Equilibrium Spectra of Water Waves Forced by Intermittent Wind Turbulence

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(Manuscript received 18 February 1992, in final form 25 January 1993)

ABSTRACT

With the help of fractal geometry used to model the intermittency of energy input from wind to wave components, the theoretical spectra of the equilibrium range in wind-generated gravity waves proposed by Phillips are refined.

On account of the intermittency, it is proven that the classical frequency spectral exponent 4 must be replaced by $4 + (2 - D)$, where D is the informational entropy dimension of the support subset, upon which the energy input from the wind to the gravity waves in the equilibrium range is concentrated. To a first approximation, it is found that $D \approx 1.88$ and $4 + (2 - D) \approx 4.12$. The variation of the Toba constant is found to be proportional to $(u_*^2/gL_0)^{(2-D)/2}$, where L_0 is the wavelength of the longest wave component in the equilibrium range, that is, the lower limit wavenumber above which the processes of energy input from wind, spectral flux divergence, and loss by breaking are all significant and proportional. The refined wavenumber spectrum is less sensitive to wind strength than the original.

1. Introduction

With reference to the high wavenumber (and frequency) regime of wind-generated gravity waves, Phillips (1958), from the consideration of a limiting configuration of surface waves, showed that an equilibrium range may exist that is independent of wind stress. He derived upper-limit spectral asymptotes in the gravity wave range of

$$\Psi(\mathbf{k}) \propto f(\theta)k^{-4} \quad (1.1)$$

for the wavenumber spectrum and

$$\Phi(\sigma) \propto g^2\sigma^{-5} \quad (1.2)$$

for the frequency spectrum from dimensional consideration, where θ is the angle between the wind and the wavenumber $\mathbf{k}$. Data collected mainly in the 1960s seems to support the relation (1.2), [e.g., summarized in the work of Hess et al. (1969)].

Subsequent reliable measurements, including the first measurements by Toba (1973), however, lend support to the forms

$$\Psi(\mathbf{k}) = \beta \cos^p\theta u_* g^{-1/2}k^{-7/2} \quad (1.3)$$

for the wavenumber spectrum in the equilibrium range, and

$$\Phi(\sigma) = \alpha u_* g\sigma^{-4} \quad (1.4)$$

for the frequency spectrum, where $p = 1/2$, u_* is the wind friction velocity, α is the Toba constant, and β is the numerical coefficient in the wavenumber spectrum. Kitaigorodskii (1983) arrived at (1.3) and (1.4) theoretically from dynamical considerations. More recently, Phillips (1985) reexamined the nature of the equilibrium range based on dynamical insights into wave–wave interactions, energy input from the wind, and wavebreaking. He arrived at Eq. (1.3) and Eq. (1.4) with the assumption that all of these three processes are important in the equilibrium range. The work of Phillips is significant in that the derivation of Eq. (1.3) and Eq. (1.4) is based on the concept of an equilibrium instead of that of saturation, which underlied his derivation of Eq. (1.1) and Eq. (1.2) in 1958. The validity of Eq. (1.3) and Eq. (1.4) (for fetch-limited conditions) may be considered as established at the present stage of development of ocean wave theory.

As indicated by Phillips (1985), there are still some discrepancies between the measurements and the spectra (1.3) and (1.4). Where the frequency spectrum (1.3) is concerned, the value α is of persistent uncertainty and the spectral exponent value 4 is also not exact. For example, in the spectra series of Kawai et al. (1977) the mean exponent and standard deviation are 4.13 ± 0.2. From the measurement results of Mitsuyasu (1980), Kawai et al. (1977), Forristall (1981), and Tang and Shemdin (1983), we can conclude that the Toba constant, α, is not likely to be a true universal constant but rather a variable, and the value of the frequency spectral exponent is between 4 and 5 rather than 4 exactly, which implies that the wavenumber spectral exponent will have a value between 7/2 and

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4. The problem is to understand the mechanics and to determine analytically the properties of such variability in observed equilibrium spectra.

Theoretical results about the fractal geometry of wind-generated gravity wave surfaces, such as Barenblatt and Leykin (1981), Glazman (1986), Glazman and Weichman (1989), and Stiassnie et al. (1991), suggest that when the Hausdorff dimension of wave surface strictly exceeds 2, the equilibrium spectra should be functions of the nondimensional fetch and the wave surface fractal codimension, μ. One of the necessary conditions for $\mu > 0$ is that the wind fetch should be long enough so that the inverse cascade and direct cascade could be developed. The results are applicable only for sufficient long fetch conditions. Experimental studies that relate the variability of the equilibrium spectra to the directionality of the waves, the wave breaking of longer wave components, Doppler shifting, wind drift, and ambient current were presented systematically by Banner (1990a) and Banner et al. (1989). Their findings challenged the correctness of the equilibrium range model formulated by Phillips (1985).

In this paper, the variability of the equilibrium spectra related to the intermittency of the turbulent energy input from wind is studied. The motive is based on two facts: first, the wind energy input is a principal forcing term in the equilibrium range; and second, the energy dissipation intermittency is a universal feature in high Reynolds number turbulent flows. We want to know whether or not the intermittency can be seen explicitly in the equilibrium spectra. We intend to take advantage of the results, developed since 1970, of the fractal geometry and its applications to approach the dissipation intermittency in turbulent wind flows in order to place the description of the wind wave spectra in the equilibrium range on a firmer theoretical basis. In doing so, we are led to a refined formula for the degree of saturation (see Phillips 1985) and then to refinements of the wind wave spectra in the equilibrium range of Phillips (1985). A formula describing the variation of the Toba constant is also obtained. Our results seem to support the equilibrium range model formulated by Phillips (1985), in which the predicted mechanics may be considered as the most essential one.

2. Intermittency and fractal structure of the rate of energy input from wind

a. Background

The two-dimensional wavenumber spectrum is given by

$$\Psi(\mathbf{k}) = (2\pi)^{-2} \int < \zeta(\mathbf{x})\zeta(\mathbf{x}+\mathbf{r}) > \exp(i\mathbf{r}\mathbf{k})d\mathbf{r}, \tag{2.1}$$

which is the Fourier transform of the instantaneous spatial covariance of the surface displacement ζ. Our study is limited to the equilibrium range

$$k_0 < k < k_1, \tag{2.2}$$

where k_0 is the wavenumber lower limit, which is somewhat larger than g/c_0 (c_0 the phase velocity of the dominant waves), and k_1 is the upper limit, which is much less than $(g/T)^{1/2}$ and less than g/u_*^2 (T being the ratio of surface tension to water density, and u_* the wind friction velocity); see Phillips (1985).

The dynamics of the wave field in the equilibrium range is expressed by the balance of the action spectral density, defined as

$$N(\mathbf{k}) = \frac{g}{\sigma}\Psi(\mathbf{k}),$$
$$= \left(\frac{g}{k}\right)^{1/2}\Psi(\mathbf{k}). \tag{2.3}$$

The gravity wave dispersion relationship ($\sigma^2 = gk$) has been applied in arriving at the latter form.

Phillips (1985) demonstrated that when $\mathbf{k}$ is restricted to the equilibrium range (2.2), the wave components will approach a state of statistical equilibrium (rather than saturation) determined by a balance among the net gain of the spectral action density through resonant wave–wave interactions, the rate of spectral input from the wind, and the loss by wave breaking, that is

$$-\nabla_k \cdot \mathbf{T}(\mathbf{k}) + S_w - D = 0. \tag{2.4}$$

After introducing the degree of saturation

$$B(\mathbf{k}) = k^4\psi(\mathbf{k}) = g^{-1/2}k^{9/2}N(\mathbf{k}), \tag{2.5}$$

and using dynamical insights into wave–wave interaction, energy input from the wind and wave breaking based on developments since his first model (1958), Phillips (1985) demonstrated

$$-\nabla_k \cdot \mathbf{T}(\mathbf{k}) = gk^{-4}B^3(\mathbf{k}), \tag{2.6}$$

$$S_w = m\cos^{2p}\theta g k^{-4}\left(\frac{u_*}{c}\right)^2 B(\mathbf{k}), \tag{2.7}$$

and

$$D = gk^{-4}f(B(\mathbf{k})), \tag{2.8}$$

where $B(\mathbf{k})$ is a dimensionless function and p, m are two universal constants. With the assumption in the equilibrium range that the processes of energy input from wind, spectral flux divergence, and loss by breaking are all of importance, Phillips (1985) showed that for wavenumber k well inside the equilibrium range (2.2), all the three quantities in Eq. (2.4) must be proportional. If we represent this proportionality as

$$\frac{S_w}{-\nabla_k \cdot \mathbf{T}(\mathbf{k})} = \frac{m}{\beta^2}, \tag{2.9}$$

where β is some constant, we find, on substituting (2.6) and (2.7) in (2.9), that the degree of saturation is given by

$$B(\mathbf{k}) = \beta \cos^p\theta\left(\frac{u_*}{c}\right). \quad (2.10)$$

The wavenumber spectrum (1.3) was obtained from relations (2.10) and (2.5).

It is worthy of note that the spectral energy input from wind (2.7) is an extremely important term that determines the expression of the degree of saturation $B(\mathbf{k})$, which in turn determines the exact form of the wavenumber spectrum $\Psi(\mathbf{k})$.

b. Intermittency of the mean wind stress

In the equilibrium range, the phase velocity c of each wave component is less than $10u_*$, so the matched layer is inside the viscous sublayer (e.g., see Phillips 1977), where the turbulent wind flow is highly rotational. Accordingly, the stretching of vortex filaments implies that the small-scale turbulent fluctuations are constantly intermittent there. This implies that the intensity of the wind stress acting upon each wave component in the equilibrium range is distributed in a nonuniform manner over the water surface. Hence, an intermittency in the air turbulence will be transformed to the water surface.

As demonstrated in Eq. (2.4) of Phillips, the energy input from wind is one of the three terms that maintains the equilibrium range of the wind waves. Its importance, especially in higher frequencies, has been confirmed experimentally by many authors, such as Schule et al. (1971), Mitsuyasu and Honda (1982) and Plant (1982). If the intensity of the mean wind stress is considerably intermittent on the air–water interface, the wave components at higher frequencies will also be considerably intermittent and will result in a spatially inhomogeneous distribution instead of being almost uniform (which is the basic hypothesis in the wind wave theory referred to above).

The inhomogeneity of the wave components at higher frequencies can be tested by the wavelet transform method. The wavelet transform $T\zeta$ of a function ζ with respect to the wavelet g is a function over the half-plane H parameterized by (τ, α), $\tau, \alpha \in R$, $\alpha > 0$:

$$T\zeta(\tau, \alpha) = \frac{1}{\alpha}\int \bar{g}\left(\frac{t-\tau}{\alpha}\right)\zeta(t)dt. \quad (2.11)$$

The wavelet $g(\epsilon L^2 \cap L^1)$ satisfies the following admissible condition:

$$\int |\hat{g}(\omega)|^2 \frac{d\omega}{\omega} < \infty \quad (2.12)$$

where $\hat{g}(\omega)$ is the Fourier transform of g.

It has been shown that the transform (2.11) can be inverted for a large class of functions. This transformation is a sort of mathematical microscope whose magnification is $1/a$, whose position is τ, and whose optics is given by the choice of the specific wavelet, g. For more details we refer to Daubechies (1988a,b).

We shall consider restrictions of $T\zeta(\tau, \alpha)$ to a fixed discrete value of the scale parameter. A restriction $T\zeta(\tau, \alpha_i)$ (α_i fixed) is called a voice.

Figure 1 shows the wavelet transforms of a wind wave elevation dataset measured by a capacitance wave gauge suspended from an oil platform located at 39°15′N, 119°50′E, in the Bohai Sea, which is semiclosed. The measured water surface displacement analyzed is shown at the bottom of this figure. Respectively, seven of the real parts of the voices are shown in the upper part of this figure. Numbering from bottom to top, the normalized angular frequencies of voice σ/σ_0 (σ_0 being the peak frequency) are correspondingly 1, 2, 8/3, 3, 10/3, 4, and 5.

The intermittency and inhomogeneity of the wave components at higher frequencies are evident in Fig. 1. This fact permits us to conclude that the mean wind

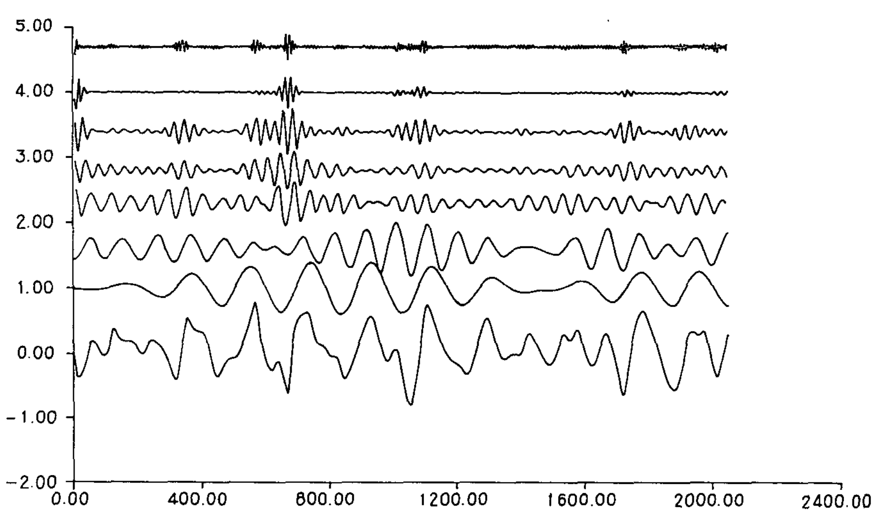

FIG. 1. The wavelet transforms of the observed wind wave elevation. The analyzing wavelet is the modulated Gaussian $\exp(i5t - 0.5t)$.

stress acting upon each wave component in the equilibrium range is considerably intermittent.

c. *Fractal structure of* S_w

The intermittency of energy dissipation in turbulent fluid movements has been the subject of many theoretical and experimental investigations over the past 15 years. Meneveau and Sreenivasan (1987), after analyzing the experimental data of grid turbulence, the wake of a circular cylinder, boundary-layer turbulent flow, and atmospheric turbulence, demonstrated that the turbulent energy dissipation distribution is considerably intermittent in space, that the geometrical structure of the intermittency is a multifractal (Frisch and Parisi 1985), and the multifractal spectra $f(\gamma)$ (where γ is the Lipschitz–Holder exponent) for one-dimensional sections through the dissipation fields of these four different turbulent flows are the same within experimental accuracy. They approximated the multifractal spectra based on a hypothesis that the turbulent cascade was a binomial multiplicative process with the fraction $p = 0.3$, that is

$$f(\xi) = -\frac{\xi \ln\xi + (1-\xi)\ln(1-\xi)}{\ln 2}, \quad (2.13)$$

with

$$\xi = \frac{\ln(1-p) + \gamma \ln 2}{\ln(1-p) - \ln p}; \quad (2.14)$$

see Fig. 2.

The binomial multiplicative model can be summarized as follows: In the turbulence field, we choose arbitrarily a unit line segment $S = [0, 1]$, and first divide it into two parts of equal length $\delta = 2^{-1}$. The left part is given a fraction p (=0.3) of the total dissipation on S, and the right hand is given the remaining fraction $q = 1 - p = 0.7$. Next consider an increased resolution $\delta = 2^{-2}$ locked at each new fraction of the line. The multiplicative process divides the dissipation in each part in the same way. Four pieces arise with the fractions of the dissipation given by

$$\{\mu_i\} = pp, pq, qp, qq. \quad (2.15)$$

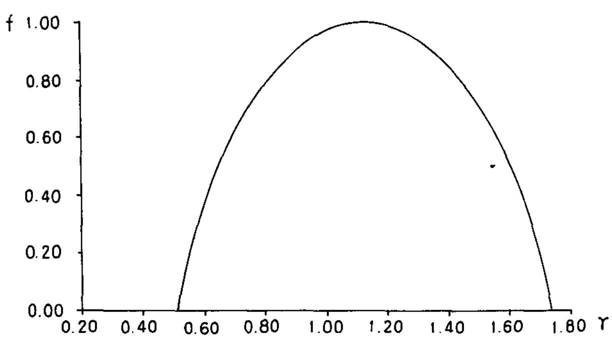

FIG. 2. The multifractal spectrum $f(\gamma)$ of the binomial multiplicative model with the fraction $p = 0.3$.

As this process of redistribution is iterated, it produces shorter and shorter segments that contain less and less of the total dissipation. After n steps there are $N = 2^n$ segments with the length $\delta = 2^{-n}$. We have then with $\xi = k/n$ and $k = 0, 1, \cdots, n$,

$$N_n(\xi) = \frac{n!}{(\xi n)!((1-\xi)n)!} \quad (2.16)$$

segments on which the fraction of the dissipation is

$$\mu_\xi = p^k q^{n-k} = (p^\xi q^{1-\xi})^n. \quad (2.17)$$

The Lipschitz–Holder exponent γ is defined by the equation

$$\mu_\xi = \delta^\gamma. \quad (2.18)$$

There is a one to one correspondence between the parameters ξ and γ. In the nth resolution, $N_n(\xi)$ line segments (with the length $\delta = 2^{-n}$) have the same fraction of the dissipation. These segments form a subset S_γ of the unit interval $S = [0, 1]$:

$$S = \bigcup_\gamma S_\gamma. \quad (2.19)$$

In the limit $n \to \infty$, we find that for each given γ (or ξ), the subset S_γ is a fractal set of points and the fractal dimension of S_γ is given by Eq. (2.13). In the nth resolution, we have, by the definition,

$$N_n(\xi) \propto \delta^{-f(\gamma)}. \quad (2.20)$$

The most important property of the binomial multiplicative model we apply in this paper is that the overwhelming bulk of the dissipation is concentrated essentially on "the set of concentration" S_{γ_s} with fractal dimension given by the entropy dimension Γ (Mandelbrot 1982):

$$\Gamma = \gamma_s = f(\gamma_s) = -\frac{p \ln p + q \ln q}{\ln 2} \approx 0.88, \quad (2.21)$$

and the corresponding fractal codimension is

$$\mu = 1 - \Gamma \approx 0.12. \quad (2.22)$$

In other words, we find that a fraction of dissipation arbitrarily close to 100% is contained in sets that have $\xi \approx p$ for which $N(\xi)\mu_\xi$ is near its maximum. It is this fact of concentration (which is called curdling) that is the topology of the intermittency of the energy dissipation in turbulent flows. We note that the entropy dimension Γ is only one of the measures of the complexity of turbulence.

Although the energy dissipation intermittency is a universal feature in high Reynolds number turbulent flows, whether their fractal structures depend on boundary conditions and external forces is still a challenge to both theory and experiment. Kerman (1993) established that the spatial distribution of breaking waves is a multifractal process. He found that the codimension for the support of breaking waves, which constitute the locations of dissipation, is essentially 0.12

[the same value as Meneveau and Sreenivasan's (1987)]. Kerman's work is important in that it suggests that although the appearance of breaking waves over the air–water interface changes radically the large-scale structure of the wind flow in the matched layer (Banner 1990b), its influence on the small-scale dissipation intermittency (curdling) would be of secondary particularity. As a hypothesis, we anticipate that the result of Meneveau and Sreenivasan (1987) can be applied in our case as a first approximation. It follows that the entropy dimension D of the support subset on which the energy input from the wind to gravity waves in the equilibrium range is mainly concentrated is (see Mandelbrot 1972, 1982)

$$D = 1 - [p \ln p + (1-p)\ln(1-p)]/\ln 2 \approx 1 - \mu,$$
$$= 1.88. \qquad (2.23)$$

We note that this entropy dimension D, being less than its topological dimension (=2), is a measure of the intermittency and fractal structure of the energy input from wind to wave components in the equilibrium range. Generally,

$$1 < D < 2. \qquad (2.24)$$

In Phillips's arguments leading to the formulation of Eq. (2.7), the input of the turbulent wind vortex momentum (and its energy as well) into the gravity wave component of wavenumber **k** was assumed to occur in a (statistical) uniform manner over the whole water surface. Nevertheless, we believe that such is not the case in the presence of intermittent turbulence. The action input S_w is highly intermittent, with large magnitudes concentrated in rather small regions. Consider a unit surface Π (more specifically a unit square) on which S_w is to be calculated. Let us divide Π into 2^{2n} subsquares, with side $\delta = 1/2^n$. According to the multifractal model presented above, the action input from wind to the wave component of length scale δ is mainly concentrated on

$$N_\delta \propto \frac{1}{\delta^D} \qquad (2.25)$$

subsquares interspersed in Π. On the rest $2^{2n} - N_\delta$ subsquares, although there are some individual locations upon which the intensity of action input is quite high, the sum of the input is nearly zero. To simplify the algebraic calculation we shall neglect this minor effect and assume that the action input S_w is totally concentrated on the

$$N_\delta = \frac{1}{\delta^D} \qquad (2.26)$$

subsquares. In Eq. (2.26) we have set the proportionality coefficient to be equal to 1. In fact, it does not change our formulation of Eq. (3.6) below.

3. Refinement to the degree of saturation $B(\mathbf{k})$

The intermittency discussed previously requires an adjustment to Eq. (2.7), which expresses the rates of spectral action input from wind to the wave components in the equilibrium range as a function of wavenumber k.

Consider the longest wave component (length scale $L_0 = 2\pi/k_0$) in the equilibrium range. Suppose that at a given time the process of action input from wind to the wave component of length scale $L_0 = 2\pi/k_0$ occurs uniformly on water surface cells of size L_0 and of area L_0^2. Then each of the initial cells breaks up into T^2 subcells, and the action input from wind to the wave component of length scale l_1 $(= L_0/T = 2\pi/Tk_0)$ is essentially concentrated within N of these T^2 subcells. The fraction τ of the surface occupied by the active energy input from the wind is

$$\tau = \frac{N}{T^2}. \qquad (3.1)$$

After infinite steps, the fractal subset $\Pi'(\subset\Pi)$ will be achieved, of which the entropy dimension is D. The number N can be defined with the aid of the entropy dimension D [see Eq. (2.26)]: $N = T^D$, and

$$\tau = T^{-\mu}, \qquad (3.2)$$

where μ is the fractal codimension

$$\mu = 2 - D. \qquad (3.3)$$

The essence of the present paper is to assume that *formulation of Eq. (2.7) in Phillips is valid only in the active regions.* At step q of the mentioned process, the corresponding wave length scale will be

$$l_q = \frac{L_0}{T^q} = \frac{2\pi}{k_0 T^q}. \qquad (3.4)$$

Then the mean rate of the action input from wind over the unit water surface will be the local Phillips formulation prorated for the active area

$$S_w = \tau^q m \cos^{2q}\theta g k^{-4}\left(\frac{u_*}{c}\right)^2 B(\mathbf{k}), \qquad (3.5)$$

which, with the aid of Eq. (3.2), can be written as

$$S_w = (L_0 k)^{-\mu} m \cos^{2p}\theta g k^{-4}\left(\frac{u_*}{c}\right)^2 B(\mathbf{k}), \qquad (3.6)$$

since $kL_0 = L_0/l_q = T^q$.

On substituting Eq. (3.6) and Eq. (2.6) in Eq. (2.9), we found the refined degree of saturation

$$B(\mathbf{k}) = (L_0 k)^{-\mu/2}\beta \cos^p\theta\left(\frac{u_*}{c}\right). \qquad (3.7)$$

Expression (3.7) is different from (2.10) by an amplitude factor $(L_0k)^{-\mu/2}$. It results from the fractal intermittency for the turbulent energy input discussed previously. Accordingly we expect the degree of saturation to be intermittent also.

4. Refined spectra in the equilibrium range

In the Phillips (1985) formulation, the wave spectrum was related to the degree of saturation $B(\mathbf{k})$, as discussed in section 2. In the spirit of the dynamics associated with the expression of wave spectral energy, albeit now locally, we consider a refinement, determined by the fractal intermittency, of the averaged wave spectrum based on the degree of saturation (3.7) averaged over local active regions.

Accordingly from Eqs. (3.7) and (2.5) we are led to a refined wavenumber spectrum in the equilibrium range

$$\begin{aligned}\Psi(\mathbf{k}) &= k^{-4}B(\mathbf{k}) \\ &= L_0^{-\mu/2}\beta \cos^p\theta\left(\frac{u_*}{c}\right)k^{-4-\mu/2}, \\ &= L_0^{-\mu/2}\beta \cos^p\theta u_* g^{-1/2}k^{-7/2-\mu/2}, \quad (4.1)\end{aligned}$$

where in view of the relation (2.24)

$$0 < \frac{\mu}{2} = \frac{2-D}{2} < \frac{1}{2}, \quad (4.2)$$

and L_0 is the wavelength of the longest wave component in the equilibrium range.

The refined wavenumber spectrum (4.1) is different from (1.3) in Phillips in that the exponent of Eq. (4.1), H_k say, is

$$\frac{7}{2} < H_k\left(=\frac{7}{2}+\frac{\mu}{2}\right) < 4 \quad (4.3)$$

instead of being 7/2, and the wavelength of the longest wave component in the equilibrium range L_0 appears explicitly in the refined wavenumber spectrum (4.1).

For a given fetch, the value L_0 will increase with the friction velocity u_*. Because the refined wavenumber spectrum (4.1) is proportional to $u_* L_0^{-\mu/2}$, it is expected that the refined $\Psi(\mathbf{k})$ is less sensitive to wind strength than Phillips' original spectrum (1.3). The experiment results of Banner (1990a) and Banner et al. (1989) confirm qualitatively this tendency as well as the relation (4.3).

The refined frequency spectrum in the equilibrium range can be found from Eq. (4.1) by integration over all wavenumbers at constant frequency σ:

$$\begin{aligned}\Phi(\sigma) &= 2\int_{-\pi/2}^{\pi/2} k\Psi(\mathbf{k})\left(\frac{\partial\sigma}{\partial k}\right)^{-1} d\theta\,|_{k=\sigma^2/g}, \\ &= 4\beta L_0^{-\mu/2} I(p) u_* g^{1+\mu/2}\sigma^{-4-\mu}, \\ &= \alpha L_0^{-\mu/2} u_* g^{1+\mu/2}\sigma^{-4-\mu}; \quad (4.4)\end{aligned}$$

where $\sigma_0 \ll \sigma \ll (4\pi s)^{-1}\sigma_0$, $s = \langle\zeta^2\rangle^{1/2}/\lambda_0$, σ_0 is the peak frequency, α the Toba constant, λ_0 the dominant wavelength, and

$$I(p) = \int_{-\pi/2}^{\pi/2} \cos^p\theta d\theta. \quad (4.5)$$

Similarly, the refined frequency spectrum in the equilibrium range is different from Phillips's Eq. (1.4) in that, first, the wavelength of the longest wave component in the equilibrium range L_0 appears explicitly in the refined frequency spectrum (4.4), and, second, the exponent of Eq. (4.4), H_σ say, is equal to $4 + \mu$ instead of being 4 and its precise value is between 4 and 5,

$$4 < H_\sigma\,(=4+\mu) < 5. \quad (4.6)$$

To a first approximation, the value of D is given by (2.23). It follows from Eq. (3.3) that the frequency exponent

$$H_\sigma + 4 + \mu \doteq 4.12. \quad (4.7)$$

The theoretical frequency and wavenumber spectra in the equilibrium range are of essential importance to the whole theory of wind-generated gravity waves. We point out that every theoretical or empirical spectrum of the wind-generated gravity wave has to be reconciled with the equilibrium range spectrum over large σ or k and this reconciliation begins generally from the frequency or wavenumber, which is equal to about twice that of spectral peaks. If the Toba constant is of persistent uncertainty, then the spectra will be undetermined not only in the equilibrium range but also in the energy-containing regime. To verify the expression for the refined frequency spectrum in the equilibrium range, that is, Eq. (4.4), we first rewrite it in dimensionless form, that is

$$\frac{\Phi(\sigma)g^3}{u_*^5} = \alpha_*(\sigma u_*/g)^{-(4+\mu)} \quad (4.8)$$

with

$$\alpha_* = \alpha\left(\frac{u_*^2}{gL_0}\right)^{\mu/2}. \quad (4.9)$$

The term α_* is dimensionless. If the entropy dimension D was set equal to 2, that is, the codimension $\mu = 0$, α_* would be the same as the Toba constant α, which is the case considered by Phillips, and correspondingly Eq. (4.8) would be identified with the classical frequency spectrum (1.4). We shall call α_* *Toba's variable,* which is determined not only by the friction velocity u_* and gravity g, but also the wavelength of the longest wave component in the equilibrium range L_0 (by the entropy dimension D), all scaled by the fractal codimension μ. Here L_0 represents the lower limit of the range of wave components over which the processes of energy input from wind, spectral flux divergence, and loss by breaking are all of importance and proportional. Therefore, L_0 will be determined by two conditions: first, the internal nature of the equilibrium range, characterized by u_* and g as demonstrated by Phillips (1985) and second, the external nature of the equilibrium range, that is, the conditions of the dominant waves, particularly their nonlinear wave–wave interactions. The relation (4.9) demonstrates that the dependent parameter L_0 is a central parameter to determine the wave spectra in the equilibrium range.

When we look for the dependence of L_0 (more specifically the geometry of the initial surface cells mentioned at the beginning of section 3) on external conditions, intuitively, it should depend on the wave age, the directionality of the wave field, the state of wave breaking, the wind drift, the ambient current, etc. A systematic experimental investigation on the influences of these processes was performed by Banner (1990a) and Banner et al. (1989), who proposed a more detailed empirical equilibrium subrange model.

Measured values of the Toba constant α vary widely. Examples of such values are $\alpha = 0.02$ (Toba 1973), 0.062 (Kawai et al. 1977), 0.14 (Mitsuyasu 1980), 0.11 (Kahma 1981), 0.11 (Forristall 1981), and 0.13 (Battjes et al. 1987), of which the relative error is greater than 85%. In view of such variability, Mitsuyasu (1980), comparing the JONSWAP spectrum with their measurements in the East China Sea and sea regions adjacent to Japan, constructed a formula for α_* that can be rewritten as

$$\alpha_* = 8.44 \times 10^{-2}\tilde{\sigma}_0^{-1/7}, \quad = 5.57 \times 10^{-2}\tilde{X}^{1/21}, \qquad (4.10)$$

where $\tilde{\sigma}_0 = \sigma_0 U_{10}/g$, $\tilde{X} = gX/U_{10}^2$, X is the fetch. Similarly, based on the measurements of Donelan et al. in Lake Ontario, Battjes et al. (1987) proposed the relation

$$\alpha_* = 0.155\tilde{\sigma}_0^{-0.457}, \quad = 0.0514\tilde{X}^{1/10}. \qquad (4.11)$$

To reconcile the relation (4.9) with Eq. (4.10) or Eq. (4.11), L_0 would be

$$L_0 \propto \frac{g}{\sigma_0^2}. \qquad (4.12)$$

In other words, L_0 would be determined only by the gravity wave dispersion relation of the dominant waves.

Phillips (1985) examined various observational results and concluded that for the Toba constant α, there seemed to be no systematic variations with "significant slope" $s\ (=\sqrt{\langle \zeta^2 \rangle}/\lambda_0)$, gX/U_{10}^2, or $\sigma_0 u_*/g$. Because of the lack of reliable measurements of the parameter L_0, an examination of the validity of Eq. (4.9) and Eq. (4.12) is not possible. In fact, to determine the intrinsic dependence of L_0 is beyond the objective of the present paper. It is suggested that future experimental work will confirm our prediction that the Toba constant, which determines the spectral levels in the equilibrium range, is not an absolute constant, but a variable whose dependence is governed by Eq. (4.9).

In the refined frequency spectrum (4.4) the frequency exponent $H_\sigma = 4 + (2 - D)$ is a number between 4 and 5, instead of exactly 4 as in Phillips model. To a first approximation, $H_\sigma = 4.12$; see Eq. (4.7). This result is confirmed as realistic by many data measurements, such as those of Mitsuyasu (1980), Kawai et al. (1977), Forristall (1981), Tang and Shemdin (1983), Battjes et al. (1987), and Wen (1990). For example, in the spectra series of Kawai the mean exponent and standard deviation are 4.13 ± 0.2. Thus, the relative difference between our result and that of Kawai's best estimate is only 0.24%. [Correspondingly, the wavenumber exponent $H_k = 7/2 + \mu/2$ of the refined wavenumber spectrum (4.1) is a number between 7/2 and 4, which is more realistic than being 7/2 exactly.]

The more reliable value of the entropy dimension D of the support subset, on which the energy input from the wind to the gravity waves in the equilibrium range is concentrated, will depend on further direct measurements of the multifractal spectrum for one-dimensional sections through the interacting wind fields over water surface.

To conclude we point out that:

1) The present paper seems to support the equilibrium range model of Phillips (1985), in which the predicted mechanics may be considered as the most essential one.

2) The fractal intermittency of the wind energy input and of wave breaking has much profound significance both to wind wave theory and to observations: the present paper suggests that the singularities of the wind wave equation caused by high Reynolds number is not negligible when we describe wind waves by wave spectra; and that the singularities in wind waves cannot be described wholly by classical spectral methods.

Acknowledgments. The authors thank Prof. O. M. Phillips and two reviewers (Prof. B. R. Kerman and anonymous) whose thoughtful comments and constructive suggestions, especially the detailed annotations of Prof. Kerman, resulted in major improvements to the initial version of this paper. This work was supported by the National Natural Science Foundation of China under Grant NSFC-48900036. The authors gratefully acknowledge this support.

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