropriate in view of the increased sophistication of observational methods available for defining such motions. An especially attractive application of the theory from the viewpoint of the observationalist is the description of the wake of a hurricane moving over a stratified ocean (Geisler, 1970).

We consider here an initial momentum input by a zonal wind stress prescribed as a function of latitude. The solution with constant Coriolis parameter, as first obtained by Cahn, is a front of high-frequency gravity waves propagating away in latitude and trailing an inertia-gravity wave tail. At large times, an inertial oscillation remains in the region decaying as $t^{-\frac{1}{2}}$. In view of the actual variation of the Coriolis parameter with latitude, some features of Cahn's solution are of doubtful applicability. The intent of this paper is to explore in the simplest possible context the modification of horizontally propagating inertia-gravity waves by the variation of the Coriolis parameter. Where the square of the Coriolis parameter f occurs as a constant in Cahn's theory, we add a term linear in y representing the next term in a Taylor series expansion of f^2. The range of validity of this approximation has been analyzed rigorously for single-frequency waves by Munk and Phillips (1968). They have used the approximation to discuss in detail the importance of variable f for inertia-gravity waves excited by random stationary wind stresses. Jacobs (1967) has used techniques of geometric optics to provide a general analysis of the propagation of inertia-gravity waves and Rossby waves including variable f. By solving "exactly" a much simpler model equation than that considered by Jacobs, we are able to obtain a more explicit description of the effects of variable f on inertia-gravity waves which have been impulsively excited.

Variable Coriolis parameter introduces a turning point into the differential equation for single-frequency waves. Component waves traveling to the north are reflected at the latitude where their frequency is equal to the Coriolis parameter. The resulting solutions of the initial value problem differ from those of the constant Coriolis parameter case in that: (i) the latitude-inde-

[1] The National Center for Atmospheric Research is sponsored by the National Science Foundation.

pendent inertial oscillation of the constant f case is not approached at large time; rather, (ii) the phase difference between oscillations at any two latitudes grows with time and the latitudinal wavelength decreases; (iii) the oscillation frequency at a point does not monotonically descend to the local Coriolis frequency as in the constant f case, but reaches a minimum value at a certain time depending on the distance from the source point, then begins to increase as waves reflected from higher latitudes become greater in amplitude than the waves arriving directly from the source; and (iv) the response to source scales large compared to a radius of deformation is concentrated on the equatorward side of the source.

The solution for a line source of momentum acting at the initial time is evaluated by an asymptotic expression obtained by the method of stationary phase. This expression has been used as a Green's function to synthesize the solution for an extended source. More general finite-difference solutions to the governing equation have also been obtained. These have been used to check the validity of the asymptotic approximations but are not otherwise described here. The effect of the variation of the Coriolis parameter depends on two nondimensional parameters: the ratio of the source scale to the radius of deformation and the ratio of the radius of deformation to the length scale over which the Coriolis parameter changes, which can be taken to be the radius of the earth.

We establish the model in Section 2 and outline the mathematical solution in Section 3. Further details of the analysis are given in the Appendix. We show in Section 4 that the residual geostrophically balanced current system is essentially unchanged in allowing variation of f of the form considered here. The question of the wake set up by a moving hurricane is formulated in Section 5. The analysis for the response to an impulsive source given in Section 3 is shown to apply also to the moving hurricane problem with minor modifications. Readers primarily interested in the analysis of the wake of a moving hurricane may turn directly to Section 5 referring to earlier sections only where necessary by reference.

2. Governing equations

We consider motion in a shallow fluid driven by a wind stress acting on the fluid as a body force independent of depth. A rotating Cartesian (x,y) geometry with variable Coriolis parameter f is assumed. Where f^2 occurs in the governing wave equation we shall retain only the first two terms in a Taylor series expansion, i.e.,

$$f^2=f_0^2(1+2\beta y/f_0). \tag{1}$$

Symbols appearing in (1)–(6) are defined as follows:

u, v	eastward, northward velocity components
τ_x, τ_y	eastward, northward stress components
t	time
η	free surface elevation
g	gravitational constant
f_0	Coriolis parameter at $y=0$
β	$\partial f/\partial y$ at $y=0$
H	depth of the fluid
c	long wave speed, $(gH)^{\frac{1}{2}}$
ρ	density
R_D	radius of deformation, c/f_0
R_E	radius of the earth
θ	latitude

The linearized governing equations are

$$\frac{\partial u}{\partial t^*}-fv=-g\frac{\partial \eta}{\partial x^*}+\frac{\tau_x}{\rho H}, \tag{2}$$

$$\frac{\partial v}{\partial t^*}+fu=-g\frac{\partial \eta}{\partial y^*}+\frac{\tau_y}{\rho H}, \tag{3}$$

$$\frac{\partial \eta}{\partial t^*}+H\left(\frac{\partial u}{\partial x^*}+\frac{\partial v}{\partial y^*}\right)=0. \tag{4}$$

We introduce the following scaling and nondimensional parameters:

$$t=f_0t^*,\quad x=R_D^{-1}x^*,\quad y=R_D^{-1}y^*, \tag{5}$$

$$\epsilon=\frac{\beta}{f_0}R_D=\frac{R_D}{R_E}\cot\theta. \tag{6}$$

The reduction of (2)–(4) to a single differential equation governing v is standard (e.g., Longuet-Higgins, 1965). In terms of the above scaling, this equation is

$$\left\{\frac{\partial^2}{\partial t^2}\nabla^2-\left[\frac{\partial^4}{\partial t^4}+(1+2\epsilon y)\frac{\partial^2}{\partial t^2}\right]+\epsilon\frac{\partial^2}{\partial x\partial t}\right\}v$$
$$=-\frac{1}{f_0\rho H}\left\{\left[\frac{\partial^3\tau_y}{\partial t^3}-(1+\epsilon y)\frac{\partial^2\tau_x}{\partial t^2}\right]\right.$$
$$\left.-\frac{\partial^2}{\partial x\partial t}\left[\frac{\partial\tau_y}{\partial x}-\frac{\partial\tau_x}{\partial y}\right]\right\}. \tag{7}$$

Terms of $O(\epsilon^2)$ have been dropped as in (7). The above referenced paper may be referred to by the reader for the application of techniques similar to those of the present paper, but for the analysis of the Rossby wave components of (7).

In the first part of the paper, we solve (7) for a zonal wind stress that is a function of the north-south coordinate y only. After two integrations in time, the governing equation is

$$\left[\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial t^2}-(1+2\epsilon y)\right]v(y,t)=\frac{(1+\epsilon y)}{f_0\rho H}\tau_x(y,t). \tag{8}$$

We regard ϵ as a quantity of order 0.1 and $\tau_x(y,t)$ to be localized in a region where y is order unity. The term $(1+\epsilon y)$ on the right-hand side of (8) can then be replaced by unity with no significant effect on the results.

3. Response to an impulsive stress

In this section, we first derive a Fourier integral representation for the Green's function of the problem under consideration. This expression is evaluated asymptotically by the method of stationary phase and used to discuss the nature of the waves excited by sources narrow and wide compared to a radius of deformation.

Let the initial conditions be $v=\partial v/\partial t=0$ and introduce zonal momentum at the instant $t=0$. It can be seen from integration of the zonal momentum equation

$$\frac{\partial u}{\partial t}-(1+\epsilon y)v=\frac{\tau_x(y,t)}{f_0\rho H}, \tag{9}$$

through the initial time that this is accomplished by a zonal stress of the form

$$\tau_x(y,t)=T(y)\delta(t). \tag{10}$$

The Green's function for the differential operator in (8) is defined by

$$\left[\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial t^2}-(1+2\epsilon y)\right]G(y,y',t)=\delta(y-y')\delta(t), \tag{11}$$

$$G(y,y',0)=\frac{\partial G}{\partial t}(y,y',0)=0, \tag{12}$$

and the solution for the stress given by (10) is

$$v(y,t)=\frac{1}{f_0\rho H}\int G(y,y',t)T(y')dy', \tag{13}$$

where the integral is over the domain of the stress function $T(y)$.

Solutions of (11) are sought in terms of a contour integral representation with Laplace kernal,

$$G(y,y',t)=\frac{1}{2\pi i}\int_c \hat{G}(y,y',\sigma)\exp(\sigma t)d\sigma. \tag{14}$$

The function $\hat{G}(y,y',\sigma)$ satisfies the ordinary differential equation

$$\left\{\frac{d^2}{dy^2}-[\sigma^2+1+2\epsilon y]\right\}\hat{G}(y,y',\sigma)=\delta(y-y'). \tag{15}$$

Substitution of the independent variable

$$z=(2\epsilon)^{-\frac{2}{3}}[\sigma^2+1+2\epsilon y], \tag{16}$$

reduces (15) to a form of Airy's differential equation

$$\left[\frac{d^2}{dz^2}-z\right]\hat{G}(z,z',\sigma)=(2\epsilon)^{-\frac{1}{3}}\delta(z-z'). \tag{17}$$

Linearly independent solutions of the homogeneous form of (17) are the Airy functions

$$\mathrm{Ai}\{z\},\quad \mathrm{Ai}\{z\exp(2\pi i/3)\},\quad \mathrm{Ai}\{z\exp(-2\pi i/3)\}. \tag{18}$$

A solution of (17) is required to satisfy

$$\hat{G}(z,z',\sigma)\exp(\sigma t)\to 0, \tag{19}$$

when $t<0$ on the infinite semicircle in the right half of the complex σ-plane. Linearly independent pairs of Airy functions which will satisfy (19) are

$$\left.\begin{array}{lll}\mathrm{Ai}\{z\}, & \mathrm{Ai}\{z\exp(-2\pi i/3)\}, & \mathrm{Im}(\sigma)>0\\ \mathrm{Ai}\{z\}, & \mathrm{Ai}\{z\exp(2\pi i/3)\}, & \mathrm{Im}(\sigma)<0\end{array}\right\}, \tag{20}$$

and the Green's function constructed from them is

$$\hat{G}(z,z',\sigma)=-2\pi(2\epsilon)^{-\frac{1}{3}}\times\begin{cases}\exp(-\pi i/6)\ \ \mathrm{Ai}\{z\}\ \ \mathrm{Ai}\{z'\exp(-2\pi i/3)\}, & \mathrm{Im}(\sigma)>0\\ \exp(\pi i/6)\ \ \mathrm{Ai}\{z\}\ \ \mathrm{Ai}\{z'\exp(2\pi i/3)\}, & \mathrm{Im}(\sigma)<0\end{cases} \tag{21}$$

The above is for $z>z'$; for $z<z'$ interchange z and z'. The factor $2\pi\exp(\pm\pi i/6)$ comes from the Wronskian of the respective pairs. Examination of the asymptotic form of Airy functions is required to show that (21) satisfies (19). This has been included in Section 1 of the Appendix.

The path for the contour integral (14) has four pieces,

$$\int_C=\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}, \tag{22}$$

as shown in Fig. 1. With these integration paths and

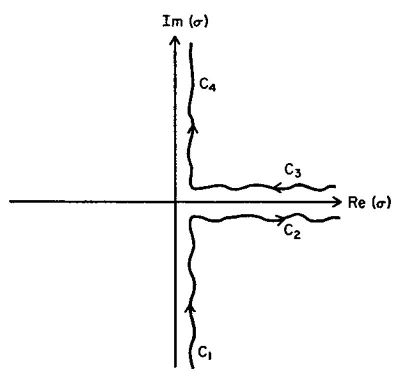

FIG. 1. Integration path for contour integral representation of solution.

(19), the causality condition $G(y,y',t)=0$ for $t<0$ follows. The pieces C_2 and C_3 arise from the branch line present on the $\text{Im}\sigma=0$ axis which is necessary to separate the two pieces of the Green's function defined by (21). The branch line contribution is of no physical interest. It represents a faster than exponential growth that occurs for large time resulting from inertial instability of regions where f^2 as defined by (1) is negative and the Taylor series expansion is no longer valid. We consider further only the other part of the solution describing gravity waves propagating in the presence of variable f. The pieces C_1 and C_4 combine into the Fourier integral

$$G(z,z',t) = -2(2\epsilon)^{-\frac{1}{3}} \int_0^\infty \text{Re}[\text{Ai}\{z\}\,\text{Ai}\{z'\exp) - 2\pi i/3)\} \times \exp\{i\omega t - \pi i/6\}]d\omega, \quad (23)$$

with the proviso that z and z' be interchanged when $z<z'$.

The function $G(z,z',t)$ given by (23) is the response to an impulsive stress with delta function latitudinal dependence centered at the latitude y'. Since information cannot travel faster than the long wave speed c, the solution is zero outside the wedge $|y-y'|<ct$. We examine the solution in two regions characterized as "close to" or "away from" the lines $|y-y'|=ct$, hereafter referred to as "the front." In our nondimensional variables, $c=1$.

The solution to (11) for constant Coriolis parameter, i.e., $\epsilon=0$, is (Cahn, 1945)

$$v(y,y',t)=\begin{cases} -\frac{1}{2}J_0[\{t^2-(y-y')^2\}^{\frac{1}{2}}], & |y-y'|\leqslant t \\ 0, & |y-y'|>t \end{cases} \quad (24)$$

With zero Coriolis parameter, the solution reduces to

$$v(y,y',t)=\begin{cases} -\frac{1}{2}, & |y-y'|\leqslant t \\ 0, & |y-y'|>t \end{cases} \quad (25)$$

Since (24) tends to (25) as $|y-y'|\rightarrow t$, it is clear that the solution near the front cannot be influenced by variation of Coriolis parameter. In terms of Fourier components, the very high-frequency modes are not affected by rotation and all propagate nondispersively as a front. Frequencies near the inertial frequency are dispersed and account for the J_0 Bessel function "tail" in (24). It is the functional form of this tail that is changed when f varies with latitude.

The lowest-order asymptotic behavior of the solution in the region away from the front can be obtained by evaluating the integral in (23) by the method of stationary phase. We assume z is large as would be the case with small ϵ [see Eq. (16)]. The stationary phase analysis is carried out in the Appendix, where we also establish that (23) reduces to (24) as $\epsilon\rightarrow 0$ and time is fixed.

The stationary phase evaluation of the integral in (23) gives a result of the form

$$G(y,y',t) \sim \begin{cases} A_d(\omega_{sp})\cos[\varphi_d(\omega_{sp})t-\pi/4], & t<t_c(y,y') \\ A_r(\omega_{sp})\cos[\varphi_r(\omega_{sp})t-\pi/4], & t>t_c(y,y') \end{cases} \quad (26)$$

where $t_c=(1+\epsilon y_>)^{\frac{1}{2}}(2\epsilon)^{-\frac{1}{2}}(y_>-y_<)^{\frac{1}{2}}$, $y_>$ being the larger of y or y', and $y_<$ the smaller of y or y'. The expressions for the factors in (26) are rather complicated and not necessary to follow the qualitative interpretation given below, so their derivation and presentation are relegated to the Appendix (Section 2). For $y>y'$, amplitudes A_d and A_r are given by (A26) and (A29); phases φ_d and φ_r are given by (A27) and (A30); and the stationary phase frequency ω_{sp} is given by (A24). For fixed ω, we interpret (26) as follows: the two terms are each locally plane waves with wavenumber $\partial(\varphi t)/\partial y$ and frequency $-\partial(\varphi t)/\partial t$ (identically equal to ω). The first term in (26) (subscript d) is the contribution at observation point y from a wave that has traveled directly from the source point at y'. The second term (subscript r) is the contribution from a wave that has traveled from source point to a reflection point at a higher latitude and back again to the observation point. The reflection point is where $\omega^2=(1+2\epsilon y)$, the square of local inertial frequency. Travel time for the wave is given by

$$t=\int dy \bigg/ \frac{\partial W}{\partial k}(k,y), \quad (27)$$

where the integral is over the path from y' to y. Here W is the frequency for plane waves where locally the square of the Coriolis parameter is $1+2\epsilon y$, i.e.,

$$W(k,y)=[(1+2\epsilon y)+k^2]^{\frac{1}{2}}, \quad (28)$$

and k is defined to be $\partial(\varphi_d t)/\partial y$ or $\partial(\varphi_w t)/\partial y$, according to whether t is less than or greater than the time required to reach the reflection point.

The frequency of the asymptotic wave is the stationary phase frequency and obtained as a function of time from (A24). What is found is that at an observation point situated at a latitude higher than that of the source, ω of the direct wave decreases with time from the rather high values near the front to the local inertial frequency. Waves reflected from increasingly higher latitudes beyond the observation point then begin to arrive, giving a reflected wave whose frequency increases with time. At an observation point situated at a latitude below that of the source, ω decreases to the inertial frequency at the source point, then increases again. After the minimum frequency has been reached, there is no longer a stationary phase point for the direct wave. The direct wave continues to ring with the local Coriolis frequency (source Coriolis frequency if the observation

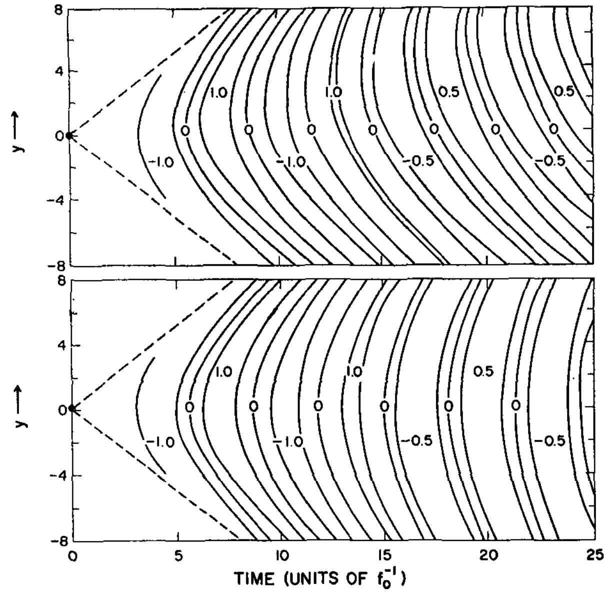

FIG. 2. Contours of the meridional component of velocity $v(y,t)$ after impulsive addition of easterly momentum with delta function dependence on y (the narrow source). Unit of meridional distance is the radius of deformation, time is in units of f_0^{-1}, and contour units are arbitrary. Top half of figure shows response when f varies; bottom half is response when f is constant. There is no response outside the region bounded by the dashed line. The asymptotic solutions are not valid close to these lines.

latitude is below the source latitude), but the time decay of this oscillation is faster than that given by the stationary phase expression and the reflected wave dominates. Likewise, the reflected wave increases from zero to its stationary phase amplitude through the contribution of nonstationary phase point frequencies very near the local Coriolis frequency. These components are elementary waves being reflected locally before waves beyond the observation point have started to arrive.

Contour diagrams of the meridional velocity component $v(y,t)$ are presented in Figs. 2 and 3. For these calculations, we have chosen $\epsilon = 0.05$. The response to the delta function source is shown in Fig. 2. In the top half of the figure is the response for variable f [Eq. (26)] and in the bottom half the response for constant f [Eq. (24)]. The stress impulse is negative (easterly momentum source). In both cases, the amplitude of the oscillation decays as $t^{-\frac{1}{2}}$ for large time. The principal effect of variable f is to alter the symmetry about the source point exhibited by the constant f case. Note also that for constant f the curvature of the phase fronts decreases with time; that is, the oscillation frequency approaches constant f at all values of y.

The behavior in Fig. 2 is essentially unchanged for any localized, smoothly varying stress with a scale much less than the radius of deformation. To illustrate the effects for a wider source, we choose the stress function $T(y)$ in (10) to be

$$T(y) = -\pi^{-\frac{1}{2}}\alpha \exp(-\alpha^2 y^2), \tag{29}$$

with $\alpha = 0.4$, that is, a Gaussian source of easterly momentum with a half-width 2.5 times the radius of deformation. The normalizing factor in (29) gives unit area under the curve, the same as in the previous case where $T(y)$ was a delta function. The response for variable f is found by evaluating the integral over the source (13) numerically using (26) for the Green's function. To evaluate the constant f case for comparison, we use the asymptotic form of (24),

$$-\tfrac{1}{2}J_0\{[t^2-(y-y')^2]^{\frac{1}{2}}\} \sim -(2\pi)^{-\frac{1}{2}}[t^2-(y-y')^2]^{-\frac{1}{4}} \times \cos\{[t^2-(y-y')^2]^{\frac{1}{2}}-\pi/4\}, \tag{30}$$

as the Green's function. Contour diagrams of $v(y,t)$ for the variable f case and the constant f case are shown, respectively, in the top and bottom halves of Fig. 3. In contrast with Fig. 2, it can be seen that the response to the wide source is characterized by some localization of amplitude in the region of the source, as evidenced by the closed contours. The top half of Fig. 3 exhibits an asymmetry in amplitude and shows that when the source is wide compared to the radius of deformation, including the variation of Coriolis parameter reduces considerably the energy in inertia-gravity wave oscillations to the north of the source region. This reduction occurs because of the dominance in the solution of low-frequency components which have already been reflected from the region of increasing Coriolis parameter to the north.

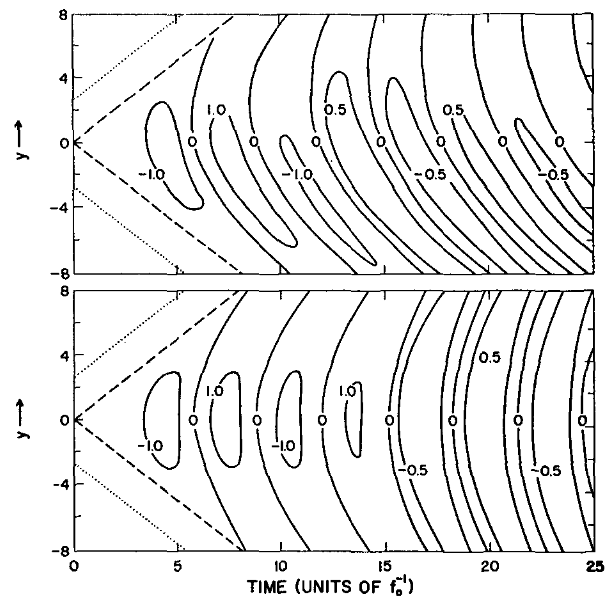

FIG. 3. Same as Fig. 1, except that the momentum source has a Gaussian dependence on y (the wide source). The dashed line indicates the arrival of the initial signal from the center of the source. The dotted line indicates the arrival of the initial signal from the closer side of the source at a distance from the center of one source width.

4. Effect of variation of f on the steady geostrophic current

We discuss here the steady geostrophic component of the motion, showing that for small ϵ the variation of f has essentially no effect on this component.

We have thus far concentrated on the meridional component of the velocity field which ultimately decays to zero. Such transient oscillations also appear in the zonal component $u(y,t)$, but this ultimately tends to a steady geostrophically balanced flow. This steady flow is found by integrating (9) to infinite time. The Laplace transform of this equation gives

$$\sigma\hat{u}(y,\sigma)=(1+\epsilon y)\hat{v}(y,\sigma)+\frac{T(y)}{f_0\rho H}. \quad (31)$$

With use of the theorem from Laplace transform theory that the value of a function $F(t)$ at infinite time is related to its transform as

$$F(\infty)=\lim_{\sigma\to 0}\sigma\hat{F}(\sigma),$$

we have from (31) and the Laplace transform of (13)

$$u(y,\infty)=\frac{1}{f_0\rho H}\left[T(y)+\int\hat{G}(y,y',0)T(y')dy'\right]. \quad (32)$$

We evaluate $G(z,z',0)$ as one-half the sum of the two terms in (21) for $\sigma=0$ or, equivalently, as the argument of (23) for $\omega=0$. For an easterly source $T(y')$ characterized by a width L that is less than the radius of deformation,

$$\int\hat{G}(y,y',0)T(y')dy'\approx-\hat{G}(y,0,0)T(0)L$$

$$T(0)L=\begin{cases}\pi(2\epsilon)^{-\frac{1}{3}}\,\mathrm{Ai}\{(2\epsilon)^{-\frac{2}{3}}(1+2\epsilon y)\}\,\mathrm{Bi}\{(2\epsilon)^{-\frac{2}{3}}\}, & y>0\\ \pi(2\epsilon)^{-\frac{1}{3}}\,\mathrm{Ai}\{(2\epsilon)^{-\frac{2}{3}}\}\,\mathrm{Bi}\{(2\epsilon)^{-\frac{2}{3}}(1+2\epsilon y)\}, & y<0\end{cases}$$

$$\sim T(0)L\exp(-|y|) \quad (33)$$

where $\mathrm{Bi}(z)$ is defined as

$$\mathrm{Bi}(z)=\exp(\pi i/6)\,\mathrm{Ai}\{z\exp(2\pi i/3)\}+\exp(-\pi i/6)\,\mathrm{Ai}\{z\exp(-2\pi i/3)\}. \quad (34)$$

For $\epsilon=0.05$, the asymptotic form gives correct values to within a few percent. In contrast, the wave component of the solution is always very different from that obtained with constant f at large enough time. Thus, for a source narrow compared to the radius of deformation, the steady zonal current decays exponentially to either side of the source region on an e-folding scale equal to the radius of deformation—the same result as obtained for a constant f-plane. Since the response to a wider source is formally the superposition of responses to a distribution of small sources [with $T(0)\to\infty$ and $L\to 0$ in such a way that $T(0)L$ is constant], we conclude that the zonal current forced by a smoothly varying $T(y)$ wide compared to the radius of deformation has a functional dependence on y that is very close to that of $T(y)$ itself.

Blumen and Washington (1969) have derived a potential vorticity equation which can be used to obtain the steady balanced current evolving from arbitrary initial conditions with the complication of horizontal shears.

5. Wake of a moving hurricane

We consider the application of results derived in the previous section to the problem of the inertia-gravity wave response of a stratified ocean to a moving hurricane. This problem was treated previously (Geisler, 1970) in the context of a two-layer ocean with constant Coriolis parameter. The wake is defined as the inertia-gravity wave train which is steady in the coordinate frame moving with the forcing function. Here we consider a shallow homogeneous fluid, recognizing that the response in a stratified ocean is described in terms of a sum of vertical structure normal modes, each of which satisfies the homogeneous fluid equations for some equivalent depth.

The equations are modified to govern the steady response in a frame traveling in the negative x direction with speed U by replacing $\partial/\partial t$ with $(U/c)\partial/\partial x$. After two integrations in x, (7) becomes

$$\left[\frac{\partial^2}{\partial y^2}-\left(\frac{U^2}{c^2}-1\right)\frac{\partial^2}{\partial x^2}-(1+2\epsilon y)\right]v(x,y)=F(x,y), \quad (35)$$

where the forcing function with $O(\epsilon)$ terms again dropped is

$$F(x,y)=-\frac{1}{f_0\rho H}\left[\left(\frac{U}{c}\frac{\partial\tau_y}{\partial x}-\tau_x\right)-\frac{c}{U}\left(\frac{\partial\tau_y}{\partial x}-\frac{\partial\tau_x}{\partial y}\right)\right]. \quad (36)$$

The operator $\epsilon\partial^2/\partial x\partial t$ in (7), previously dropped because of the assumption of no dependence on x, would give rise to a term $(\epsilon c/U)$ inside the brackets in (35). We consider only the case $U>c$ and $\epsilon\sim O(0.1)$. We omit this term so that our equations do not describe the Rossby wave dispersion of the geostrophically balanced current system far behind the hurricane, which would occur for $(\epsilon c/U)>1$.

We regard the hurricane as a localized region of stress having positive curl and no divergence following Geisler (1970), who showed that the width of the region in the direction of translation controls the magnitude of the baroclinic mode inertia-gravity wave wake. Specifically, if the dimensions of the storm and translation speed are such that the stress is put in over an inertial day

($2\pi f^{-1}$ sec) or longer, then the wake is minimal. Since we are interested here only in the modification of the *form* of the wake by a variable Coriolis parameter, we make the mathematically convenient approximation that the width of the storm in the direction of propagation is very narrow. We let the positive curl be provided by a stress $\tau_x = T(y)$ that has a delta function character in x, converting (36) to

$$F(x,y) = -\frac{1}{f_0 \rho H}\left[\frac{c}{U}\frac{\partial}{\partial y} - 1\right] T(y)\delta(x). \quad (37)$$

Finally, we consider only the case $U > c$, since for $U < c$ the differential operator in (35) becomes elliptic and there is no wake. This restriction confines the applicability of our results to internal modes for typical hurricanes moving a few meters per second. Since we are interested only in the horizontal structure of each mode, we still discuss the results in terms of the shallow fluid with a free surface model.

The most interesting variable in this problem is $w = U(\partial\eta/\partial x)$, the vertical velocity of the free surface. To relate this to v, we replace $\partial/\partial t$ by $U\partial/\partial x$ so that (2) and (4) become

$$\frac{U}{c}\frac{\partial u}{\partial x} - (1+\epsilon y)v = -\frac{g}{f_0 U}w + \frac{\tau_x}{f_0 \rho H}, \quad (38)$$

$$w + \frac{f_0 H}{c}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \quad (39)$$

In (38) and (39), x and y are the nondimensional coordinates defined by (5) with primes suppressed. Solving for w, we have

$$w = -\frac{U f_0 H}{(U^2 - c^2)}\left\{\left[(1+\epsilon y) + \frac{U}{c}\frac{\partial}{\partial y}\right]v + \frac{\tau_x}{f_0 \rho H}\right\}. \quad (40)$$

Let L denote the differential operator in (35) and L^{-1} its inverse, so that the solution of (35) can be written formally as

$$v(x,y) = L^{-1}F(x,y). \quad (41)$$

It can be shown by expressing w and v as an ordered expansion in ϵ that for lowest order the operator L is commutative with respect to the operator enclosed by brackets in (40). Writing τ_x as $LL^{-1}\tau_x$, we then have (40) in the form

$$w = -\frac{U f_0 H}{(U^2 - c^2)} L^{-1}\left\{\left[(1+\epsilon y) + \frac{U}{c}\frac{\partial}{\partial y}\right]F(x,y) + \frac{L\tau_x}{f_0 \rho H}\right\}, \quad (42)$$

and after carrying out the indicated operations on $F(x,y)$ and τ_x, and neglecting a contribution from the stress divergence, we find

$$w = -\frac{1}{\rho c}L^{-1}\left(\frac{\partial \tau_x}{\partial y}\right) = -\frac{1}{\rho c}L^{-1}\left(\frac{\partial T}{\partial y}\delta(x)\right). \quad (43)$$

Let $\tilde{G}(y,y',x)$ be the Green's function for the operator L. This is related to the Green's function $G(y,y',t)$ for the operator discussed in Section 3 [Eq. (8)] as

$$\tilde{G}(y,y',x) = \gamma G(y,y',\gamma x), \quad (44)$$

where

$$\gamma = \left(\frac{U^2}{c^2} - 1\right)^{-\frac{1}{2}}. \quad (45)$$

Eq. (43) can therefore be written

$$w(\gamma x,y) = -\frac{\gamma}{\rho c}\int G(y,y',\gamma x)\frac{\partial T}{\partial y'}dy'. \quad (46)$$

This expression is the same as (13) from which the results shown in the figures were derived, except that the forcing in (46) is the stress curl rather than the stress itself. The limiting case $U \gg c$ can be more simply derived using (39) with neglect of $\partial u/\partial x$ to replace v with w in (13).

To model the hurricane as a concentrated source of positive wind stress curl, we choose $\partial T/\partial y$ to have the same functional form as (29), that is, the negative of a Gaussian with a half-width equal to 2.5 times the radius of deformation. If the signs on the contours are changed, Fig. 3 gives contours of vertical velocity on the free surface in response to the moving positive wind stress curl pattern. The constant f pattern is very similar in form to the corresponding figure in Geisler (1970). The change in the pattern brought about by variation of f (with the choice of $\epsilon = 0.05$) can be seen by comparing the top and bottom parts of Fig. 3.

These oscillations are superimposed on a trough in the free surface which accompanies the geostrophically balanced current system which remains along the storm track after the oscillations have dispersed. According to the results of Section 4, the shape of the trough will be essentially the same as that of the wind stress curl pattern.

6. Discussion

The analysis of the preceding section applies to a storm translating with speed U over a homogeneous ocean that is sufficiently shallow that $U > c$. In the open ocean, the condition $U > c$ is met only by the internal modes. The horizontal structure equation for each of these modes is the same as that for a homogeneous ocean except that the long wave speed and hence radius of deformation involves the eigenvalue of the vertical structure equation. The relative contribution of each normal mode depends on the details of the stratification and how the stress is distributed as a function of depth. The numerical results described in the preceding section were obtained for a value of $\epsilon=0.05$, corresponding to a radius of deformation that is, for example, 70 km at 15° latitude and about 140 km at 30° latitude. Our results thus indicate that if the dominant vertical modes excited by the hurricane have a radius of deformation of this size or larger, then the pattern of the wake will exhibit the asymmetry seen in the top half of the figures. Modes with much smaller radius of deformation will not feel the effect of variable Coriolis parameter for the time scale of a few days appropriate to the figures and the wake will have the symmetric form seen in the bottom half of the figures.

The choice of Gaussian wind stress curl pattern is not especially realistic in the case of an actual hurricane, since it does not model the broad region of weak negative stress curl that surrounds the narrow region of strong positive curl. We have determined the response for a number of cases using a curl function which is obtained from the second derivative of a Gaussian, but incorporating a scaling factor that allows the shape and magnitude of the negative curl region to be changed while the positive curl region is held constant. The results are somewhat sensitive to these changes, the most conspicuous modification to the constant f case being the appearance in some cases of a series of closed contours trailing behind the two zones of maximum negative curl. However, our conclusions as to the dependence of symmetry of the wake upon the magnitude of the radius of deformation are not altered by the addition of this feature.

Acknowledgments. The authors are grateful for the assistance of F. Clare and R. Pearson in writing the computer programs used in conjunction with this study. A. Lundberg typed the manuscript and W. Blumen read it and offered helpful suggestions. J. Geisler was supported during this work by the National Science Foundation under Grant GA-26452.

APPENDIX

1. Asymptotic behavior of the Airy function product

We examine first the asymptotic behavior of the product of Airy functions in (21) for large σ in the right half of the complex plane. We write $\sigma=|\sigma|\exp(i\theta)$ and since $z\rightarrow|\sigma|^2\exp(2i\theta)$ for σ large, we examine the product

$$P(\sigma,\theta)=\mathrm{Ai}\{\lambda\exp(2i\theta)\}\,\mathrm{Ai}\{\lambda\exp(2i\theta\mp2\pi i/3)\},\qquad\text{(A1)}$$

where $\lambda=(2\epsilon)^{-\frac{2}{3}}|\sigma|^2$ is real. It is necessary to show that (A1) satisfies (19), that is, for $t<0$

$$\lim_{|\sigma|\rightarrow\infty}P(\sigma,\theta)\exp(\sigma t)=0\qquad\text{(A2)}$$

in the right half-plane. The minus sign in (A1) is taken for $\mathrm{Im}(\sigma)>0\,(0<\theta<\pi/2)$ and the plus sign for $\mathrm{Im}(\sigma)<0\,(-\pi/2<\theta<0)$. Unless otherwise noted by an absolute value sign, we assume throughout the treatment in the Appendix that $y>y'$; for $y<y'$, interchange y and y' in the results. All asymptotic forms used are taken from Abramowitz and Stegun (1964). A particular form is never valid over the whole plane (Stokes phenomenon) and it is therefore necessary to consider various sectors separately.

Consider first $0<\theta<\pi/3$. Retaining only the lowest-order term in the asymptotic expansions, we have

$$\mathrm{Ai}\{\lambda\exp(2i\theta)\}\sim\tfrac{1}{2}\pi^{-\frac{1}{2}}\lambda^{-\frac{1}{4}}\exp(-\zeta-i\theta/2),\qquad\text{(A3)}$$

$$\mathrm{Ai}\{\lambda\exp(2i\theta-2\pi i/3)\}\sim\tfrac{1}{2}\pi^{-\frac{1}{2}}\lambda^{-\frac{1}{4}}\times\exp(-\zeta'-i\theta/2+\pi i/6),\qquad\text{(A4)}$$

where

$$\zeta=\tfrac{2}{3}\lambda^{\frac{3}{2}}\exp(3i\theta),\quad\zeta'=\tfrac{2}{3}\lambda^{\frac{3}{2}}\exp(3i\theta-\pi i).\qquad\text{(A5)}$$

Multiplication of (A3) and (A4) gives

$$P(\sigma,\theta)\sim\tfrac{1}{4}\pi^{-1}\lambda^{-\frac{1}{2}}\exp(-i\theta+\pi i/6),\qquad\text{(A6)}$$

which satisfies condition (A2).

For $\pi/3<\theta<\pi/2$, we have for lowest-order terms

$$\mathrm{Ai}\{\lambda\exp(2i\theta)\}=\mathrm{Ai}\{-\lambda\exp(2i\theta-\pi i)\}\sim\tfrac{1}{2}\pi^{-\frac{1}{2}}\lambda^{-\frac{1}{4}}[\exp(i\zeta-i\theta/2)+i\exp(-i\zeta-i\theta/2)],\qquad\text{(A7)}$$

$$\mathrm{Ai}\{\lambda\exp(2i\theta-2\pi i/3)\}\sim\tfrac{1}{2}\pi^{-\frac{1}{2}}\lambda^{-\frac{1}{4}}\exp(-\zeta'-i\theta/2+\pi i/6),\qquad\text{(A8)}$$

where

$$\zeta=\tfrac{2}{3}\lambda^{\frac{3}{2}}\exp(3i\theta-3\pi i/2),\quad\zeta'=\tfrac{2}{3}\lambda^{\frac{3}{2}}\exp(3i\theta-\pi i).\qquad\text{(A9)}$$

Multiplication of (A7) and (A8) gives

$$P(\sigma,\theta)=\tfrac{1}{4}\pi^{-1}\lambda^{-\frac{1}{2}}[\exp(-i\theta+\pi i/6)-\exp(-2\zeta'-i\theta+\pi i/6)].\qquad\text{(A10)}$$

Since $\mathrm{Re}(\zeta)>0$ for all values of θ in the range considered, $P(\sigma,\theta)$ satisfies (A2).

For $\mathrm{Im}(\sigma)<0$, we select the plus sign before the $2\pi i/3$ in (A1). The treatment is the same as that given above, with θ replaced by $-\theta$ and π by $-\pi$. Both (A6) and (A10) still satisfy condition (A2), so the result is established also for $\mathrm{Im}(\sigma)<0$.

We consider next the product of Airy functions on the real axis appearing in (23). For $y>y'$

$$P(\omega)=\mathrm{Ai}\{z\}\,\mathrm{Ai}\{z'\exp(-2\pi i/3)\},\qquad\text{(A11)}$$

where

$$\left.\begin{aligned}z&=(2\epsilon)^{-\frac{2}{3}}(-\omega^2+1+2\epsilon y)\\z'&=(2\epsilon)^{-\frac{2}{3}}(-\omega^2+1+2\epsilon y')\end{aligned}\right\}.\qquad\text{(A12)}$$

Lowest-order terms in the asymptotic expansions are, for $z<0$ (i.e., $\omega^2>1+2\epsilon y$)

$$\mathrm{Ai}\{z\}=\mathrm{Ai}\{-|z|\}\sim\frac{1}{2i}\pi^{-\frac{1}{2}}|z|^{-\frac{1}{4}}[\exp(i\tfrac{2}{3}|z|^{\frac{3}{2}}+i\pi/4)-\exp(-i\tfrac{2}{3}|z|^{\frac{3}{2}}-i\pi/4)],\qquad\text{(A13)}$$

and for $z'<0$ (i.e., $\omega^2>1+2\epsilon y'$)

$$\mathrm{Ai}\{z'\exp(-2\pi i/3)\}=\mathrm{Ai}\{|z'|\exp(\pi i/3)\}$$
$$\sim\tfrac{1}{2}\pi^{-\frac{1}{2}}|z'|^{-\frac{1}{4}}\exp(-i\tfrac{2}{3}|z'|^{\frac{3}{2}}-i\pi/12). \quad \text{(A14)}$$

Lowest-order terms for $z>0$ (i.e., $\omega^2<1+2\epsilon y$) are

$$\mathrm{Ai}\{z\}\sim\tfrac{1}{2}\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\exp(-\tfrac{2}{3}z^{\frac{3}{2}}), \quad \text{(A15)}$$

and for $z'>0$ (i.e., $\omega^2<1+2\epsilon y'$)

$$\mathrm{Ai}\{z'\exp(-2\pi i/3)\}=\mathrm{Ai}\{-z'\exp(\pi i/3)\}$$
$$\sim\frac{1}{2i}\pi^{-\frac{1}{2}}z'^{-\frac{1}{4}}\exp(-\pi/12)$$
$$\times[\exp(-\tfrac{2}{3}z'^{\frac{3}{2}}+i\pi/4)-\exp(\tfrac{2}{3}z'^{\frac{3}{2}}-i\pi/4)]. \quad \text{(A16)}$$

Let $\zeta=\frac{2}{3}|z|^{\frac{3}{2}}$, $\zeta'=\frac{2}{3}|z'|^{\frac{3}{2}}$. Multiplication of (A13) and (A14) gives

$$P(\omega)\sim\tfrac{1}{4}\pi^{-1}(\tfrac{3}{2}\zeta)^{-1/6}(\tfrac{3}{2}\zeta')^{-1/6}\exp(\pi i/6)$$
$$\times\{\exp[i(\zeta-\zeta')-i\pi/2]+\exp[-i(\zeta+\zeta')]\}, \quad \text{(A17)}$$

valid when $\omega^2>(1+2\epsilon y)$. Multiplication of (A15) and (A16) gives

$$P(\omega)\sim\tfrac{1}{4}\pi^{-1}(\tfrac{3}{2}\zeta)^{-1/6}(\tfrac{3}{2}\zeta')^{-1/6}\exp(\pi i/6)$$
$$\times\{\exp[(\zeta'-\zeta]-\exp[-(\zeta+\zeta')+i\pi/2]\}, \quad \text{(A18)}$$

valid when $\omega^2<(1+2\epsilon y')$. Finally, multiplication of (A14) and (A15) gives

$$P(\omega)\sim\tfrac{1}{4}\pi^{-1}(\tfrac{3}{2}\zeta)^{-1/6}(\tfrac{3}{2}\zeta')^{-1/6}$$
$$\times\exp[-i\zeta'-\zeta-i\pi/12], \quad \text{(A19)}$$

valid when $\omega^2<(1+2\epsilon y)$, $\omega^2>1+2\epsilon y'$.

2. Stationary phase analysis

Substitution of the asymptotic form (A17) into (23) gives

$$G(z,z',t)=-\frac{1}{2\pi}(2\epsilon)^{-\frac{1}{3}}[I_d+I_r], \quad \text{(A20)}$$

where

$$I_d=\int_{\omega 0}^{\infty}(\tfrac{3}{2}\zeta)^{-1/6}(\tfrac{3}{2}\zeta')^{-1/6}$$
$$\times\cos[(\zeta-\zeta')+\omega t+\pi/2]d\omega, \quad \text{(A21)}$$

$$I_r=\int_{\omega 0}^{\infty}(\tfrac{3}{2}\zeta)^{-1/6}(\tfrac{3}{2}\zeta')^{-1/6}\cos[(\zeta+\zeta')-\omega t]d\omega, \quad \text{(A22)}$$

using $\omega_0=(1+2\epsilon y)^{\frac{1}{2}}$. The stationary phase analysis for integrals of this form gives (Jeffreys, 1962, p. 40)

$$I=\int_{\omega 0}^{\infty}g(\omega)\cos[\varphi(\omega)t]d\omega$$
$$\sim g(\omega_{sp})[2\pi t|\varphi''|]^{-\frac{1}{2}}\cos[\varphi t\pm\pi/4], \quad \text{(A23)}$$

where all functions are evaluated at the stationary phase point $\omega_{sp}>\omega_0$ and the plus or minus sign corresponds to the sign of φ''.

The stationary phase point occurs where $\varphi'=0$. After some algebra involving elimination of square root terms by squaring, we obtain for (A21) a quadratic expression for ω^2 whose real root is given by

$$\omega_{sp}^2=\Omega_{sp}^2\tfrac{1}{2}\{1+\epsilon(y+y')$$
$$+\{[1+\epsilon(y+y')]^2+\epsilon^2[t^2-(y-y')^2]\}]^{\frac{1}{2}}, \quad \text{(A24)}$$

where $\Omega_{sp}=t/[t^2-(y-y')^2]^{\frac{1}{2}}$ is the local frequency for the constant f solution. We find the same stationary phase point for (A22). However, spurious roots may be introduced in the processes of squaring, so it is necessary to substitute (A24) into $\varphi'=0$ to establish valid roots. This substitution shows that for small time and $y>y'$, ω is a stationary phase point for (A21) and that (A22) has no stationary phase point. With passage of time, ω^2 given by (A24) decreases to ω_0^2, the endpoint of the integral (and local f^2) at $t=t_c=(1+2\epsilon y)^{\frac{1}{2}}(2\epsilon)^{-\frac{1}{2}}(y-y')^{\frac{1}{2}}$. At greater times, (A24) is a stationary phase point for (A22). For $y<y'$, ω^2 given by (A24) decreases to $(1+2\epsilon y')$ (f^2 at the source point) and then becomes a stationary phase point for (A22).

It is not possible to determine simply an asymptotic expression for the integrals when the stationary phase point coalesces with the endpoint. Furthermore, the approximation to the Airy function (A17) is not valid very near the endpoint. However, for $t\gg t_c$, an endpoint contribution to (A21) can be determined proportional to $t^{-1}e^{i\omega_0 t}$ as compared to $O(t^{-\frac{1}{2}})$ decay of the stationary phase contribution. Likewise, (A22) for $t\ll t_c$ gives a contribution oscillating with the local Coriolis frequency but small compared to the stationary phase contribution obtained for $t>t_c$.

We consider in detail only the stationary phase contributions. Evaluating terms in (A23) at the stationary phase point, we find

$$I_d\sim A_d(\omega)\cos\{\varphi_d(\omega)t-\pi/4\}, \quad t<t_c, \quad \text{(A25)}$$

where

$$A_d(\omega)=\pi(2\epsilon)^{\frac{1}{3}}\{[\omega^2-(1+2\epsilon y)]^{\frac{1}{2}}-[\omega^2-(1+2\epsilon y')]^{\frac{1}{2}}\}^{-\frac{1}{2}}$$
$$\times\{[\omega^2-(1+2\epsilon y)]^{\frac{1}{2}}$$
$$\times[\omega^2-(1+2\epsilon y')]^{\frac{1}{2}}-\omega^2\}^{-\frac{1}{2}}, \quad \text{(A26)}$$

$$\varphi_d(\omega)=\tfrac{2}{3}(2\epsilon t)^{-1}\{[\omega^2-(1+2\epsilon y)]^{\frac{3}{2}}$$
$$-[\omega^2-(1+2\epsilon y')]^{\frac{3}{2}}\}+\omega, \quad \text{(A27)}$$

$$I_r\sim A_r(\omega)\cos[\varphi_r(\omega)t-\pi/4], \quad t>t_c, \quad \text{(A28)}$$

where

$$A_r(\omega)=\pi(2\epsilon)^{\frac{1}{3}}\{[\omega^2-(1+2\epsilon y)]^{\frac{1}{2}}+[\omega^2-(1+2\epsilon y')]^{\frac{1}{2}}\}^{-\frac{1}{2}}$$
$$\times\{[\omega^2-(1+2\epsilon y)]^{\frac{1}{2}}$$
$$\times[\omega^2-(1+2\epsilon y')]^{\frac{1}{2}}+\omega^2\}^{-\frac{1}{2}}, \quad \text{(A29)}$$

$$\varphi_r(\omega)=-\tfrac{2}{3}(2\epsilon t)^{-1}\{[\omega^2-(1+2\epsilon y)]^{\frac{3}{2}}$$
$$+[\omega^2-(1+2\epsilon y)]^{\frac{3}{2}}\}+\omega. \quad \text{(A30)}$$

Above is for $y>y'$. For $y<y'$, we interchange y and y' in (A26) and (A27).

3. Reduction to constant f case

It is clear from (A24) that for $y\sim y'\sim O(1)$, $\epsilon t\ll 1$, the phase of direct wave reduces to that of the asymptotic expression for the solution with constant f as given in (30). We show here in greater detail how we can recover the known solution for constant f (14) when $\beta\rightarrow 0$, i.e., $\epsilon\rightarrow 0$, but t is fixed. For small ϵ, z and z' in (23) become large and we may replace the product of Airy functions by its asymptotic form. Consider $y>y'$. When $\omega^2>(1+2\epsilon y)$, the correct form is (A17). Now from definition of ζ as it appears in (A17), for small ϵ,

$$\zeta=\tfrac{2}{3}(2\epsilon)^{-1}[\omega^2-(1+2\epsilon y)]^{\frac{3}{2}}$$
$$\approx\tfrac{2}{3}(\omega^2-1)^{\frac{3}{2}}(2\epsilon)^{-1}\left[1-\frac{3}{2}\frac{2\epsilon y}{(\omega^2-1)}\right], \quad \text{(A31)}$$

and similarly, for ζ' so that in (A17) as $\epsilon\rightarrow 0$

$$\zeta-\zeta'\rightarrow -(\omega^2-1)^{\frac{1}{2}}|y-y'|, \quad \text{(A32)}$$
$$\zeta+\zeta'\rightarrow \tfrac{4}{3}(2\epsilon)^{-1}(\omega^2-1)^{\frac{3}{2}}. \quad \text{(A33)}$$

When $\omega^2<(1+2\epsilon y')$, the correct form is (A18). For small ϵ

$$\zeta=\tfrac{2}{3}(2\epsilon)^{-1}[-\omega^2+1+2\epsilon y]^{\frac{3}{2}}$$
$$\approx\tfrac{2}{3}(1-\omega^2)^{\frac{3}{2}}(2\epsilon)^{-1}\left[1+\frac{3}{2}\frac{(2\epsilon y)}{(1-\omega^2)^{\frac{3}{2}}}\right], \quad \text{(A34)}$$

and similarly for ζ' so that in (A18) as $\epsilon\rightarrow 0$

$$\zeta'-\zeta\rightarrow -(1-\omega^2)^{\frac{1}{2}}|y-y'|, \quad \text{(A35)}$$
$$\zeta+\zeta'\rightarrow \tfrac{4}{3}(2\epsilon)^{-1}(1-\omega^2)^{\frac{3}{2}}. \quad \text{(A36)}$$

Then as $\epsilon\rightarrow 0$, (A32) and (A33) apply for $\omega^2>1$, (A35) and (A36) for $\omega^2<1$, and (23) becomes

$$G(y,y',t)\approx -\{I_1+I_2+I_3+I_4\}. \quad \text{(A37)}$$

We do not include here the contribution from (A19) since the width of the band of frequencies over which this expression is valid is proportional to ϵ and it follows that the contribution to (23) decreases more rapidly than ϵ. The other contributions are

$$I_1=\frac{1}{2\pi}\int_0^1 (1-\omega^2)^{-\frac{1}{2}} \times\exp[-(1-\omega^2)^{\frac{1}{2}}|y-y'|]\cos(\omega t)d\omega, \quad \text{(A38)}$$

$$I_2=-\frac{1}{2\pi}\int_1^\infty (\omega^2-1)^{-\frac{1}{2}} \times\sin[(\omega^2-1)^{\frac{1}{2}}|y-y'|-\omega t]d\omega, \quad \text{(A39)}$$

$$I_3=\frac{1}{2\pi}\int_0^1 (1-\omega^2)^{-\frac{1}{2}} \times\exp[-\tfrac{4}{3}(2\epsilon)^{-1}(1-\omega^2)^{\frac{3}{2}}]\sin(\omega t)d\omega, \quad \text{(A40)}$$

$$I_4=\frac{1}{2\pi}\int_1^\infty (\omega^2-1)^{-\frac{1}{2}}\cos[\tfrac{4}{3}(2\epsilon)^{-1}(\omega^2-1)^{\frac{3}{2}}-\omega t]d\omega. \quad \text{(A41)}$$

Integrals in (A38) and (A39) are given in Magnus and Oberhettinger (1949, pp. 118–119) and the result is

$$I_1+I_2=\tfrac{1}{2}J_0\{[t^2-(y-y')^2]^{\frac{1}{2}}\}. \quad \text{(A42)}$$

The integrals I_3 and I_4 are the contribution from waves reflected back from higher latitudes. But since $\epsilon\rightarrow 0$, we can for any frequency $\omega^2>1$ choose ϵ small enough that no reflected wave of this frequency has yet reached the observation point in time t. We have evaluated I_3 and I_4 asymptotically with $(2\epsilon)^{-1}$ as the large parameter. The contribution for $\omega\approx 1$ goes to zero as $O(\epsilon^{-\frac{1}{3}})$. The stationary phase contribution to I_4 goes to zero as $O((\epsilon t)^{-1})$.

REFERENCES

Abramowitz, M., and I. A. Stegun, 1964: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Washington, D. C., National Bureau of Standards, 1046 pp.

Blumen, W., 1972: Geostrophic adjustment. *Rev. Geophys. Space Phys.*, **10** (in press).

——, and W. M. Washington, 1969: The effect of horizontal shear flow on geostrophic adjustment in a barotropic fluid. *Tellus*, **21**, 167–176.

Cahn, A., 1945: An investigation of the free oscillations of a simple current system. *J. Meteor.*, **2**, 113–119.

Geisler, J. E., 1970: Linear theory of the response of a two-layer ocean to a moving hurricane. *Geophys. Fluid Dyn.*, **1**, 249–272.

Jacobs, S. J., 1967: An asympotic solution of the tidal equations. *J. Fluid Mech.*, **30**, 417–438.

Jeffreys, H., 1962: *Asymptotic Approximations*. Oxford University Press, 141 pp.

Longuet-Higgins, M. S., 1965: The response of a stratified ocean to stationary or moving wind systems. *Deep-Sea Res.*, **12**, 923–973.

Magnus, W., and F. Oberhettinger, 1949: *Formulas and Theorems for the Functions of Mathematical Physics*. New York, Chelsea Publ. Co., 172 pp.

Munk, W., and N. Phillips, 1968: Coherence and band structure of the inertial motion in the sea. *Rev. Geophys. Space Phys.*, **6**, 447–472.

Rossby, C. G., 1938: On the mutual adjustment of pressure and velocity distributions in certain simple current systems, II. *J. Marine Res.*, **1**, 239–263.

Veronis, G., 1956: Partition of energy between geostrophic and non-geostrophic oceanic motions. *Deep-Sea Res.*, **3**, 157–177.

Washington, W. M., 1964: A note on the adjustment towards geostrophic equilibrium in a simple fluid system. *Tellus*, **16**, 530–533.