Theoretical Aspects of Wind-Driven Currents in a Sea or Lake of Variable Depth with No Horizontal Mixing

G. E. BIRCHFIELD

Dept. of Engineering Sciences and Geological Sciences, Northwestern University, Evanston, Ill. 60201

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ABSTRACT

Steady-state wind-driven circulations in a basin of variable depth, with only vertical turbulent mixing, are considered. The geostrophic flow away from the coast is determined for an arbitrary bottom topography consisting of only closed depth contours and with depth at the coast small but larger than $E^{\frac{1}{2}}$, where E is the Ekman number. Conditions under which cross-contour geostrophic flow can occur are investigated. A coastal boundary layer of horizontal width $E^{\frac{1}{2}}$ is required to bring the normal transport to zero on the coast. Although the layer cannot incorporate upwelling explicitly, it does bring about mass balance by transporting mass along the coast from regions of mass surplus to regions of mass deficit. A brief comparison of conventional transport models and this three-dimensional model is made.

1. Introduction

This paper is concerned with the steady-state dynamics of wind-driven currents in a homogeneous body of water sufficiently large for the earth's rotation to be important but small enough to insure that the variation of the local component of the earth's rotation with latitude is unimportant. For this scale of basin the scale of the wind disturbance can for many purposes be assumed large compared to that of the basin. It is the purpose here to examine, for a basin of this scale, the role of bottom topography and vertical turbulent mixing in the overall three-dimensional mass balance of wind-generated circulations. Since the effects of horizontal turbulent mixing are neglected, only the total normal transport is brought to zero at the coast.

Observation of coastal currents in the Great Lakes as shown by Birchfield and Davidson (1967) and more recently with considerably more spatial detail by Csanady (1972), exhibit a fairly well defined barotropic coastal longshore flow with maximum speeds on the order of 50 cm sec^{-1} and with a width on the order of 10 km. Csanady's data indicate the center of the current is between 3 and 10 km from the coast. Since coastal slopes are $O(10^{-3})$ it is evident that the coastal current is in water of depth of several tens of meters, that is, in water at least as deep as the appropriate Ekman depth (10 m, say). It is plausible then to use boundary layer methods in the vertical resolution of the flow field and to assume a small but nonvanishing depth at the coast.

This analysis will be concerned primarily with a basin of arbitrary but smooth bottom and shore topography, with the important restrictions that all depth contours be closed and the coastal depth be

greater than $E^{\frac{1}{2}}$ times the characteristic depth, where E is the Ekman number. With the simplified coastal boundary condition of zero normal transport the model is essentially equivalent to the transport streamfunction models that have been used many times in studies of steady-state ocean dynamics (e.g., Welander, 1968) and more recently in numerical models of lake circulations (e.g., Gedney, 1971; Rao and Murty, 1970). By the use of asymptotics the mechanics of mass balance can be understood more completely than from the streamfunction formulation, particularly with regard to the exchange of mass between the coastal region and the interior of the basin; and to some extent in terms of coastal upwelling dynamics.

The general steady-state topographic-dependent solution away from the coastal region is examined in some detail. Because this interior solution in general does not satisfy the boundary conditions at the coast, a coastal boundary layer is required, the thickness of which depends on the vertical Ekman number. An example solution is presented for the circular parabolic basin.

2. Formulation of the model equations

The dimensional equations of motion and continuity are

$$f^* \hat{\mathbf{K}} \times \mathbf{V}^* = -\frac{1}{\rho^*} \nabla^* p^* + A_1 \nabla_z^2 \mathbf{V}^* + A_2 \frac{\partial^2 \mathbf{V}^*}{\partial z^{*2}},$$

$$\nabla^* \cdot \mathbf{V}^* \equiv \hat{\mathbf{K}} \cdot \text{curl}(\hat{\mathbf{K}} \times \mathbf{V}^*) + \hat{\mathbf{K}} \cdot \nabla^*(\mathbf{V}^* \cdot \hat{\mathbf{K}}) = 0,$$

where $f^* = 2\Omega \sin \varphi$ is the Coriolis parameter, $\hat{\mathbf{K}}$ is the unit vector normal to a level surface, Ω the earth's angular velocity, φ the latitude, $\mathbf{V}^*$ the three-dimen-

sional velocity vector, ρ^* the density, p^* the pressure, ∇^* the three-dimensional gradient operator, ∇_2^{*2} the two-dimensional Laplacian operator in the plane normal to $\hat{\mathbf{K}}$, and A_1, A_2 are the horizontal and vertical eddy viscosity coefficients. The basin bottom is given by $z^* = -h^*(x^*, y^*)$, and $\mathbf{n}$ is the unit vector normal to the bottom. With $z^* = 0$ at the undisturbed free surface, the boundary conditions are:

$$z^* = 0: \quad A_2 \hat{\mathbf{K}} \cdot \nabla^* \mathbf{V}_h^* = \tau^*(x^*, y^*), \quad \hat{\mathbf{K}} \cdot \mathbf{V}^* = 0$$

$$\mathbf{V}_h^* \equiv -\hat{\mathbf{K}} \times (\hat{\mathbf{K}} \times \mathbf{V}^*)$$

$$z^* = -h^*: \quad \mathbf{V}^* = 0$$

$$\text{On the coast:} \quad \mathbf{V}^* = 0$$

The model equations are scaled by the characteristic velocities U^*, W^* , the horizontal basin dimension L^* , the depth h_M^* , and the wind stress τ_0^* , where the aspect ratio

$$\delta \equiv h_M^*/L^* = O(10^{-3});$$

and

$$W = \delta U^*, \quad U^* = \frac{\tau_0^*}{\rho^*} / (A_2 f/2)^{1/2}.$$

Thus,

$$\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + \delta w\hat{\mathbf{K}}.$$

The pressure is scaled $p = 2p^*/(\rho^* U^* L^* f^*)$ and the stress $\tau = \tau^*/\tau_0^*$. The two Ekman numbers are defined by

$$E_1 = 2A_1/(L^{*2} f^*) \quad \text{and} \quad E_2 = 2A_2/(D^{*2} f^*).$$

On setting $E_1 = 0, E_2 = E$, the model equations become

$$+2\hat{\mathbf{K}} \times \mathbf{V} = -\nabla p + E(\hat{\mathbf{K}} \cdot \nabla) \delta^2 \mathbf{V}, \quad (2.1)$$

$$\nabla \cdot \mathbf{V} = \hat{\mathbf{K}} \cdot \text{curl}(\hat{\mathbf{K}} \times \mathbf{V}) + (\hat{\mathbf{K}} \cdot \nabla)(\hat{\mathbf{K}} \cdot \mathbf{V}) = 0, \quad (2.2)$$

where $\mathbf{V} \cdot \hat{\mathbf{K}} = \delta w$, $\hat{\mathbf{K}} \cdot \nabla = \delta^{-1} \partial/\partial z$. Note that (2.1) is independent of δ . Further, the vertical equation is

$$0 = -\hat{\mathbf{K}} \cdot \nabla p + E \delta^2 (\hat{\mathbf{K}} \cdot \nabla)^2 (\mathbf{V} \cdot \hat{\mathbf{K}}),$$

and therefore $\partial p/\partial z = O(\delta^2 E)$. Thus, ∇p , to any order considered here, is a horizontal vector. Eq. (2.1) can then be regarded as having only horizontal components.

The boundary conditions become

$$\left. \begin{aligned} z=0: \quad \delta E^{1/2} (\hat{\mathbf{K}} \cdot \nabla) (\hat{\mathbf{K}} \times \mathbf{V}) &= \hat{\mathbf{K}} \times \boldsymbol{\tau}, \quad \hat{\mathbf{K}} \cdot \mathbf{V} = 0 \\ z=-h: \quad \mathbf{V} &\equiv -\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{V}) + (\hat{\mathbf{n}} \cdot \mathbf{V}) \hat{\mathbf{n}} = 0 \\ \text{At the coast:} \quad \hat{\mathbf{i}} \cdot \int_{-h_0}^0 \hat{\mathbf{K}} \times (\hat{\mathbf{K}} \times \mathbf{V}) dz &\equiv \hat{\mathbf{i}} \cdot \mathbf{M} = 0 \end{aligned} \right\}, \quad (2.3)$$

where $\hat{\mathbf{i}}$ is normal to the coastal wall of height h_0 , and $\mathbf{M}$ is the horizontal mass transport. Since $\delta \ll 1$, the

unit normal vector may be written to $O(\delta^2)$

$$\hat{\mathbf{n}} = \hat{\mathbf{K}} + \delta \nabla h;$$

therefore, the boundary condition at $z = -h$, in (2.3), may be written approximately

$$\left. \begin{aligned} \hat{\mathbf{n}} \cdot \mathbf{V} &= (\hat{\mathbf{K}} + \delta \nabla h) \cdot \mathbf{V} = 0 \\ \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{V}) &\approx \hat{\mathbf{K}} \times (\hat{\mathbf{K}} \times \mathbf{V}) = 0 \end{aligned} \right\}, \quad (2.3a)$$

the latter being valid to $O(\delta)$. Thus, the tangential boundary condition may be replaced by a condition on the horizontal velocity, while the normal boundary condition cannot be modified.

The procedure for solution is to assume for the dependent variables a composite solution of the form

$$\mathbf{V} \sim \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 + \hat{\mathbf{V}}_1 + \hat{\mathbf{V}}_3, \quad (2.4)$$

where $\mathbf{V}_1$ is the inviscid or interior solution, $\mathbf{V}_2$ is valid near the free surface, $\mathbf{V}_3$ is valid near the bottom, $\hat{\mathbf{V}}_1$ is valid at a distance $O(E^{1/2})$ from the coast, and $\hat{\mathbf{V}}_3$ is valid at a distance $O(E^{1/2})$ from the coast, near the bottom. The pressure is similarly represented. It is assumed that the component solutions become transcendentally small away from their region of validity. Each component solution is represented by an asymptotic expansion in E . In order to adequately discuss the global mass balance it is necessary to obtain the solution through $O(E^{1/2})$.

3. The solution away from the coast

An approximate solution valid away from the coast will be obtained for an arbitrary smooth bottom topography provided all depth contours are closed. The method is simply that of Greenspan (1968), applied to a geophysical problem. The no-slip boundary condition used here will be extended to a linear slip condition in the next section. It is assumed that

$$[\mathbf{V}_1(x, y), \mathbf{V}_2(x, y, \zeta), \mathbf{V}_3(x, y, \zeta')] = \sum_{j=1}^N E^{(j-1)/4} [\mathbf{V}_{1j}, \mathbf{V}_{2j}, \mathbf{V}_{3j}]$$

where

$$z = E^{1/2} \zeta, \quad (z+h) = E^{1/2} \zeta'.$$

On substitution of the expansion in (2.1), (2.2) and ordering with respect to E , it is found that

$$2\hat{\mathbf{K}} \times \mathbf{V}_{1j} = -\nabla p_{1j} \quad (3.1)$$

$$\left. \begin{aligned} \hat{\mathbf{K}} \cdot \text{curl}(\hat{\mathbf{K}} \times \mathbf{V}_{1j}) &= \frac{\partial w_{1j}}{\partial z} = 0 \end{aligned} \right\} \quad j=1, 2, 3. \quad (3.2)$$

The bottom boundary condition (2.3) requires

$$\nabla h \cdot \mathbf{V}_{11} = 0,$$

that is, the low-order geostrophic flow is along depth contours. Since ∇p_{1j} is horizontal and perpendicular

to $\mathbf{V}_{1j}$, we have $p_{11} = p_{11}(h)$ and from (3.1)

$$\mathbf{V}_{11} = -\frac{1}{2} \frac{\partial p_{11}}{\partial h} \hat{\mathbf{K}} \times \nabla h. \quad (3.3)$$

The surface Ekman layer solution $\mathbf{V}_2$ is obtained in the usual manner and will not be described. The Ekman suction velocity determines

$$\hat{\mathbf{K}} \cdot \mathbf{V}_{13} = \frac{1}{2} \delta \hat{\mathbf{K}} \cdot \text{curl} \boldsymbol{\tau}. \quad (3.4)$$

The horizontal mass transport in the surface layer is

$$\mathbf{M}_2 = E^{\frac{1}{2}} \int_{-\infty}^0 \mathbf{V}_{21} d\zeta' = \frac{1}{2} E^{\frac{1}{2}} \boldsymbol{\tau} \times \hat{\mathbf{K}}. \quad (3.5)$$

The continuity equation near the bottom may be written to a high degree of approximation (see Greenspan, 1968) as

$$\hat{\mathbf{n}} \cdot \text{curl}(\hat{\mathbf{n}} \times \mathbf{V}) + (\hat{\mathbf{n}} \cdot \nabla)(\hat{\mathbf{n}} \cdot \mathbf{V}) = 0.$$

If the normal distance off the bottom is $z' = E^{\frac{1}{2}} \zeta'$, we have

$$\left. \begin{aligned} 2\hat{\mathbf{K}} \times \mathbf{V}_{31} &= \hat{\mathbf{n}} \frac{\partial p_{33}}{\partial \zeta'} + \frac{\partial^2 \mathbf{V}_{31}}{\partial \zeta'^2} + O(\delta^2) \\ \hat{\mathbf{n}} \cdot \text{curl}(\hat{\mathbf{n}} \times \mathbf{V}_{31}) + \delta^{-1} \frac{\partial}{\partial \zeta'} (\hat{\mathbf{n}} \cdot \mathbf{V}_{33}) &= 0 \\ \zeta' = 0: \mathbf{V}_{11} + \mathbf{V}_{13} &= \hat{\mathbf{n}} \cdot (\mathbf{V}_{33} + \mathbf{V}_{13}) = 0 \end{aligned} \right\} \quad (3.6)$$

From the solution the mass flux along the bottom is found to be

$$\begin{aligned} \mathbf{M}_3 &= E^{\frac{1}{2}} \int_0^\infty \mathbf{V}_{31} d\zeta' = \frac{1}{2} E^{\frac{1}{2}} (\hat{\mathbf{n}} \times \mathbf{V}_{11} - \mathbf{V}_{11}) \\ &\approx \frac{1}{2} E^{\frac{1}{2}} (\hat{\mathbf{K}} \times \mathbf{V}_{11} - \mathbf{V}_{11}), \end{aligned} \quad (3.7)$$

while at $z' = 0$:

$$\hat{\mathbf{n}} \cdot \mathbf{V}_{33} = -\frac{1}{2} \delta \hat{\mathbf{n}} \cdot \text{curl} \mathbf{V}_{11} \approx -\frac{1}{2} \delta \hat{\mathbf{K}} \cdot \text{curl} \boldsymbol{\tau}. \quad (3.8)$$

The normal boundary condition yields the vorticity equation for the interior flow:

$$\frac{1}{2} \hat{\mathbf{K}} \cdot \text{curl} \boldsymbol{\tau} - \frac{1}{2} \hat{\mathbf{K}} \cdot \text{curl} \mathbf{V}_{11} + \nabla h \cdot \mathbf{V}_{13} = 0. \quad (3.9)$$

Since $\mathbf{V}_{13}$ is nondivergent we may write $\mathbf{V}_{13} = \text{curl} \mathbf{B}$, $\mathbf{B} = \mathbf{B}(x, y)$; integrating over the free surface Σ_s enclosed by a depth contour h , we have

$$\int_{\Sigma_s} \hat{\mathbf{K}} \cdot \mathbf{V}_{13} d\Sigma = \int_{\Sigma_s} \hat{\mathbf{K}} \cdot \text{curl} \mathbf{B} d\Sigma = \oint_h \mathbf{B} \cdot d\mathbf{s} \quad (3.10)$$

using Stokes theorem. If Σ_b is the bottom surface enclosed by h , we have

$$\int_{\Sigma_b} \mathbf{n} \cdot \mathbf{V}_{13} d\Sigma = \int_{\Sigma_b} \mathbf{n} \cdot \text{curl} \mathbf{B} d\Sigma = \oint_h \mathbf{B} \cdot d\mathbf{s}.$$

From (3.6) and (3.8)

$$\begin{aligned} \int_{\Sigma_b} \mathbf{n} \cdot \mathbf{V}_{13} d\Sigma &= - \int_{\Sigma_b} \mathbf{n} \cdot \mathbf{V}_{33} d\Sigma = \frac{1}{2} \delta \int_{\Sigma_b} \hat{\mathbf{K}} \cdot \text{curl} \mathbf{V}_{11} d\Sigma \\ &= \frac{1}{2} \delta \oint_h \mathbf{V}_{11} \cdot d\mathbf{s}. \end{aligned}$$

Equating integrals and using (3.4) we have

$$\Gamma(h) \equiv \oint_h \mathbf{V}_{11} \cdot d\mathbf{s} = \int_{\Sigma_s} \hat{\mathbf{K}} \cdot \text{curl} \boldsymbol{\tau} d\Sigma = \oint_h \boldsymbol{\tau} \cdot d\mathbf{s}. \quad (3.11)$$

Geostrophic circulation Γ is generated only if there is a nonzero average wind stress curl on the free surface enclosed by h .

Finally, integrating (3.3) around h we have

$$\oint_h \mathbf{V}_{11} \cdot d\mathbf{s} = -\frac{1}{2} \frac{\partial p_{11}}{\partial h} \oint_h \hat{\mathbf{K}} \times \nabla h \cdot d\mathbf{s} = \oint_h \boldsymbol{\tau} \cdot d\mathbf{s}, \quad (3.12)$$

$$\frac{\partial p_{11}}{\partial h} = 2 \oint_h \boldsymbol{\tau} \cdot d\mathbf{s} / \oint_h \hat{\mathbf{K}} \times \nabla h \cdot d\mathbf{s}. \quad (3.13)$$

The $O(1)$ pressure and velocity field is now determined in terms of the wind stress curl and the topography. The cross-contour $O(E^{\frac{1}{2}})$ velocity can be determined from (3.9). To find the component along contours let

$$\mathbf{V}_{13} = u_{13} \frac{\nabla h}{|\nabla h|} + v_{13} \frac{\hat{\mathbf{K}} \times \nabla h}{|\nabla h|} + w_{13} \hat{\mathbf{K}}.$$

Since $\text{div} \mathbf{V}_{13} = 0$ and u_{13} is known, an equation for v_{13} is found. Following an argument similar to that used in obtaining (3.12), a further constraint on v_{13} is

$$\oint_h \mathbf{V}_{13} \cdot d\mathbf{s} = 0,$$

that is, the $O(E^{\frac{1}{2}})$ horizontal flow has zero circulation.

4. Discussion of the interior flow

The total steady-state mass transport is simply

$$\begin{aligned} \mathbf{M} &= \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = h(\mathbf{V}_{11} + E^{\frac{1}{2}} \mathbf{V}_{13}) \\ &\quad + \frac{1}{2} E^{\frac{1}{2}} \boldsymbol{\tau} \times \hat{\mathbf{K}} + \frac{1}{2} E^{\frac{1}{2}} (\hat{\mathbf{K}} \times \mathbf{V}_{11} - \mathbf{V}_{11}) + \dots, \end{aligned} \quad (4.1)$$

the sum of the geostrophic transport, the surface Ekman transport, and the bottom Ekman transport, respectively. If the wind stress curl vanishes, the only remaining steady-state horizontal transport is the $O(E^{\frac{1}{2}})$ surface transport 90° *cum sole* to the wind stress, since $\mathbf{V}_{11}$, $\mathbf{V}_{13}$ vanish. The flow is then independent of the topography. Since the normal transport component will not in general vanish at the coast, a coastal boundary layer is required.

If the wind stress curl is non-vanishing, a much larger geostrophic flow is required to drive the bottom Ekman layer which exchanges mass with the surface layer via Ekman suction. If $W \equiv \frac{1}{2} \hat{\mathbf{K}} \cdot \text{curl} \boldsymbol{\tau}$ is uniform, using (3.3), (3.12) and (4.1), the geostrophic transport is

$$\mathbf{M}_1 = W h G(h) \hat{\mathbf{K}} \times \nabla h + E^{\frac{1}{2}} h \mathbf{V}_{13},$$

$$G(h) \equiv -2A(h) / \oint |\nabla h| ds,$$

where $A(h)$ is the surface area enclosed by contour h . If $W > 0$, a local valley in bottom topography has a counterclockwise geostrophic circulation, a local hill has clockwise circulation, that is, the circulation is such that deep water is always on the left, facing downstream. A local saddle has clockwise circulation around the ridges and counterclockwise circulation around the troughs. From (4.1) it is seen that the bottom $O(E^{\frac{1}{2}})$ transport is, for $W > 0$, 135° to the left of the current, that is, toward deeper water.

Contrasting with a uniform depth basin, there is no $O(E^{\frac{1}{2}})$ interior geostrophic flow in the absence of a wind stress curl, and hence surface displacement is confined to regions of shallow water. With a nonzero wind-stress curl the cross-contour geostrophic flow is at most $O(E^{\frac{1}{2}})$. The vorticity equation (3.9) states that, in general, the curl of the bottom stress, $\hat{\mathbf{K}} \cdot \text{curl} \mathbf{V}_{11}$, does not exactly balance the generation of vorticity by the wind stress, balance being achieved by a displacement of columns across depth contours. Using (3.13) and (3.3) the bottom stress curl can be evaluated and (3.9) solved for u_{13} . If $W \equiv \frac{1}{2} \hat{\mathbf{K}} \cdot \text{curl} \boldsymbol{\tau}$ is again assumed uniform, we have

$$\left. \begin{aligned} \hat{\mathbf{K}} \cdot \text{curl} \mathbf{V}_{11} &= 2W \left[G \nabla^2 h + \frac{dG}{dh} |\nabla h|^2 \right] \\ u_{13} &= \frac{W}{\beta} \left[G \nabla^2 h + \frac{dG}{dh} |\nabla h|^2 - 1 \right] \end{aligned} \right\}, \quad (4.2)$$

where $\beta = |\nabla h|$. Since the right side of (4.2) now depends only on the bottom topography, it can be computed for any particular basin. Evaluated near the coast it can indicate regions of maximum horizontal exchange (if any) of water between the coastal boundary layer and the interior of the basin associated with wind stress curl circulations.

One can show that for any exactly elliptical basin, if $W = \text{constant}$, $u_{13} = 0$ for all h ; hence, the generation of vorticity by wind stress curl is balanced by the bottom stress curl. A basin topography for which this is not true, however, is a nearly elliptical basin, e.g.

$$h = h_0(1 - \mu), \quad \mu = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) [1 + \epsilon f(\theta)], \quad \epsilon \ll 1,$$

where $f(\theta)$ is an arbitrary periodic function of polar coordinate θ , and r is the radius vector. If $f(\theta)$ has zero mean then by a lengthy but straightforward calculation it can be shown that $G(h)$ is independent of h to $O(\epsilon^2)$:

$$G = \frac{-a^2 b^2}{a^2 + b^2} \frac{1}{2h_0} + O(\epsilon^2) \equiv G_0 + O(\epsilon^2).$$

Hence the bottom stress curl is

$$\hat{\mathbf{K}} \cdot \text{curl} \mathbf{V}_{11} = -2W G_0 \nabla^2 h + O(\epsilon^2), \quad (4.3)$$

and

$$u_{13} = -\frac{W}{\beta} [G_0 \nabla^2 h - 1] + O(\epsilon^2). \quad (4.4)$$

For the special case

$$f(\theta) = \cos(\theta - \theta_0)$$

after further calculations it is found that

$$\frac{2u_{13}h_0}{a^2W} = \frac{\epsilon f(\theta)}{g(\theta)} \left[1 + \gamma^{-1} \left(g + \frac{\epsilon f'(\theta) \sin 2\theta}{f} \right) \right] + O(\epsilon^2), \quad (4.5)$$

where

$$f' = \frac{df}{d\theta}, \quad g = 1 + e \sin^2 \theta, \quad e = (a^2 - b^2)/b^2, \quad \gamma = \frac{1}{2} \left(1 + \frac{a^2}{b^2} \right).$$

From (4.3) it is seen that the bottom stress curl depends only on the *local* Laplacian of the depth field. Hence, from (4.4) if $\nabla^2 h \geq (1/G_0)$ locally, then $u_{13} \geq 0$, in order to conserve vorticity. For a nearly circular basin

$$g = 1, \quad e = 0, \quad \gamma = 1$$

$$\frac{u_{13}h_0}{a^2W} = \epsilon \cos(\theta - \theta_0)$$

Assuming $W > 0$, cross-contour flow occurs toward deeper water ($u_{13} > 0$) with a maximum at $\theta = \theta_0$, the point of maximum outward bulge of the basin; maximum flow to shallow water occurs at $\theta_0 + \pi$. For a nearly elliptical basin of eccentricity $\alpha = 3$, the cross-contour flow is shown in Fig. 1 for three different values of θ_0 . If the maximum basin distortion occurs on the major axis ($\theta_0 = \pi$, say) fairly sharp maxima in u_{13} occur likewise on the major axis, again with flow to deeper water occurring at the point of maximum perturbation. If the maximum basin distortion occurs on the minor axis ($\theta_0 = \pi/2$), a broad cross flow occurs, with weak maxima near $\theta = 0, \pi$.

If one takes the horizontal divergence of (4.1), the vorticity equation (3.9) is obtained. If, however, Eq. (4.1) is solved for $\mathbf{V}_1$ and then $\mathbf{V}_1$ eliminated by taking

the horizontal divergence, an equation for the transport alone is obtained:

$$\nabla^2 \psi - \frac{2}{h} \nabla \psi \cdot \nabla h - 2E^{-1} \hat{\mathbf{K}} \times \nabla \psi \cdot \nabla h = h \hat{\mathbf{K}} \cdot \text{curl} \boldsymbol{\tau} + \nabla h \cdot \hat{\mathbf{K}} \times \boldsymbol{\tau} + O(E^{\frac{1}{2}}), \quad (4.6)$$

where $\mathbf{M} = \hat{\mathbf{K}} \times \nabla \psi$. This is the deep water form of the equation derived by Felsenbaum (1956) and used by Gedney (1971) in a numerical model of Lake Erie. The first and second terms on the left are from the bottom Ekman transport, the terms on the right from the surface Ekman transport. In the absence of wind stress curl a balance occurs between the last term on the left and the last term on the right. Because of the shallowness of Lake Erie and the corresponding large Ekman number such a simple flow away from the coast is not observed in the numerical integration of Gedney.

For nonvanishing wind stress curl the third term on the left forces the $O(1)$ flow to be along depth contours; the $O(E^{\frac{1}{2}})$ transport will come from a balance between the third terms on the left and the right sides. Thus, the bottom transport terms remain negligible. Again for a shallow lake this must be modified significantly.

An equation of the same form as (4.6) is obtained by setting the bottom stress proportional to the average velocity in a column of water. The bottom transport is then

$$\mathbf{M}_3 = \frac{1}{2} \kappa \hat{\mathbf{K}} \times \mathbf{M} / h,$$

where $\kappa = 2kC^*/(h_M^* f^*)$, C^* is a characteristic bottom speed, and k a nondimensional constant. If $\kappa = E^{\frac{1}{2}}$, taking $\text{div}(\mathbf{M}/h)$ of the transport equation (4.1) gives exactly equation (4.6) to $O(E^{\frac{1}{2}})$.

In (2.3) the no-slip boundary condition has been imposed at the bottom of the basin. This may be relaxed by demanding instead that at the bottom the stress is parallel to the bottom velocity:

$$z = -h: \quad \hat{\mathbf{K}} \times (\hat{\mathbf{K}} \times \mathbf{V}) = \gamma E^{\frac{1}{2}} \frac{\partial}{\partial z} [\hat{\mathbf{K}} \times (\hat{\mathbf{K}} \times \mathbf{V})],$$

where γ is an $O(1)$ constant, which must be determined empirically. The solution follows through with little change. The vorticity equation (3.9) now becomes

$$\frac{1}{2} \hat{\mathbf{K}} \cdot \text{curl} \boldsymbol{\tau} - \frac{1}{2} \frac{(1+2\gamma)}{(1+\gamma)^2 + \gamma^2} \hat{\mathbf{K}} \cdot \text{curl} \mathbf{V}_{11} + \nabla h \cdot \mathbf{V}_{13} = 0,$$

Eq. (3.13) becomes

$$\frac{\partial \phi_{11}}{\partial h} = \frac{2[(1+\gamma)^2 + \gamma^2]}{1+2\gamma} \oint_h \boldsymbol{\tau} \cdot d\mathbf{s} / \oint_h \hat{\mathbf{K}} \times \nabla h \cdot d\mathbf{s},$$

the bottom transport equation (3.7) becomes

$$\mathbf{M}_3 = - \frac{1}{2} \frac{1}{(1+\gamma)^2 + \gamma^2} [(1+2\gamma) \hat{\mathbf{K}} \times \mathbf{V}_{11} - \mathbf{V}_{11}],$$

FIG. 1. Geostrophic transport across depth contours at the coast, in a nearly elliptical basin with major axis along polar coordinate $\theta = 0, \pi$. Position on the basin wall is indicated by θ . Position of maximum distortion of basin topography is at $\theta = \theta_0, \theta_0 + \pi$.

and Eq. (4.6) becomes

$$\frac{(1+2\gamma)}{(1+\gamma)^2 + \gamma^2} \left[\nabla^2 \psi - \frac{2}{h} \nabla \psi \cdot \nabla h \right] - 2E^{-1} \hat{\mathbf{K}} \times \nabla \psi \cdot \nabla h = h \hat{\mathbf{K}} \cdot \text{curl} \boldsymbol{\tau} + \nabla h \cdot \hat{\mathbf{K}} \times \boldsymbol{\tau} + O(E^{\frac{1}{2}}).$$

Since $\gamma = O(1)$ the effect of bottom slip is seen to reduce the size of the "effective" Ekman number and hence the width of the coastal boundary layer to be discussed next.

5. The coastal boundary current

Since the normal component of the transport $\mathbf{M}$ in (4.1) does not vanish at the coast to $O(E^{\frac{1}{2}})$ even in the case of a uniform wind stress, a coastal boundary layer is required. One could treat directly the total transport equation (4.6) as a boundary layer problem, but more information is obtained by considering the original equations (2.1), (2.2) and boundary conditions (2.3). The boundary layer at the coast has thickness $O(E^{\frac{1}{2}})$; in order for the boundary layer correction variables to satisfy the boundary conditions at the basin bottom an additional Ekman layer is required at $z = -h$.

For the coastal region axes $\hat{\mathbf{i}}$ normal and inward to

the basin wall, and $\hat{\mathbf{j}} = \hat{\mathbf{K}} \times \hat{\mathbf{i}}$ are chosen. On introduction of $x = E^{\frac{1}{2}}\eta$, we have in components

$$\left. \begin{aligned} -2v &= -E^{-\frac{1}{2}} \frac{\partial p}{\partial \eta} + E \frac{\partial^2 u}{\partial z^2} \\ 2u &= -\frac{\partial p}{\partial y} + E \frac{\partial^2 v}{\partial z^2} \\ E^{-\frac{1}{2}} \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \end{aligned} \right\} \quad (5.1)$$

The boundary conditions require at the coast $\hat{\mathbf{i}} \cdot \mathbf{M} = 0$ and at $z = -h$, $\mathbf{V} = 0$. Eqs. (2.4) are substituted in the above after assuming:

$$\begin{aligned} \hat{u}_1 &= E^{\frac{1}{2}} \hat{u}_{12} + \dots \\ \hat{v}_1 &= E^{\frac{1}{2}} \hat{v}_{12} + \dots \\ \hat{w}_1 &= E^{\frac{1}{2}} \hat{w}_{12} + \dots \\ \hat{p}_1 &= E^{\frac{1}{2}} \hat{p}_{13} + \dots \end{aligned}$$

The resulting equations are

$$\left. \begin{aligned} -2\hat{v}_{12} &= -\frac{\partial \hat{p}_{13}}{\partial \eta} \\ 2\hat{u}_{13} &= -\frac{\partial \hat{p}_{13}}{\partial y} \\ \frac{\partial \hat{u}_{23}}{\partial \eta} + \frac{\partial \hat{v}_{12}}{\partial y} + \frac{\partial \hat{w}_{12}}{\partial z} &= 0 \end{aligned} \right\} \quad (5.2)$$

that is, the horizontal flow is geostrophic and $\hat{w}_{12} = 0$.

Near the bottom, we let again $(z+h) = E^{\frac{1}{2}}\zeta'$, giving

$$\begin{aligned} \hat{u}_3 &= E^{\frac{1}{2}} \hat{u}_{32} + \dots \\ \hat{v}_3 &= E^{\frac{1}{2}} \hat{v}_{32} + \dots \\ \hat{w}_3 &= E^{\frac{1}{2}} \hat{w}_{33} + \dots \\ \hat{p}_3 &= O(E) \end{aligned}$$

On substitution Eqs. (5.1) reduce to the Ekman equations

$$\left. \begin{aligned} \hat{v}_{32} &= \hat{u}_{32} \zeta' \\ -\hat{u}_{32} &= \hat{v}_{32} \zeta' \\ \frac{\partial \hat{u}_{32}}{\partial \eta} + \delta^{-1} \frac{\partial}{\partial \zeta'} (\mathbf{V}_{33} \cdot \hat{\mathbf{n}}) &= 0 \end{aligned} \right\} \quad (5.3)$$

The boundary conditions at $\zeta' = 0$ are

$$\left. \begin{aligned} \hat{v}_{32} + \hat{v}_{12} &= 0 \\ \hat{u}_{32} &= 0 \\ \hat{\mathbf{n}} \cdot (\mathbf{V}_{13} + \mathbf{V}_{33}) &= 0 \end{aligned} \right\} \quad (5.4)$$

From (5.3), (5.4) it is found at $\zeta' = 0$ that

$$\hat{\mathbf{n}} \cdot \mathbf{V}_{33} = -\frac{1}{2} \delta \frac{\partial \hat{v}_{12}}{\partial \eta}.$$

Hence, the equation for vorticity balance is

$$\nabla h \cdot \hat{\mathbf{V}}_{13} - \frac{1}{2} \frac{\partial \hat{v}_{12}}{\partial \eta} = 0. \quad (5.5)$$

Thus, transport across depth contours, necessary for bringing the interior normal flow to zero, is made possible by having the stretching term balanced by vorticity dissipation at the bottom.

From (5.2) we have

$$\hat{p}_{13} \eta \eta + 2\beta \hat{p}_{13} y = 0.$$

The boundary conditions at the coast are, using (4.1),

$$h_0 \hat{u}_{13} + \mathbf{M} \cdot \hat{\mathbf{i}} = 0,$$

or

$$\hat{p}_{13}(0, y) = \frac{1}{h_0} \int_0^y \left(\frac{1}{2} \tau \times \hat{\mathbf{K}} \cdot \hat{\mathbf{i}} + u_{13} - v_{11} \right) dy = P(y). \quad (5.6)$$

Further $\hat{p}_{13}(\eta, y) \rightarrow 0$, $\eta \rightarrow \infty$, and $\hat{p}_{13}$ is periodic in y . Having determined $P(y)$ from the interior solution, $\hat{p}_{13}$ may be found from the straightforward solution of the above equations, and hence $\hat{v}_{12}$, $\hat{u}_{13}$ from (5.2).

Since the longshore transport is $O(E^{\frac{1}{2}})$, a local mass surplus is carried around the coast to a region of mass deficit. This is simplest in the uniform stress case, where a mass surplus created on the right-hand coast, facing downwind, is carried around the coast to the mass deficit region on the left side of the basin. For a circular parabolic basin with a uniform stress τ_0 along $\theta = \pi/2$, we have

$$P(\theta) = \frac{\tau_0}{h_0} \sin \theta,$$

$$\hat{p}_{13}(\eta, \theta) = \tau_0 \sin \left(\theta - \frac{1}{\sqrt{2}} \eta \right) \exp \left(-\frac{1}{\sqrt{2}} \eta \right).$$

The integrated coastal layer transport longshore is then

$$\mathbf{M} \cdot \hat{\mathbf{j}} = E^{\frac{1}{2}} h_0 \int_0^\infty \hat{v}_{12} d\eta = -\frac{1}{2} E^{\frac{1}{2}} \hat{p}_{13}(0, \theta).$$

The longshore transport is zero on the left and right sides of the basin and maximum at the upwind and downwind sides of the basin, with coastal transport from right to left.

An important feature of this coastal layer is the absence of friction-induced upwelling on the $E^{\frac{1}{2}}$ scale because the boundary layer flow is geostrophic and hence nondivergent to at least $O(E^{\frac{1}{2}})$. The dynamics

required to return the bottom transport and the geostrophic transport $h_0 u_{13}$ to the surface is absent. The necessary upwelling can possibly be produced by the incorporation of lateral turbulent mixing. If the horizontal Ekman number $E_1 = O(E)$, then the largest horizontal scale of upwelling is $E_1^{1/2}$ and $w = O(E_1^{1/2})$. As in the uniform depth basin (Birchfield, 1972), an additional and narrower horizontal scale $O(E_1^{1/2})$ is required with $w = O(1)$.

Janowitz (1970) has shown, for a constant depth basin, that if $E_1 < E^{1/2} \ll 1$ the two horizontal scales are $E_1^{1/2}/E^{1/2}$ and $E_1^{1/2}$, respectively. This analysis probably is unrealistic for a variable depth basin, since for horizontal scales $< E^{1/2}$, the depths are such that vertical friction is important throughout the water column. In fact, if the coastal depth $h_0 = O(E^{1/2})$, the vertical-friction-produced horizontal divergence is sufficiently large to provide the necessary upwelling. A third mechanism for upwelling could come from the convective terms in the equations of motion. This requires the Rossby number to be $O(1)$, again inside the coastal region where vertical friction dominates in the entire water column.

The dynamics in our model can profitably be compared to that of Welander (1968) for a homogeneous ocean of variable depth. In his model the geostrophic contours or lines of constant potential vorticity, f^*/h^* , include f^* variation with latitude; in the oceanic model the contours are not closed, whereas in our model only closed contours are assumed present. In the sea model, geostrophic cross-contour flow u_{13} is analogous to the Sverdrup transport in the ocean model, in both cases driven by the wind stress curl. Although the coastal boundary layer in both models invokes a large bottom stress curl to balance vorticity generation by columnar stretching, there are significant differences. In the ocean model, the coastal jet has a large component along the gradient of potential vorticity; thus, the large vorticity changes require a relatively large bottom stress curl, which in turn requires a narrow boundary layer, of thickness $O(E^{1/2})$, as can be deduced by balancing the first and third terms in (4.6) if h is replaced by f/h . In the sea or lake model the coastal jet is *exactly* normal to the gradient of potential vorticity; the relatively small flow across contours requires smaller bottom stress curl and hence the $O(E^{1/2})$ boundary layer thickness. It is clear that a relatively small deviation of the jet away from the normal could produce significant vorticity changes; hence, the neglect of the variation of f with latitude becomes a more delicate matter than appears in the uniform depth basin, particularly for larger lakes or seas.

6. Summary

The above discussion presents an asymptotic solution for steady wind-driven currents in a variable topography basin, provided the Ekman number is sufficiently

small to insure that $E^{1/2}$ is small. The coastal depth is small (but larger than $E^{1/2} h_M^*$) and constant.

A relatively simple circulation is generated by a uniform wind stress. Away from the coast the free surface remains flat and there is thus no significant geostrophic flow or bottom stress. The only mass flux is in the surface Ekman layer, to the right of the wind stress. A coastal boundary layer of thickness $E^{1/2} L^*$ is established to bring the transport normal to the coast to zero and to transport mass around the coast from surplus regions to deficit regions. To accomplish these things a geostrophic flow is established with an $O(E^{1/2})$ velocity along the coast and $O(E^{1/2})$ component normal to the coast. The vorticity change of a fluid column moving across depth contours is now balanced by changes due to bottom stress. The sum of the geostrophic transport and the bottom Ekman transport normal to the coast then cancels the surface Ekman normal transport. The convergence of mass moving normal to the coast is balanced by a divergence in the tangential direction, generating the longshore current. An explicit mechanism for upwelling is absent on the $E^{1/2}$ scale and hence must occur on a smaller horizontal scale. The overall effect of topography is, through the Taylor-Proudman constraint, to force the geostrophic return flow (which, in the uniform depth basin, moves directly across the basin) to be returned in the relatively broad boundary layer around the coast.

For a uniform wind stress curl, taken negative say, in addition to the rightward surface Ekman transport, the downward $O(E^{1/2})$ flux of mass out of the surface Ekman layer forces a large $O(1)$ geostrophic flow directed along contours such that deep water is always on the right, facing downstream. The geostrophic flow forces a bottom boundary layer which accepts the downward mass flux from the surface layer. The $O(E^{1/2})$ return flow in the bottom layer moves outward to the coast. Depending on the local topography, in the deep water, there may be an $O(E^{1/2})$ geostrophic cross-contour flow required for local vorticity balance. Hence, as the coast is approached, the normal components of the surface transport, the bottom transport, and the geostrophic $O(E^{1/2})$ transport force a geostrophic boundary layer normal flow to reduce the total normal transport to zero, and to establish an $O(E^{1/2})$ longshore mass flux to bring about horizontal mass balance.

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