Balance Equations Based on Momentum Equations with Global Invariants of Potential Enstrophy and Energy

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ABSTRACT

An approximate model for small Rossby number ϵ that is close to the balance equations (BE) but that is based on approximate momentum equations is formulated for a rotating, continuously stratified fluid governed by the hydrostatic, Boussinesq, inviscid, adiabatic primitive equations with spatially variable Coriolis parameter. This model, referred to as BEM (balance equations based on momentum equations), conserves volume integrals of an appropriate energy density and also conserves potential vorticity on fluid particles and thus volume integrals of potential enstrophy density. The fact that, unlike the BE model which is derived from equations for the vertical component of vorticity and for the horizontal divergence, BEM is based on approximate momentum equations is important for two reasons. It allows the derivation of equations for the horizontal components of vorticity that are needed in the subsequent derivation of an equation for the potential vorticity and it allows the consistent formulation of boundary conditions at rigid surfaces. As is the case for BE, the BEM equations filter out high-frequency internal-gravity waves and remain valid for motion over O(1) variations in bottom topography and for flows with O(1) variations in the height of density surfaces. The governing equations for BEM may be conveniently expressed in a form similar to BE involving a vorticity and a divergence (balance) equation. In this formulation, the BE and BEM models involve identical equations for continuity, vorticity, and heat with differences represented only by the presence of additonal higher order terms in the balance equation for BEM. Methods for the numerical solution of BEM and for the application of boundary conditions are presented.

1. Introduction

Intermediate models (McWilliams and Gent 1980) contain physics between that in the primitive equations and that in the quasi-geostrophic approximation. They are derived under the assumption that the Rossby number ϵ is small and they filter out high-frequency gravity–inertial waves. They are capable of representing subinertial frequency flows over O(1) variations in bottom topography for which the quasi-geostrophic approximation is not formally valid. We are interested in intermediate models for their possible application to the study of mesoscale physical oceanographic flow fields over the continental shelf and slope. The balance equations (Charney 1955, 1962; Bolin 1955, 1956; Lorenz 1960) are one intermediate model that has received a substantial amount of study. Recent investigations have concerned the formulation and properties of various balance equation models for continuously stratified oceanic and atmospheric motions (Gent and McWilliams 1983a), the development of methods for their numerical solution (Norton et al. 1986), and applications to the study of vortex evolution on a β-plane (McWilliams et al. 1986; McWilliams and Gent 1986) and to wind-driven ocean circulation (McWilliams et al. 1990).

Motivated by the possible application of intermediate models to coastal flow fields, we have recently completed a set of comparative studies of a large number of different intermediate models applied to the *f*-plane shallow water equations (SWE) for flows over bottom topography with O(1) variations (Allen et al. 1990a,b; Barth et al. 1990). These studies involve comparisons of intermediate model solutions, along with those from a quasi-geostrophic model (QG), to solutions from the primitive SWE. The solutions are obtained both from exact analytical analyses for special cases and from a set of numerical experiments for initial-value problems. The results of the numerical experiments are quite conclusive regarding the relative accuracy of different intermediate models and are consistent with the less general analytical results. The balance equations (BE) and the closely related BEM model (balance equations based on momentum equations) give consistently accurate approximate solutions to the shallow-water equations over a wide range of parameter values. A slightly simpler LQBE model (potential vorticity conserving linear balance equations) also produces good results, with solution errors usually only slightly larger than BE and BEM. Solutions from

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the other intermediate models generally give substantially larger errors while the solutions from QG are the most inaccurate. The BEM and LQBE models were formulated and introduced in Allen et al. (1990a). These results are consistent with the high accuracy found in the studies of the balance equations applied to continuously stratified fluids by McWilliams et al. (1986).

One central issue regarding the properties of different intermediate models (McWilliams and Gent 1980) concerns the possession, as is the case for the primitive equations, of corresponding conservation equations for potential vorticity on fluid particles and for energy. Although it has been argued that retention of analogues of both of these conservation laws is likely to be important (Salmon 1983) or advocated that possession of at least one of these conservation relations is desirable (Gent and McWilliams 1984), a conclusive demonstration of the validity of those statements has not been given, in general, and that issue is not settled. It is clear, however, from the intermediate-model numerical solution comparisons for the shallow-water equations (Barth et al. 1990; Allen et al. 1990b), that intermediate models that possess analogues of both forms of these conservation laws can give inaccurate results, while others that possess only one can produce quite accurate solutions. For example, the most accurate models in these experiments, BE, BEM and LQBE, all imply conservation of some form of potential vorticity following fluid particles, but none of those have identified conservation equations for energy. On the other hand, the geostrophic momentum (GM) approximation (Hoskins 1975) and Salmon's (1983) model, both of which have conservation statements for different analogues of potential vorticity and of energy, generally give considerably less accurate solutions than BE, BEM, and LQBE. These results indicate that the possession of conservation laws for both energy and potential vorticity by itself does not imply that an approximate model will give accurate solutions.

The results obtained from the shallow-water equation studies regarding the high accuracy of balance-equation–type models are so conclusive that, taken in conjunction with the excellent results found for BE in the set of studies by McWilliams, Gent and Norton (e.g., McWilliams et al. 1986), we are encouraged to conclude that balance-equation–type models are clearly the prime candidates among intermediate models to be developed further for application to continental shelf and slope flow fields. One nagging point remains to be clarified, however, and that is related to the fact that for the shallow-water equations (SWE) the BE conserve potential vorticity on fluid particles, but do not have a conservation law for energy, whereas in the continuously stratified case in physical coordinates, the BE conserve energy (Lorenz 1960), but not potential vorticity. The question naturally arises as to what extent the accuracy of BE, BEM, and LQBE for the SWE depends on the fact that these models conserve analogues of potential vorticity Q on fluid particles. We recall that this property implies conservation of volume integrals of not only potential enstrophy density Q^2, but also of arbitrary functions of Q and thus results in an infinite number of conserved quantities. In that sense, the property of potential vorticity conservation on fluid particles may be more fundamental to model performance than that of conservation of volume integrals of energy and could possibly be a basic reason why the balance-equation-type models do well for the SWE. Thus, it would be useful in the continuously stratified case to assess the importance of the lack of potential vorticity conservation for BE. Further, it would be desirable to develop an extension of BE that possesses an analogue of potential vorticity conservation on fluid particles.

In this paper, we show that for a rotating, continuously stratified fluid governed by the hydrostatic, Boussinesq, inviscid, adiabatic primitive equations (PE) with spatially variable Coriolis parameter, a model very close to the balance equations can be formulated based on approximate momentum equations such that an analogue of potential vorticity is conserved on fluid particles and volume integrals of an appropriate energy density are conserved. This formulation is somewhat similar to that used in Allen et al. (1990a) to derive the BEM model for the SWE [see also Allen et al. (1990b)] and we refer to the present set of equations for the continuously stratified case also as a BEM model (balance equations based on momentum equations). The governing equations for BEM may be conveniently expressed in a form similar to that for BE involving a vorticity and a divergence (balance) equation. In this formulation, the BE and BEM models involve identical equations for continuity, vorticity, and heat with differences represented only by the presence of additional higher order terms in the balance equations for BEM. General procedures for the numerical solution of BE and BEM can be formulated in a common manner and the computational effort involved in solving both is similar. As a result, future comparisons of BE and BEM model numerical solutions with each other and with PE solutions should be able to provide detailed information on the importance of balance-equation–type models possessing an analogue of potential vorticity conservation.

We note that the balance equations are derived (Gent and McWilliams 1983a) by a systematic truncation of the equations for the vertical component of vorticity and for the divergence such that $O(1)$ and $O(\epsilon)$ terms are retained. The approximate momentum equations that correspond to the truncated vorticity and divergence equations do not follow directly from approximations to the PE momentum equations, but involve implicitly defined force potential correction terms and are referred to as the equivalent momentum equations (Gent and McWilliams 1983a). The for-

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mulation of BEM for the SWE in Allen et al. (1990a) was motivated by a desire to retain explicit momentum equations for use in the consistent specification of vertical-wall boundary conditions so that ageostrophic, coastally trapped waves would be properly represented. The same motivation applies here also, of course, but in the continuously stratified case there is an additional important reason to base the model on approximate momentum equations. That reason follows from the fact that the potential vorticity Q involves the horizontal components of vorticity as well as the vertical component. To obtain an equation for an analogue to Q, approximate momentum equations are necessary for use in the derivation of relevant equations for the horizontal components of vorticity.

The outline of this paper is as follows. In section 2, we record the primitive equations and the resulting relations for potential vorticity and energy conservation. In section 3, the BEM model is formulated. Solution procedures and methods of boundary condition application for BEM are discussed in section 4. The behavior of the BEM equations in limiting cases is examined in section 5 and a summary of the results in given in section 6.

2. Primitive equations–PE

We consider the motion of rotating, continuously stratified fluid governed by the hydrostatic, Boussinesq, adiabatic, inviscid, primitive equations (PE) with a spatially variable Coriolis parameter. Dimensionless variables are utilized so that in Cartesian coordinates (x, y, z) the PE are

$$\nabla_3 \cdot \mathbf{u}_3 = 0, \tag{2.1a}$$

$$\epsilon \frac{D\mathbf{u}}{Dt} + f\mathbf{k} \times \mathbf{u} = -\nabla\phi, \tag{2.1b}$$

$$0 = -\phi_z + \theta, \tag{2.1c}$$

$$\frac{D\theta}{Dt} + Sw = 0, \tag{2.1d}$$

where, for some applications, it is convenient to utilize, in place of (2.1c,d), the equivalent equations,

$$0 = -p_z - \rho, \tag{2.1c'}$$

$$\frac{D\rho}{Dt} = 0, \tag{2.1d'}$$

and where

$$\mathbf{x} = (x, y, z), \tag{2.2a}$$

$$\mathbf{u}_3 = (u, v, \epsilon w), \quad \mathbf{u} = (u, v, 0), \tag{2.2b,c}$$

$$\nabla_3 = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right), \tag{2.2d,e}$$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{u}_3 \cdot \nabla_3\right), \tag{2.2f}$$

and $\mathbf{k}$ is the unit vector in the vertical z direction. Note that the three-dimensional velocity vector $\mathbf{u}_3$ and gradient operator ∇_3 are labeled with a subscript 3, whereas the horizontal velocity vector $\mathbf{u}$ and gradient operator ∇ are not subscripted.

Dimensionless variables are formed using the characteristic values (L, H, U_0, f_0) for, respectively, a horizontal length scale, vertical depth scale, horizontal velocity, and Coriolis parameter. The Rossby number

$$\epsilon = U_0/f_0 L. \tag{2.3}$$

With dimensional variables denoted by primes, we have

$$(x, y) = (x', y')/L, \quad z = z'/H, \tag{2.4a,b}$$

$$(u, v) = (u', v')/U_0, \quad \epsilon w = w'L/(U_0 H), \tag{2.4c,d}$$

$$t = t'U_0/L, \quad f(x, y) = f'(x', y')/f_0, \tag{2.4e,f}$$

so that $(u, v, \epsilon w)$ are the dimensionless velocity components in the (x, y, z) directions, t is time, and f is the dimensionless Coriolis parameter. The total dimensional density is given by

$$\rho_T' = \rho_0 + \bar{\rho}'(z') - \theta'(\mathbf{x}', t'), \tag{2.5}$$

where ρ_0 is a constant reference density, $\bar{\rho}'(z')$ the basic undisturbed z'-dependent field, and θ' the negative of the density fluctuation. We define

$$\bar{\rho}(z) = \bar{\rho}'(z')/\rho_C, \quad \theta = \theta'/\rho_C, \tag{2.6a,b}$$

and

$$\rho = \bar{\rho}(z) - \theta, \tag{2.7}$$

where $\rho_C = p_C/(Hg)$, $p_C = \rho_0 U_0 f_0 L$, and g is the acceleration of gravity. In addition, we write

$$\bar{\rho}_z = -S(z)/\epsilon, \quad S(z) = N^2(z)H^2/(f_0^2 L^2), \tag{2.8a,b}$$

where

$$N^2(z) = -g\bar{\rho}_{z'}'/\rho_0, \tag{2.8c}$$

is the square of the basic Brunt–Väisälä frequency. Subscripts (x, y, z, t) denote partial differentiation. Pressure variables p, $\bar{p}$ and ϕ are defined by nondimensionalizing with p_C such that

$$p = \bar{p}(z) + \phi, \tag{2.9}$$

where

$$\bar{p}_z = -\bar{\rho}, \quad \nabla p = \nabla\phi. \tag{2.10a,b}$$

We are primarily interested in the limit of small Rossby number,

$$\epsilon \ll 1, \tag{2.11}$$

with, in general, $S = O(1)$. It will also be assumed that the variations in f satisfy

$$|\nabla f| = O(\epsilon). \tag{2.12}$$

It is useful to express (2.1b) in the equivalent form,

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$$\epsilon \mathbf{u}_t + \epsilon^2 w \mathbf{u}_z + (\epsilon\zeta + f)\mathbf{k} \times \mathbf{u} = -\nabla \mathcal{B}, \quad (2.13)$$

where the vertical component of vorticity ζ, the Bernoulli function $\mathcal{B}$, and the horizontal kinetic energy density ϵK are given by

$$\zeta = \zeta^{(z)} = v_x - u_y, \quad (2.14a)$$

$$\mathcal{B} = \phi + \epsilon K, \quad (2.14b)$$

$$K = \frac{1}{2}(u^2 + v^2) = \frac{1}{2}\mathbf{u}\cdot\mathbf{u} = \frac{1}{2}\mathbf{u}^2. \quad (2.14c)$$

The horizontal velocity vector may be represented as the sum of rotational and divergent components, such that

$$\mathbf{u} = \mathbf{k} \times \nabla\psi + \epsilon\nabla\chi, \quad (2.15a)$$

$$= \mathbf{u}_R + \epsilon \mathbf{u}_D, \quad (2.15b)$$

where

$$\mathbf{u}_R = (u_R, v_R, 0) = (-\psi_y, \psi_x, 0), \quad (2.15c)$$

$$\mathbf{u}_D = (u_D, v_D, 0) = (\chi_x, \chi_y, 0), \quad (2.15d)$$

and

$$\zeta = \zeta^{(z)} = v_x - u_y = v_{Rx} - u_{Ry}$$

$$= \nabla^2\psi = \psi_{xx} + \psi_{yy}, \quad (2.16)$$

$$D = u_x + v_y = \epsilon(u_{Dx} + v_{Dy}) = \epsilon\nabla^2\chi, \quad (2.17)$$

where D is the horizontal divergence. The continuity equation (2.1a) then gives

$$w_z = -\nabla^2\chi. \quad (2.18)$$

Boundary conditions for (2.1) at rigid boundaries, such as at the surface, bottom, or sidewalls are

$$\mathbf{u}_3 \cdot \mathbf{n}_3 = 0 \quad \text{at} \quad \mathbf{x} = \mathbf{x}_S, \quad (2.19)$$

where $\mathbf{n}_3 = \mathbf{n}_3(\mathbf{x}_S)$ is the outward-pointing unit vector normal to the boundary surface at $\mathbf{x} = \mathbf{x}_S$.

The equation for the vertical component of vorticity (2.16) derived from (2.13) is

$$\frac{D}{Dt}(\epsilon\zeta + f) + \epsilon(\epsilon\zeta + f)\nabla^2\chi + \epsilon^2(w_x v_z - w_y u_z) = 0, \quad (2.20a)$$

or

$$\zeta_t + J(\psi, \zeta) + f\nabla^2\chi + \epsilon^{-1}J(\psi, f) + \epsilon\nabla\cdot[w\nabla\psi_z + \zeta\nabla\chi] + \nabla\chi\cdot\nabla f + \epsilon^2 J(w, \chi_z) = 0, \quad (2.20b)$$

where the operator $J(a, b) = a_x b_y - a_y b_x$ is the Jacobian. The equation for the horizontal divergence derived form (2.13) is

$$\nabla^2\phi - \nabla\cdot(f\nabla\psi) - \epsilon 2J(\psi_x, \psi_y) + \epsilon J(\chi, f) + \epsilon^2[\nabla^2\chi_t + \nabla^2 J(\psi, \chi) - J(\zeta, \chi) - J(w, \psi_z)] + \epsilon^3\nabla\cdot[w\nabla\chi_z + \frac{1}{2}\nabla(|\nabla\chi|^2)] = 0. \quad (2.21)$$

The primitive equations (2.1a, b, c′, d′) imply the conservation of potential vorticity Q on fluid particles,

$$\frac{D}{Dt}Q = 0, \quad (2.22)$$

where

$$Q = (\epsilon\boldsymbol{\zeta}_3 + f\mathbf{k})\cdot\nabla_3\rho, \quad (2.23)$$

$$\boldsymbol{\zeta}_3 = \nabla_3 \times \mathbf{u} = (\zeta^{(x)}, \zeta^{(y)}, \zeta^{(z)}) = (-v_z, u_z, v_x - u_y), \quad (2.24)$$

and where ρ is defined in (2.7). The relation (2.22) may be combined with (2.1a) to give the following conservation equation,

$$Q_t^2 + \nabla_3\cdot(\mathbf{u}_3 Q^2) = 0, \quad (2.25)$$

for potential enstrophy density Q^2. A similar equation may be derived for an arbitrary function $Fn(Q)$ which replaces Q^2 in (2.25). In bounded regions with boundary condition (2.19), (2.25) implies conservation of volume integrals of Q^2.

The PE (2.1a, b, c′, d′) also imply the energy conservation equation,

$$E_t + \nabla_3\cdot[\mathbf{u}_3(p + E)] = 0, \quad (2.26)$$

$$E = \frac{\epsilon}{2}\mathbf{u}^2 + (z - z_0)\rho, \quad (2.27)$$

where E is the energy density, z_0 is a constant reference level, ρ is defined in (2.7), and p is defined in (2.9). In bounded regions with boundary condition (2.19), (2.26) implies the conservation of volume integrals of E.

3. Balance equations (based on momentum equations)—BEM

We formulate here a set of approximate balance-type equations BEM to (2.1) for small Rossby number (2.11). These are very close to the balance equations BE (Gent and McWilliams 1983a) with the exception that the BEM are derived from approximate momentum equations in which, in addition to O(1) and O(ϵ) terms, some O(ϵ^2) terms are retained. This is to ensure that, as in BE, all O(1) and O(ϵ) terms are kept in the vorticity equation (2.20b) and that, further, conservation of an analogue of potential vorticity on fluid particles and conservation of volume integrals of energy are implied.

For BEM, the horizontal velocity vector is represented in terms of ψ and χ (2.15). The full continuity (2.1a), hydrostatic (2.1c), and heat equations (2.1d)

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are utilized the same as for BE. The horizontal momentum equations (2.13) are approximated as given below in (3.1b). The resulting equation set is

$$\nabla^2\chi + w_z = 0, \tag{3.1a}$$

$$\epsilon\mathbf{u}_{Rt} + \epsilon^2 w\mathbf{u}_{Rz} + (\epsilon\zeta + f)\mathbf{k}\times\mathbf{u} = -\nabla\mathcal{B}_{\mathrm{BE}}, \tag{3.1b}$$

$$0 = -\phi_z + \theta, \tag{3.1c}$$

$$\theta_t + J(\psi, \theta) + Sw + \epsilon[\nabla\cdot(\theta\nabla\chi) + (w\theta)_z] = 0, \tag{3.1d}$$

where, as with PE, (3.1c, d) may be replaced by the equivalent equations

$$0 = -p_z - \rho, \tag{3.1c'}$$

$$\frac{D\rho}{Dt} = 0, \tag{3.1d'}$$

and where

$$\mathcal{B}_{\mathrm{BE}} = \phi + \epsilon K_{\mathrm{BE}}, \tag{3.2a}$$

$$K_{\mathrm{BE}} = K_R + \epsilon K_{1B}, \tag{3.2b}$$

$$K_R = \frac{1}{2}(u_R{}^2 + v_R{}^2) = \frac{1}{2}\mathbf{u}_R{}^2, \tag{3.2c}$$

$$K_{1Bz} = u_D u_{Rz} + v_D v_{Rz} = J(\psi_z, \chi), \tag{3.2d}$$

$$K_{1B} = \int_{z_1}^{z} J(\psi_{\hat{z}}, \chi)d\hat{z} + g(x, y, t). \tag{3.2e}$$

Note that while K_{1Bz} is uniquely defined, K_{1B} contains an arbitrary function $g(x, y, t)$ and an arbitrary choice of lower limit z_1 for the integral in (3.2e). As will be shown in the discussion of the solution procedure in section 4, the total Bernoulli function $\mathcal{B}_{\mathrm{BE}} = \phi + \epsilon K_R + \epsilon^2 K_{1B}$ is uniquely defined and the explicit calculation of K_{1B} is not necessary. Thus, the arbitrariness in the definition of K_{1B} in (3.2e) is of no consequence. Comparing the z derivative of K (2.14c) from PE,

$$K_z = K_{Rz} + \epsilon K_{1z} + \epsilon^2 K_{Dz}, \tag{3.3a}$$

where

$$K_1 = J(\psi, \chi), \quad K_{1z} = J(\psi_z, \chi) + J(\psi, \chi_z), \tag{3.3b,c}$$

and

$$K_D = \frac{1}{2}(u_D{}^2 + v_D{}^2), \tag{3.3d}$$

$$K_{\mathrm{BE}z} = K_{Rz} + \epsilon K_{1Bz}, \tag{3.4}$$

we see that in $\epsilon K_{\mathrm{BE}z}$ the $O(\epsilon^3)$ term $\epsilon^3 K_{Dz}$ and part $[\epsilon^2 J(\psi, \chi_z)]$ of the $O(\epsilon^2)$ term are neglected, while $\epsilon^2 K_{1Bz} = \epsilon^2 J(\psi_z, \chi)$ is retained. This choice is motivated by the fact that, given the other retained terms in (3.1b), it is precisely the retention of K_{1B} in $\mathcal{B}_{\mathrm{BE}}$ that allows appropriate conservation statements for both potential vorticity and energy to be derived from (3.1).

The approximation (3.1b) to (2.13) consequently involves a truncation wherein, after the substitution of $\mathbf{u}$ as represented by (2.15), all $O(1)$ and $O(\epsilon)$ terms are retained and all $O(\epsilon^3)$ terms are neglected. The $O(\epsilon^2)$ terms $\epsilon^2\mathbf{u}_{Dt}$ and $\epsilon^2\nabla(K_1 - K_{1B})$ are neglected, while the $O(\epsilon^2)$ terms $\epsilon^2 w\mathbf{u}_{Rz}$, $\epsilon^2\zeta\mathbf{k}\times\mathbf{u}_D$ and $\epsilon^2\nabla K_{1B}$ are retained. The neglect of $\epsilon^2\mathbf{u}_{Dt}$ filters out high-frequency gravity–inertial waves. The retention of $\epsilon^2 w\mathbf{u}_{Rz}$ and $\epsilon^2\zeta\mathbf{k}\times\mathbf{u}_D$ is necessary to ensure that, as in the balance equations, all $O(\epsilon)$, as well as $O(1)$, terms are retained in the equation for the vertical component of vorticity ζ [(3.5) below]. The retention of $\epsilon^2\nabla K_{1B}$ and the negelct of $\epsilon^2\nabla(K_1 - K_{1B})$ ensures that conservation equations for potential vorticity and for energy, analogous to (2.22) and (2.26) for PE, follow from the equation set (3.1). This nonuniform method of treating the $O(\epsilon^2)$ terms is not very satisfying, but appears to be necessary to achieve the stated objectives. On the other hand, R. Salmon (personal communication) has pointed out that approximations made in different coordinate systems can result in equations that differ by small terms and that the approximations employed here may be more consistently applied in other variables.

Boundary conditions for (3.1) at rigid boundaries are given by (2.19) the same as for PE.

The equation for the vertical component of vorticity derived from (3.1b), to be compared with (2.20) for PE, is

$$\frac{D}{Dt}(\epsilon\zeta + f) + \epsilon(\epsilon\zeta + f)\nabla^2\chi + \epsilon^2\nabla w\cdot\nabla\psi_z = 0, \tag{3.5a}$$

or

$$\zeta_t + J(\psi, \zeta) + f\nabla^2\chi + \epsilon^{-1}J(\psi, f) + \epsilon\nabla\cdot[w\nabla\psi_z + \zeta\nabla\chi] + \nabla\chi\cdot\nabla f = 0. \tag{3.5b}$$

Note that with $|\nabla f| = O(\epsilon)$ from (2.12), $\epsilon^{-1}J(\psi, f) = O(1)$ and $\nabla\chi\cdot\nabla f = O(\epsilon)$. Thus, equation (3.5) includes all $O(1)$ and $O(\epsilon)$ terms in (2.20) and is identical to the vorticity equation of the balance equations (BE) (Gent and McWilliams 1983a).

The divergence equation derived from (3.1b), to be compared with (2.21) for PE, is

$$\nabla^2\phi - \nabla\cdot(f\nabla\psi) - \epsilon 2J(\psi_x, \psi_y) + \epsilon J(\chi, f) + \epsilon^2[\nabla^2 K_{1B} - J(\zeta, \chi) - J(w, \psi_z)] = 0. \tag{3.6}$$

No time derivative terms appear in (3.6) and it is referred to as the equation of balance. In BE (Gent and McWilliams, 1983a), the divergence equation is truncated to retain $O(1)$ and $O(\epsilon)$ terms and the equation of balance is given by the first three terms in (3.6),

$$\nabla^2\phi - \nabla\cdot(f\nabla\psi) - \epsilon 2J(\psi_x, \psi_y) = 0. \tag{3.7}$$

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Both BE and BEM consist of (3.1a, c, d) and (3.5a); BEM includes the equation of balance (3.6), whereas BE utilizes (3.7). The differences in the BEM and BE equation sets are thus completely represented by the $O(\epsilon^2)$ terms retained in (3.6) for BEM compared with (3.7) for BE.

The BEM equations (3.1a, b, c′, d′) imply the conservation of balance equation potential vorticity Q_{BE} on fluid particles moving with velocity $\mathbf{u}$ (2.15),

$$\frac{D}{Dt} Q_{BE} = 0, \tag{3.8}$$

where

$$Q_{BE} = (\epsilon \boldsymbol{\zeta}_{3R} + f\mathbf{k}) \cdot \nabla_3 \rho, \tag{3.9}$$

$$\boldsymbol{\zeta}_{3R} = (\zeta_R^{(x)}, \zeta_R^{(y)}, \zeta) = (-v_{Rz}, u_{Rz}, v_x - u_y), \tag{3.10}$$

and where ρ is defined in (2.7). Equation (3.8) may be combined with (3.1a) to give a conservation equation,

$$Q^2_{BEt} + \nabla_3 \cdot (\mathbf{u}_3 Q^2_{BE}) = 0, \tag{3.11}$$

for balance equation potential enstrophy density Q^2_{BE}. As with (2.25), a similar equation holds with an arbitrary function $Fn(Q_{BE})$ replacing Q^2_{BE} in (3.11). If the fluid is contained in a basin such that condition (2.19) $\mathbf{u}_3 \cdot \mathbf{n}_3 = 0$ applies on all boundaries, (3.11) implies the conservation of volume integrals of Q^2_{BE}. Consequently, potential enstrophy is a global invariant (McWilliams and Gent 1980) for BEM. Writing out Q_{BE} (3.9),

$$Q_{BE} = -\epsilon v_{Rz}\theta_x + \epsilon u_{Rz}\theta_y - Sf/\epsilon - S\zeta - \theta_z(\epsilon\zeta + f), \tag{3.12}$$

comparing with Q (2.23) from PE,

$$Q = -\epsilon v_z \theta_x + \epsilon u_z \theta_y - Sf/\epsilon - S\zeta - \theta_z(\epsilon\zeta + f), \tag{3.13}$$

and noting that with (2.12) $D(Sf/\epsilon)/Dt = O(1)$, we can see that Q_{BE} effectively retains all $O(1)$ and $O(\epsilon)$ terms in Q. This seems to be an additional desirable feature of BEM.

An equation for balance equation energy density may be derived from (3.1a, b, c′, d′) by taking the dot product of (3.1b) with $\mathbf{u}$, multiplying (3.1c′) by ϵw, and combining the resulting equations with the product of (3.1d′) and $(z - z_0)$:

$$E_{RT} + \nabla_3 \cdot [\mathbf{u}_3(p + E_T)] + \epsilon^2 \nabla \cdot [\chi \mathbf{u}_{Rt}] = 0, \tag{3.14}$$

where

$$E_R = \epsilon K_R + (z - z_0)\rho, \tag{3.15}$$

$$E_T = \epsilon K_{BE} + (z - z_0)\rho, \tag{3.16}$$

z_0 is a constant reference level and p is defined in (2.9). For a fluid that is contained in a basin such that (2.19) applies on all boundaries, volume integrals of E_R will be conserved if

$$\oint_{C_0(z)} \chi \mathbf{u}_{Rt} \cdot \mathbf{n} ds = 0, \tag{3.17}$$

where $\mathbf{n}$ is a unit vector aligned along the horizontal component of $\mathbf{n}_3$ and (3.17) involves integrals around closed contours $C_0(z)$ on the boundary at all relevant z levels. Procedures for obtaining solutions of BEM and for the application of boundary conditions are discussed in the next section. Boundary conditions will be formulated such that (3.17) is satisfied and volume integrals of balance equation energy density E_R are conserved. As a result, energy is also a global invariant for BEM. Note that (3.14) may be written in a form that is more similar to (2.26) for PE if we regard $\epsilon^2 K_{1B}$ as a pressure correction and define

$$p_T = p + \epsilon^2 K_{1B}, \tag{3.18}$$

so that (3.14) becomes

$$E_{Rt} + \nabla_3 \cdot [\mathbf{u}_3(p_T + E_R)] + \epsilon^2 \nabla \cdot [\chi \mathbf{u}_{Rt}] = 0. \tag{3.19}$$

4. Solution procedure and boundary conditions for BEM

With the vorticity (3.5) and divergence (3.6) equations replacing the horizontal momentum equations (3.1b), the BEM model is governed by (3.1a, c, d), (3.5), and (3.6). We discuss methods for obtaining numerical solutions and for applying boundary conditions in this section. For simplicity, we consider a β-plane-type approximation in which

$$f = 1 + \epsilon f_1(x, y). \tag{4.1}$$

Note this is consistent with, but more restrictive than, condition (2.12).

With (4.1), the vorticity equation (3.5) is

$$\zeta_t + J(\psi, \zeta + f_1) + \nabla^2 \chi + \epsilon \nabla \cdot [w \nabla \psi_z + (\zeta + f_1)\nabla \chi] = 0, \tag{4.2}$$

and the divergence, or balance, equation (3.6) may be written

$$\nabla^2(\mathcal{B}_{BE} - \psi) = \epsilon \nabla \cdot [(\zeta + f_1)\nabla \psi] + \epsilon^2 [J(\zeta + f_1, \chi) + J(w, \psi_z)]. \tag{4.3}$$

In the numerical solution procedure for BE in Norton et al. (1986), an omega equation, formed by substituting (3.1d) and (3.5) in the time derivative (3.7), is utilized in place of (3.1d). Since the time derivative of the balance equation (3.6) in BEM results in terms containing χ_t, while none of the other prognostic equations involve χ_t, that procedure is not directly applicable here. In fact, the presence of χ in (3.6) may be expected to lead to the possibility of spurious high-

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frequency time-dependent solutions similar to the situation for BEM applied to the SWE as discussed in Allen et al. (1990b). This complication may be seen most easily in the limiting case of linear β-plane motion and is discussed further in section 5. In obtaining numerical solutions to the BEM model, the choice of finite-difference time integration scheme and the method of initialization must evidently be utilized to filter out effects of the spurious high-frequency motion. Thus, presentation of a solution procedure is an important part of the model description. The numerical methods formulated and implemented in Allen et al. (1990a,b) and Barth et al. (1990) for BEM applied to the SWE appear to have been effective. A similar method is being applied successfully to the present stratified BEM model and is described here. That method utilizes an equation for the linearized potential vorticity in place of (3.1d) and is similar also to one of the solution procedures discussed for the linear balance equations in McWilliams et al. (1990).

An equation for the linearized potential vorticity is formed by eliminating $\nabla^2\chi$ from (3.1d) and (4.2):

$$[\zeta + (S^{-1}\theta)_z]_t = -J(\psi, \zeta + f_1) - \epsilon\nabla\cdot[w\nabla\psi_z + (\zeta + f_1)\nabla\chi] - \{S^{-1}[J(\psi, \theta) + \epsilon[\nabla\cdot(\theta\nabla\chi) + (w\theta)_z]]\}_z . \quad (4.4)$$

It is useful to define

$$\phi = \psi + \epsilon\phi', \quad (4.5)$$

so that the complete set of governing equations (3.1a, c), (4.2) (4.3) and (4.4) [in place of (3.1d)] may be rewritten in the following form:

$$[\nabla^2\psi + (S^{-1}\psi_z)_z]_t = -J(\psi, \zeta + f_1) - \epsilon\nabla\cdot[w\nabla\psi_z + (\zeta + f_1)\nabla\chi] - \{S^{-1}[\epsilon\phi'_{zt} + J(\psi, \theta) + \epsilon[\nabla\cdot(\theta\nabla\chi) + (w\theta)_z]]\}_z , \quad (4.6a)$$

$$\nabla^2(\phi' + K_{BE}) = \nabla\cdot[(\zeta + f_1)\nabla\psi] + \epsilon[J(\zeta + f_1, \chi) + J(w, \psi_z)], \quad (4.6b)$$

$$\nabla^2\chi = -\nabla^2\psi_t - J(\psi, \zeta + f_1) - \epsilon\nabla\cdot[w\nabla\psi_z + (\zeta + f_1)\nabla\chi], \quad (4.6c)$$

$$\theta = \phi_z = \psi_z + \epsilon\phi'_z, \quad (4.6d)$$

$$w_z = -\nabla^2\chi. \quad (4.6e)$$

We consider motion in a closed basin with a rigid top surface at $z = 1$ and with a variable bottom surface at $z = h_B(x, y)$. The boundary conditions from (2.19) are

$$\epsilon w = 0 \quad \text{at} \quad z = 1, \quad (4.7a)$$

$$\epsilon w = \mathbf{u}\cdot\nabla h_B \quad \text{at} \quad z = h_B(x, y), \quad (4.7b)$$

and

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{at vertical sidewalls.} \quad (4.7c)$$

For use in the application of (4.7), we express $\mathbf{u}$ from (3.1b) as

$$[\epsilon(\zeta + f_1) + 1]\mathbf{u} = \mathbf{k}\times\nabla\mathcal{B}_{BE} - \epsilon^2 w\nabla\psi_z - \epsilon\nabla\psi_t, \quad (4.8a)$$

and w from (3.1d) as

$$w = -S^{-1}\{\theta_t + J(\psi, \theta) + \epsilon[\nabla\cdot(\theta\nabla\chi) + (w\theta)_z]\}. \quad (4.8b)$$

Solutions to (4.6) may be obtained from a finite-difference approximation that utilizes an iteration procedure at each time step in which the terms on the right hand sides of (4.6) are regarded as known and we solve (4.6a) for ψ_t, (4.6b) for $\phi' + K_{BE}$, (4.6c) for χ, (4.6d) for θ, and (4.6e) for w. Boundary conditions for the equation (4.6a) for linearized potential vorticity are given by the substitution of $\mathbf{u}$ and w from (4.8a, b) in (4.7a, b, c). The boundary condition for the vorticity equation (4.6c) is obtained by utilizing (4.7) and the component of the momentum equation (3.1b) tangential to the boundary. For the equation of balance (4.6b), it is necessary, in order to avoid nonuniqueness of the solutions for ψ and χ, to specify a boundary condition in addition to that implied by (3.1) and (4.7) alone (Gent and McWilliams 1983a; Allen et al. 1990b). We accomplish that by choosing a condition such that (3.17) is satisfied and volume integrals of energy are conserved. The resulting boundary condition, which is essentially $0 = \mathbf{n}\cdot\epsilon\mathbf{u}_{Rt}$ minus terms in $\mathbf{n}\cdot\epsilon\mathbf{u}_{Rt}$ that, when multiplied by χ, give zero contribution to the integral in (3.17), is

$$\mathbf{n}\cdot\nabla(\phi' + K_{BE}) = (\zeta + f_1)[\mathbf{n}\cdot\nabla\psi + \epsilon\mathbf{s}\cdot\nabla\chi] + \epsilon w\mathbf{s}\cdot\nabla\psi_z, \quad \text{on the boundary} \quad \mathbf{x} = \mathbf{x}_S, \quad (4.9)$$

where $\mathbf{s}$ is a horizontal unit vector tangent to the boundary, positive in the direction of counterclockwise circuits, and normal to $\mathbf{n}$. Note that in $\mathbf{n}\cdot\epsilon\mathbf{u}_{Rt}$ the two component parts of $\mathbf{n}\cdot\mathbf{k}\times\mathbf{u} = -\mathbf{n}\cdot\nabla\psi - \epsilon\mathbf{s}\cdot\nabla\chi$ are such that the ψ component is absorbed in $\mathbf{n}\cdot\nabla\phi'$ whereas the χ term forms a perfect differential in the integrand of (3.17) and thus drops out. The condition (4.9) is readily applied to (4.6b), since it represents a balance on the boundary of all the terms obtained from a horizontal area integral of (4.6b). We note also that (4.9) reduces in the case of the f-plane SWE to the boundary condition formulated in Allen et al. (1990b) for BEM and successfully implemented there in obtaining numerical solutions in a periodic channel.

To give a specific example illustrating boundary condition application and solution procedure, we consider an idealized geometry. Assume the motion is confined to a rectangular, mid-latitude, flat-bottom, ocean basin,

$$0 \leq x \leq L^{(x)}, \quad 0 \leq y \leq 1, \quad (4.10a,b)$$

$$0 \leq z \leq 1, \quad (4.10c)$$

on a β-plane, i.e., with $f_1 = \beta y$.

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The boundary conditions

$$w = 0 \quad \text{at} \quad z = 0, 1, \qquad (4.11a,b)$$

are straightforwardly applied to (4.6a) by setting $w = 0$ in (4.8b) to obtain,

$$\psi_{zt} = -\epsilon\phi'_{zt} - J(\psi, \theta) - \epsilon[\nabla\cdot(\theta\nabla\chi) + (w\theta)_z] \quad \text{at} \quad z = 0, 1. \qquad (4.12a,b)$$

The condition $w(z = 0) = 0$ is also used in finding w from (4.6e):

$$w = -\int_0^z \nabla^2\chi dz'. \qquad (4.13)$$

The correct application of (4.12a, b) to (4.6a) ensures that (4.13) implies $w(z = 1) = 0$.

The horizontal boundary conditions

$$u = -\psi_y + \epsilon\chi_x = 0 \quad \text{at} \quad x = 0, L^{(x)}, \qquad (4.14a,b)$$

$$v = \psi_x + \epsilon\chi_y = 0 \quad \text{at} \quad y = 0, 1, \qquad (4.14c,d)$$

are applied to (4.6a, b, c) in the manner described above. The resulting conditions for (4.6a, b, c) are the following. For (4.6a), the boundary conditions are

$$\epsilon\psi_{xt} = -\epsilon^2 w\psi_{xz} - \mathcal{B}_{BEy} \quad \text{at} \quad x = 0, L^{(x)}, \qquad (4.15a,b)$$

$$-\epsilon\psi_{yt} = \epsilon^2 w\psi_{yz} - \mathcal{B}_{BEx} \quad \text{at} \quad y = 0, 1, \qquad (4.15c,d)$$

where (4.14) is used also to evaluate boundary values of terms on the right-hand-side of (4.6a). For (4.6b), the boundary conditions are

$$(\phi' + K_{BE})_x = (\zeta + f_1)(\psi_x + \epsilon\chi_y) + \epsilon w\psi_{yz} \quad \text{at} \quad x = 0, L^{(x)}, \qquad (4.16a,b)$$

$$(\phi' + K_{BE})_y = (\zeta + f_1)(\psi_y - \epsilon\chi_x) - \epsilon w\psi_{xz} \quad \text{at} \quad y = 0, 1. \qquad (4.16c,d)$$

For (4.6c), the boundary conditions are

$$\epsilon\chi_x = \psi_y \quad \text{at} \quad x = 0, L^{(x)}, \qquad (4.17a,b)$$

$$\epsilon\chi_y = -\psi_x \quad \text{at} \quad y = 0, 1, \qquad (4.17c,d)$$

where, in addition, the boundary values of the terms on the right-hand-side of (4.6c) are found from (4.14) and (4.15). To ensure unique solutions for (4.6a, b, c) we also require

$$\left\langle \int_0^1 \psi_t dz \right\rangle = 0, \qquad (4.18a)$$

$$\langle \phi'_T \rangle = 0, \quad \langle \chi \rangle = 0, \qquad (4.18b,c)$$

where

$$\phi'_T = \phi' + K_{BE}, \qquad (4.19)$$

and the angle brackets denote a horizontal area average, i.e.,

$$\langle \chi \rangle = \langle \chi \rangle(z, t) = \frac{1}{L^{(x)}} \int_0^{L^{(x)}} \int_0^1 \chi \, dxdy. \qquad (4.20)$$

Note that the relation,

$$\langle \theta \rangle_t + \epsilon \langle w\theta \rangle_z = 0, \qquad (4.21)$$

found from the area average of (3.1d) with boundary conditions (4.14), is maintained by the solution of (4.6a) with boundary conditions (4.11) and (4.14).

The solution to (4.6b) gives ϕ'_T from which we find

$$\phi'_z = \phi'_{Tz} - K_{BEz}, \qquad (4.22)$$

and then θ from (4.6d). Also, for use in the boundary conditions (4.15), we obtain $\mathcal{B}_{BE}$ directly from

$$\mathcal{B}_{BE} = \phi + \epsilon K_{BE} = \psi + \epsilon\phi'_T. \qquad (4.23)$$

A method to obtain numerical finite-difference solutions to (4.6) for the above defined problem is outlined here. The method is similar to one introduced in Allen et al. (1990a) for the BEM model applied to the f-plane SWE. It utilizes an iteration procedure at each time step that is based on a scheme developed by Norton et al. (1986) for application to BE in a formulation involving an omega equation. The method presented here has been successfully implemented to obtain numerical finite-difference solutions to (4.6) in a flat-bottom, doubly periodic domain with constant f. Studies involving solution intercomparisons for BEM, BE, QG, and PE applied to initial-value problems are in progress and will be reported on separately.

We primarily discuss the time-difference scheme, since that is the part of the numerical solution procedure that requires special treatment. Relevant spatial difference approximations are described in Norton et al. (1986) and in McWilliams et al. (1990). We mention, however, that in solving (4.6a) for ψ_t it is convenient, after applying (4.12) to (4.6a) in difference form, to utilize an expansion for ψ_t in terms of the vertical linear normal modes, i.e.,

$$\psi_t = \sum_j g_j(z)\psi_{jt}(x, y, t) \qquad (4.24)$$

where

$$(S^{-1}g_{jz})_z + \lambda_j{}^2 g_j = 0, \qquad (4.25a)$$

and

$$g_{jz} = 0 \quad \text{at} \quad z = 0, 1. \qquad (4.25b,c)$$

The time variable is discretized in the usual manner $t = n\Delta t$, where n is an integer and Δt is a constant time step. The notation $\phi^n = \phi(n\Delta t)$ is used to denote evaluation of ϕ at time level n. We assume that the variables $\psi^n, \phi_z^m, \phi_T^m, \chi^n, w^n, \psi_t{}^n, \phi_{zt}^m$ are known at time $t = n\Delta t$ and at previous time levels. An iterative solution of (4.6) is used to provide new values for these variables at the next time level $t = (n + 1)\Delta t$. First, estimate $\psi^{n+1} = \psi^{n-1} + 2\Delta t\psi_t{}^n$, $\phi_z^{m+1} = \phi_z^{m-1} + 2\Delta t\phi_{zt}^m$, $\phi_T^{m+1} = 2\phi_T^m - \phi_T^{m-1}$, $\chi^{n+1} = 2\chi^n - \chi^{n-1}$, $w^{n+1} = 2w^n - w^{n-1}$, $\phi_{zt}^{m+1} = 2\phi_{zt}^m - \phi_{zt}^{m-1}$. Find θ^{n+1} from (4.6d). Use these estimates in the right-hand-side (rhs) of

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(4.6a) and solve (4.6a) for ψ_t^{n+1}. Use this value of ψ_t^{n+1} along with the other estimates in the rhs of (4.6c). Calculate χ^{n+1} from (4.6c) and then w^{n+1} from (4.13). Correct $\psi^{n+1} = \psi^n + \Delta t(\psi_t^{n+1} + \psi_t^n)/2$. Use the new values of ψ^{n+1}, χ^{n+1}, w^{n+1} in the rhs of (4.6b). Calculate ϕ_T^{m+1} from (4.6b) and then ϕ_z^{m+1} from (4.22) and θ^{n+1} from (4.6d). Find $\phi_{zt}^{m+1} = 2(\phi_z^{m+1} - \phi_z^m)/\Delta t - \phi_{zt}^m$. Return to the step where ψ_t^{n+1} is calculated from (4.6a) and use the new estimates for the variables ψ^{n+1}, χ^{n+1}, w^{n+1}, θ^{n+1}, ϕ_{zt}^{m+1} in the rhs of (4.6a). Repeat the cycle until convergence for ψ^{n+1}, ϕ_z^{m+1}, ϕ_T^{m+1}, χ^{n+1}, w^{n+1}, ψ_t^{n+1}, ϕ_{zt}^{m+1} is obtained.

An initial-value problem calculation is started with the specification at $t = 0$ of $\psi^0 = \psi(\mathbf{x}, 0)$ and an additional assumed condition (see section 5c) that $\chi_t^0 = 0$. An iteration procedure is used to initialize the other variables, i.e., to find $\phi_z'^0$, $\phi_T'^0$, χ^0, w^0, ψ_t^0, $\phi_{zt}'^0$. Estimate $\chi^0 = w^0 = \phi_{zt}'^0 = 0$. Calculate $\phi_T'^0$ from (4.6b) and then $\phi_z'^0$ from (4.22) and θ^0 from (4.6d). Calculate ψ_t^0 from (4.6a). Use the value of ψ_t^0 along with ψ^0 and θ^0 in the rhs of (4.6c). Calculate χ^0 from (4.6c) and then w^0 from (4.13). Use these values of χ^0, w^0 along with ψ^0 in the rhs of (4.6b). Calculate $\phi_T'^0$ from (4.6b) and then $\phi_z'^0$ from (4.22) and θ^0 from (4.6d). Recalculate ψ_t^0 from (4.6a) and χ^0 from (4.6c). Calculate $\psi^1 = \psi^0 + \Delta t\psi_t^0$ and set $\chi^1 = \chi^0$ and $w^1 = w^0$. Calculate $\phi_T'^1$ from (4.6b), using ψ^1, χ^1, and w^1 for the terms on the rhs, and then $\phi_z'^1$ from (4.22). Calculate $\phi_{zt}'^0 = (\phi_z'^1 - \phi_z'^0)/\Delta t$. Return to the step where ψ_t^0 is first calculated from (4.6a) and use the new estimates for χ^0, w^0, θ^0 and $\phi_{zt}'^0$ in the rhs of (4.6a). Repeat until convergence for $\phi_z'^0$, $\phi_T'^0$, χ^0, w^0, ψ_t^0, $\phi_{zt}'^0$ is obtained. For the first time step, estimate $\psi^1 = \psi^0 + \Delta t\psi_t^0$, $\phi_z'^1 = \phi_z'^0 + \Delta t\phi_{zt}'^0$, $\phi_T'^1 = \phi_T'^0$, $\chi^1 = \chi^0$, $w^1 = w^0$, $\phi_{zt}'^1 = \phi_{zt}^0$ and proceed with the iteration as outlined above for the general $(n + 1)$ time level calculation.

We now consider the point concerning the arbitrariness in the definition of K_{1B} (3.2e). First, note that K_{1B} enters the equations (4.6) only in (4.6b) as part of K_{BE} and thus always together with ϕ' in ϕ_T' and $\mathcal{B}_{BE}$. Consequently, as indicated by (3.18) and (3.19), K_{1B} essentially acts as a pressure correction in $\mathcal{B}_{BE}$. For flow problems in doubly periodic domains, such as those investigated in McWilliams et al. (1986) and Norton et al. (1986) and the study in progress mentioned above, where side wall boundary conditions are replaced by periodicity conditions, ϕ' itself does not enter. Only the z derivative of ϕ' is used to find θ from (4.6d) and to provide ϕ_{zt}' for the rhs of (4.6a). Thus, only K_{1Bz} is needed to find ϕ_z' in (4.22) and K_{1B} is not required. Since K_{1Bz} is well defined (3.2d) and the calculation of K_{1B} is not necessary, the issue concerning the arbitrariness of K_{1B} does not arise. For flow in bounded regions, it appears at first that K_{1B} enters the problem through the boundary condtion involving $\mathcal{B}_{BE}$ (4.15). We note, however, that in (4.23) $\mathcal{B}_{BE}$ is given by ψ and ϕ_T'. Since ϕ_T' is determined uniquely by the solution to (4.6b) with boundary conditions (4.16) and with (4.18b), $\mathcal{B}_{BE}$ may be found directly from (4.23) and again the explicit calculation of K_{1B} is not necessary. As a result, we see that the arbitrariness in K_{1B} (3.2e) is not important because it is compensated for in ϕ' such that $\mathcal{B}_{BE}$ takes on a uniquely defined value.

5. Limiting cases

We examine here the behavior of the BEM model in the limiting cases (a) $\epsilon \ll 1$ with $h_B = O(\epsilon)$, (b) small-amplitude linear motion with $h_B = O(1)$ and constant $f = 1$, and (c) small-amplitude linear motion with $h_B = 0$ and $f = 1 + \epsilon\beta y$. It is convenient for case (a) to consider the geometry (4.10) of a rectangular, mid-latitude, β-plane ocean basin, with the addition of variable bottom topography $h_B \neq 0$. As a result, sidewall boundary conditions remain (4.14), (4.15), and (4.16) while the vertical boundary conditions (4.7a, b) are replaced by (4.11a, b). For case (b), we are primarily interested in how BEM represents ageostrophic coastally trapped waves in the presence of O(1) variations in h_B typical of the continental shelf and slope. The geometry (4.10) with constant $f = 1$ and $h_B = O(1)$ is sufficient for illustration of this point and will be utilized. The boundary conditions are (4.7a, b), (4.14), (4.15), and (4.16). For case (c), the geometry (4.10) with $h_B = 0$ and $f = 1 + \epsilon\beta y$ is considered so that the boundary conditions are (4.11), (4.14), (4.15), and (4.16).

a. Quasi-geostrophic limit, $\epsilon \ll 1$, $h_B = O(\epsilon)$

For the limit $\epsilon \ll 1$, $\hat{h}_B = \epsilon^{-1}h_B = O(1)$, the BEM equations (4.5) and (4.6a, d) give

$$\psi = \phi, \tag{5.1a}$$

$$[\nabla^2\phi + (S^{-1}\phi_z)_z]_t = -J(\phi, \zeta + f_1) - [S^{-1}J(\phi, \theta)]_z, \tag{5.1b}$$

$$\theta = \phi_z, \tag{5.1c}$$

while (4.8a, b) reduce to

$$\mathbf{u} = \mathbf{k} \times \nabla\phi, \tag{5.2a}$$

$$w = -S^{-1}[\theta_t + J(\phi, \theta)]. \tag{5.2b}$$

The boundary conditions (4.7a, b) become

$$w = -S^{-1}[\theta_t + J(\phi, \theta)] = 0 \quad \text{at} \quad z = 1, \tag{5.3a}$$

and

$$w = -S^{-1}[\theta_t + J(\phi, \theta)] = \mathbf{k} \times \nabla\phi \cdot \nabla\hat{h}_B \quad \text{at} \quad z = 0. \tag{5.3b}$$

The side wall boundary conditions (4.15) for (5.1b) reduce to

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$$\phi_y = 0 \quad \text{at} \quad x = 0, L^{(x)}, \tag{5.4a}$$

$$\phi_x = 0 \quad \text{at} \quad y = 0, 1. \tag{5.4b}$$

The problem posed by (5.1), (5.2), (5.3) and (5.4) is identical to that for a QG model. Thus, BEM reduces to the QG equations and boundary conditions on a β-plane in the limit $\epsilon \ll 1$, $h_B = O(\epsilon)$.

b. Small-amplitude linear motion, $h_B = O(1)$, $f = 1$

For the limit of small-amplitude linear motion with $h_B = O(1)$ and constant $f = 1$, the BEM equations (4.5) (4.6a, b, d), along with boundary condition (4.16) for (4.6b), imply

$$\psi = \phi, \tag{5.5a}$$

$$[\nabla^2\phi + (S^{-1}\phi_z)_z]_t = 0, \tag{5.5b}$$

$$\theta = \phi_z, \tag{5.5c}$$

and (4.8a, b) reduce to

$$\mathbf{u} = \mathbf{k} \times \nabla\phi - \epsilon\nabla\phi_t, \tag{5.6a}$$

$$w = -S^{-1}\theta_t. \tag{5.6b}$$

The boundary conditions (4.7a, b) become

$$w = -S\phi_{zt} = 0 \quad \text{at} \quad z = 1, \tag{5.7a}$$

and

$$\epsilon w = -\epsilon S^{-1}\phi_{zt} = [\mathbf{k} \times \nabla\phi - \epsilon\nabla\phi_t] \cdot \nabla h_B \quad \text{at} \quad z = h_B(x, y). \tag{5.7b}$$

The sidewall boundary conditions (4.15) for (5.5b) reduce to

$$\epsilon\phi_{xt} = -\phi_y \quad \text{at} \quad x = 0, L^{(x)}, \tag{5.8a}$$

$$-\epsilon\phi_{yt} = -\phi_x \quad \text{at} \quad y = 0, 1. \tag{5.8b}$$

The problem posed by (5.5), (5.6), (5.7) and (5.8) is identical to that for the standard, linear, low-frequency approximation for stratified flow with O(1) bottom topographic variations. Thus, BEM includes the dynamics of ageostrophic coastally trapped waves with the same accuracy (and errors) as the usual linear low-frequency approximation for these waves.

c. Small-amplitude linear motion, $h_B = 0$, $f = 1 + \epsilon\beta y$

This case is examined to illustrate some of the implications for BEM of the presence of χ in (3.6). The points are similar to those discussed for BEM applied to the SWE in appendix B of Allen et al. (1990b).

The linear limit of (3.1) gives

$$\nabla^2\chi + w_z = 0, \tag{5.9a}$$

$$\epsilon\mathbf{u}_{Rt} + f\mathbf{k} \times \mathbf{u} = -\nabla\phi, \tag{5.9b}$$

$$0 = -\phi_z + \theta, \tag{5.9c}$$

$$\theta_t + Sw = 0, \tag{5.9d}$$

where

$$f = 1 + \epsilon\beta y. \tag{5.10}$$

Equations (5.9) are the same as the linearized global balance equations (gBE) considered by Gent and McWilliams (1983b) and are similar to the linear "modified balance equations" examined by Moura (1976). They differ from the linear limit of the equations considered above in (a) since there the limit $\epsilon \ll 1$ results in the neglect of some of the linear terms.

The vorticity and divergence equations implied by (5.9) are

$$\zeta_t + f\nabla^2\chi + \beta v = 0, \tag{5.11a}$$

$$\nabla^2\phi - f\nabla^2\psi + \epsilon\beta u_R + \epsilon^2\beta\chi_x = 0, \tag{5.11b}$$

An omega equation is obtained by substituting (5.9d), the z derivative of the x-component of (5.9b), and the z derivative of (5.11a) in the t and z derivative of (5.11b):

$$S\nabla^2 w + f^2 w_{zz} - \beta(2fv_z - \phi_{xz}) = \epsilon^2\beta\chi_{xzt}. \tag{5.12}$$

Analysis of wave solutions to the linear gBE by Gent and McWilliams (1983b) and of the linear "modified balance equation" by Moura (1976) showed the existence of spurious, high-frequency oscillations. That property is present in (5.9) and may be seen most easily by examining modulated wave solutions for $\epsilon \ll 1$ and slowly varying f (5.10) (e.g., Grimshaw and Allen 1988). Expanding ψ, ϕ, and χ in the vertical normal modes (4.25), we find that modulated waves exist with approximate dispersion relations,

$$\omega = -\beta k/(K^2 + f^2\lambda_j^2), \tag{5.13a}$$

and

$$\omega = \frac{(K^2 + \lambda_j^2)K^2}{\epsilon^2\beta k\lambda_j^2}, \tag{5.13b}$$

where ω is the frequency, (k, l) are the wavenumbers in the (x, y) directions, and $K^2 = k^2 + l^2$. The first set of frequencies (5.13a) corresponds to the expected baroclinic Rossby wave solutions where $O(\phi/\chi) = O(\psi/\chi) = O(1)$. In the Rossby wave, the balance in (5.11a) involves all three terms and the primary balance in (5.12) is between the terms on the left-hand side. The second set of frequencies (5.13b) have $\omega = O(\epsilon^{-2})$ and correspond to the spurious high-frequency oscillations found in Gent and McWilliams (1983b) and Moura (1976). For this wave, $O(\phi/\chi) = O(\psi/\chi) = O(\epsilon^2)$ and the primary balance in (5.11a) is between ζ_t and $f\nabla^2\chi$. In (5.12), the approximate balance is between $S\nabla^2 w$, $f^2 w_{zz}$, and $\epsilon^2\beta\chi_{xzt}$ and this leads directly to the approximate dispersion relation (5.13b).

In addition, if we consider the governing equations to be (5.9a), (5.11a, b), and (5.12) we see that initial values for both ψ ($t = 0$) and χ ($t = 0$) evidently may

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be specified. This differs from BE, where the $\epsilon^2\beta\chi_x$ and $\epsilon^2\beta\chi_{xzt}$ terms are absent from (5.11b) and (5.12), respectively. In BE, only the initial value of ψ ($t = 0$) is specified and ϕ ($t = 0$) and χ ($t = 0$) are then determined from the equations. To reduce initial excitation of the spurious high-frequency oscillations in BEM, it appears from (5.9), (5.11), (5.12) and (5.13b) that BEM should be initialized by specifying ψ ($t = 0$), similar to BE, and by determining ϕ ($t = 0$) and χ ($t = 0$) such that (5.9a), (5.11a, b) and (5.12) are satisfied with χ_{xzt} ($t = 0$) = 0 in (5.12).

Behavior related to the spurious high-frequency oscillations and initial-value requirements found above in the linear β-plane problem may be expected to occur also for BEM in more general problems, even with constant f. The situation is more complicated in the general case, however, because χ enters (3.6) through the nonlinear terms $\nabla^2 K_{1B}$, $J(\zeta, \chi)$ and $J(w, \psi_z)$. In obtaining numerical solutions to the BEM model, the method of initialization of the fields and the choice of finite-difference time integration scheme must evidently be utilized to filter out effects of the spurious high-frequency motions. The initialization can be accomplished, using the above considerations of the linear β-plane problem as a guide, by the specification of ψ ($t = 0$) only, as in BE, and the subsequent determination of ϕ ($t = 0$) and χ ($t = 0$) from the equations and an assumed additional initial condition of χ_t ($t = 0$) = 0, as described in section 4. Following the initialization of the fields, the variables may be stepped forward in time using the time integration procedure given in section 4. Further study of the general implications of the possible existence of spurious high-frequency oscillations in the BEM model and of procedures for initialization and time differencing is desirable.

6. Summary

An approximate model for $\epsilon \ll 1$ that is close to the balance equations (BE) has been formulated for a rotating, continuously stratified fluid governed by the hydrostatic, Boussinesq, inviscid, adiabatic primitive equations with spatially variable Coriolis parameter. This model is formulated from approximate momentum equations and is referred to as BEM (balance equations based on momentum equations). The BEM model equations imply conservation of volume integrals of an appropriate energy density and also conservation of an analogue of potential vorticity on fluid particles and thus conservation of volume integrals of potential enstrophy density. Consequently, energy and potential enstrophy are global invariants (McWilliams and Gent 1980) for BEM. As is the case for BE, the BEM equations filter out high-frequency gravity-inertial waves and remain valid for motion over O(1) variations in bottom topography and for flows with O(1) variations in the height of density surfaces. The fact that BEM is based on approximate momentum equations is important in providing equations for horizontal vorticity components that are used in the derivation of equation (3.8) expressing conservation of potential vorticity on fluid particles. The existence of approximate momentum equations is also valuable for the formulation, based on the no-normal-flow condition (2.19), of consistent boundary conditions at rigid surfaces.

The governing equations for BEM are conveniently expressed in a form similar to BE involving a vorticity (3.5) and a divergence (balance) equation (3.6). In this formulation, the BE and BEM models involve identical equations for continuity, vorticity, and heat with differences represented only by the presence of additional higher order terms in the balance equation (3.6) for BEM. In the limit $\epsilon \ll 1$, with $h_B = O(\epsilon)$, the BEM equations and boundary conditions reduce to those for QG. For small amplitude linear motion over O(1) bottom topographic variations, the BEM model and boundary conditions reduce to the standard stratified linear model for ageostrophic topographic waves. The possibility of the existence in BEM of spurious high-frequency time-dependent oscillations is indicated by a consideration of a linear β-plane problem. Evidently, the method of initialization of the variables and the choice of time integration scheme must be utilized in BEM to filter out effects of these motions.

Methods for obtaining numerical solutions and for applying boundary conditions to BEM are presented. The governing set of equations (4.6) utilized in the numerical solution procedure includes the linearized potential vorticity (4.6a), balance (4.6b), vorticity (4.6c), hydrostatic (4.6d) and continuity (4.6e) equations. The methods of solution and of boundary condition application are similar to those formulated and applied to a BEM model for the f-plane shallow-water equations in Allen et al. (1990a,b). The problem of flow in a flat-bottom, mid-latitude, β-plane, rectangular ocean basin with vertical side-walls is considered to give a specific example of the method of boundary condition application for the equation set (4.6). The fields are initialized, similar to BE, by the specification of ψ ($t = 0$) = $\psi(\mathbf{x}, 0)$. Initial values of the other variables are determined from the equations and the additional assumption that χ_t ($t = 0$) = 0. The numerical time integration scheme utilizes an iteration procedure at each time-step that is based on one developed by Norton et al. (1986) for application to the BE formulated with an omega equation. The method presented here has been successfully implemented to obtain finite-difference solutions to (4.6) in a flat-bottom, doubly periodic domain with constant f. Results of studies in progress involving comparisons of numerical solutions from BEM, BE, QG and PE will be reported on separately.

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