A Numerical Model of Internal Tides with Application to the Australian North West Shelf

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(Manuscript received 20 May 1994, in final form 17 December 1994)

ABSTRACT

A nonlinear, primitive equation, finite-difference numerical model is applied to the problem of the generation, propagation, and dissipation of internal tides over a cross section of the continental slope and shelf topography of a region on the Australian North West Shelf. The model is forced through the specification of the offshore tidal elevation and as such the full tidal field is modeled for the M2 constituent. An energetic internal tide is produced in the model with results showing sensitivity to changes in both stratification and bathymetry. The ratio of the slope of the internal wave characteristics to the bathymetry is generally less than or close to one, producing subcritical and approximately critical conditions. Model results are compared to previously reported observations and show reasonable agreement in terms of wave structure, propagation direction, and regions of generation and energy dissipation.

The model shows a high degree of spatial variability in the amplitude and phase of internal wave currents and vertical displacements with motion tending to propagate along characteristic paths as beams of signal. However, dissipation prevents the beams from radiating large distances from the generation regions. The energy flux of the internal tide propagates both onshore and offshore and the magnitude of the flux is strongly dependent on the slope of the bathymetry with largest values occurring for steepest topography. The internal wave amplitude and hence energy flux is also found to be dependent on the magnitude of the vertical and horizontal mixing of momentum with maximum values achieved under conditions of no mixing.

1. Introduction

The action of a barotropic tide forcing stratified water to oscillate over continental slope topography is often seen to produce a long internal wave, known as a baroclinic or internal tide, that can propagate either onshore or offshore. Observations have been reported from various shelf and slope regions, for example, the Bay of Biscay (Pingree et al. 1986), around the U.K. shelf (Sherwin 1988), off the east coast of Canada (Petrie 1975), off the northwest African shelf (Huthnance and Baines 1982), and the Australian North West Shelf (NWS) (Holloway 1984, 1985). A comprehensive review of internal tide observations is given by Huthnance (1989).

Internal tides are characterized by large variations, over relatively small spatial and temporal scales, in amplitudes and phases of currents and vertical displacements. There is usually a 180° phase change in the currents over the water column depth and, with horizontal wavelengths of approximately 20–30 km around the shelf edge, phase changes rapidly with horizontal distance. Temporal variations in phase can also be caused by variations in stratification, e.g., Holloway (1994), or possibly by effects of longshelf geostrophic currents (Mooers 1975). A critical parameter in determining the existence of internal tides was suggested by Baines (1982) as the ratio α/c, where α is the slope of the seabed and

$$c = \left( \frac{\omega^2 - f^2}{N^2 - \omega^2} \right)^{1/2}$$

is the slope of the internal wave characteristics where ω is the wave frequency, f the Coriolis parameter, and N(ζ) is the buoyancy frequency, which is a function of depth (ζ). Baines suggests that generation is strongest when the ratio α/c is approximately 1—that is, when the bathymetry is at a critical slope. Under these conditions, currents are predicted to be intensified near the seabed, and such intensification has been reported from observations, for example Schott (1977) and Holloway (1985). Internal tides are also frequently observed to become strongly nonlinear as they propagate and shoal. They can be strongly distorted from a sinusoidal form and at times become unstable to form undular bores or solitons (e.g., Sandstrom and Elliot 1984; Holloway 1987; New and Pingree 1990).

The first attempts to model the generation and propagation of internal tides used analytical models with idealized topography and stratification, for example,
Rattray et al. (1969), Prinsenberg and Rattray (1975), Sandstrom (1976), and Baines (1973, 1974, 1982). While these early models made valuable contributions to our understanding of the generation process, they are limited as predictive tools and do not account for the very important effects of energy dissipation, particularly when there are very large current shears near critical bottom slopes.

Chuang and Wang (1981) used a linear finite-difference model of internal tide generation that is two-dimensional in the horizontal/vertical plane with arbitrary topography and stratification. Their model does not properly account for dissipation but includes a decay length scale to help produce numerical stability. They used the model to investigate the effect of a front on the generation and propagation of internal tides. Sherwin and Taylor (1990) have also used the Chuang and Wang model to investigate the generation of internal tides on the Rockall trough, a region with internal tides that are largely linear in their properties. They obtained a reasonable agreement between the model and observations.

Craig (1987a) published a model of internal tide generation over coastal topography utilizing a solution technique in characteristic coordinates. The model is linear, inviscid, and two-dimensional in the horizontal/vertical plane. Using a transformation to characteristic coordinates, a modal solution is sought that is matched over the model domain. For a constant buoyancy frequency (uniform stratification) an analytical solution is obtained and for variable buoyancy frequency a numerical integration is necessary (Craig 1987b). The model was applied to the NWS (Craig 1988) and produced a reasonable agreement with observed modal properties of the internal tide. However, the model could not explain some of the observed large amplitude internal tides, and of course could not tackle the problems of nonlinearity or dissipation that are so evident in the observations. However, Craig (1991) has since developed techniques for incorporating dissipation effects into this style of model.

Some studies have concentrated on the modeling of the nonlinear character of an internal tide. Pingree et al. (1983) used a simple two-layer model of internal tide generation over topography and applied it to the Celtic Sea shelf front. The model retained the nonlinear terms due to barotropic tidal advection and produced a distorted wave shape similar to that observed. Wilmont and Edwards (1987) developed a three-layer nonlinear numerical model and applied it to tidal flow over a sill in a fjord, and Serpette and Maze (1989) used a two-layer nonlinear model of internal tide generation over two-dimensional topography in the Bay of Biscay. Lamb (1994) used a two-dimensional nonlinear model to investigate the formation of bores and short-period waves from a tidally forced internal wave over a smooth bank.

In this study a numerical model of the generation, propagation, and dissipation of internal tides is applied to a slope and shelf region of the NWS. The model is fully three-dimensional but is applied neglecting long-shelf variations so that only the vertical and cross-shelf variability is investigated over a cross section. The model allows for arbitrary bathymetry and stratification and uses the Mellor–Yamada level 2.5 turbulence closure scheme to model vertical mixing.

2. The numerical model

The internal tide model is an application of the coastal ocean circulation model developed by Blumberg and Mellor (1987), also known as the Princeton Ocean Model. The model is fully three-dimensional, time stepping, solves the primitive equations, it is nonlinear with Boussinesq and hydrostatic assumptions, uses a σ-coordinate transformation where the z level is scaled according to the water depth, and has a free surface. The vertical mixing of momentum, heat, and salt are determined by a turbulence submodel known as the Mellor–Yamada level 2.5 turbulence closure scheme (see Mellor and Yamada 1974, 1982), although in this application mixing of heat and salt is excluded. With internal tides often characterized by intense currents close to the sea bed, it seems important to accurately model the vertical mixing of momentum and the strong current shear region near the seabed. The use of the turbulence closure model means that there are no adjustable constants to parameterize the vertical mixing of heat, salt, and momentum. The use of σ coordinates also allows for fine model resolution near the seabed. Horizontal mixing uses Smagorinsky diffusivity where the horizontal mixing coefficient depends on the grid size and horizontal shear as well as an arbitrary constant (see Mellor 1993). An equation of state uses potential temperature and salinity to determine density. The equations that are solved are not reproduced here but are given, along with a more extensive discussion of the features of the model, by Blumberg and Mellor (1987) and Mellor (1993).

In application to the NWS the model is applied to a particular cross section assuming that the shelf is uniform in the longshelf direction so that all longshelf gradient terms in the equations become zero. Real topography in the cross-shelf direction is used along with depth-dependent temperature and salinity profiles that determine the vertical density profile.

The model is forced through the specification of the tidal elevation at the offshore boundary and only a single harmonic, the M₄ tide with period of 12.42 h, is considered. For a tidal constituent of amplitude $\zeta_0$, phase lag $\phi$, and frequency $\omega$, the offshore boundary condition becomes

$$\zeta = \zeta_0 F \cos(\omega t + V_0 + U_0 - \phi), \quad (2)$$
where \( F \) and \( U_0 \) are the nodal modulation corrections for amplitude and phase, respectively, and \( V_0 \) is the astronomical argument for the time origin of 0000 1 January 1900, and these corrections are included to provide results consistent with tidal analysis procedures.

Values of velocity, temperature, and salinity must also be specified on the offshore open boundary, although a depth-averaged velocity is not specified as this is computed directly from the surface elevation gradient. The temperature and salinity are specified on the open boundary using upstream differencing, which is simply a solution of the linear equations for these variables. In addition, a flow relaxation scheme is used to bring the solution to the boundary value. Specification of the depth-dependent velocity is important as this must prevent offshore propagating internal gravity waves from being reflected back into the model domain. This is achieved through the use of a sponge layer that relaxes the velocity to zero as it approaches the offshore boundary. The flow relaxation scheme is discussed by Martinsen and Engedahl (1987) and summarized here. If the boundary value, at grid point \( i = 1 \), is \( V_B \) (zero in the case of the baroclinic velocity) and the unrelaxed value at grid point \( i \) in the flow relaxation zone is \( \dot{V}_i \), then the new value of the variable is given as

\[
V_i = \alpha_i V_B + (1 - \alpha_i) \dot{V}_i, \quad i = 1, 2, \ldots, \text{NL},
\]

where NL is the number of grid points in the flow relaxation zone and

\[
\alpha_i = \left( \frac{\text{NL} - i - 1}{\text{NL}} \right)^2.
\]

This condition was applied over \( \text{NL} = 5 \) grid points, while at the same time the grid spacing is stretched over the flow relaxation zone to help filter out short internal waves.

A grid spacing of 2.5 km in the offshore direction and 10 km in the longshore direction was used with only seven longshelf grid points. The time step is controlled by a CFL criterion and a value of 7.5 s for the external mode and 225 s for the internal mode was used, with a maximum water depth of 1600 m.

The use of the \( \sigma \)-coordinate transformation in the presence of steeply sloping bottom topography introduces an artificial cross-shelf baroclinic pressure gradient, which if geostrophically adjusted, will produce a longshore mean flow. This problem is discussed by Haney (1991) and Cheng (1992), who show that for given topography and stratification the pressure gradient is reduced by increasing the vertical resolution. In most runs, 31 vertical grid levels are used in the model, and experiments with no tidal forcing show that this provides sufficient vertical resolution to reduce the artificial pressure gradient to a level that produces a longshelf geostrophic flow of only a few millimeters per second. The vertical distribution of \( \sigma \) levels is uneven with finer resolution near the seabed to model the bottom boundary layer.

3. Application to the North West Shelf

The model is applied to a section of the NWS from which observations of internal tides have previously been reported. Figure 1 shows the locations of these moorings and the local bathymetry. Although it is clear that there are substantial longshelf topographic variations, in this model application, only cross-shelf topographic variations are considered. Holloway (1994) showed that the dominant direction of propagation of the internal tide was at an angle of 135° east of north. This angle is taken to define the cross-shelf bathymetric section, extending from the coastline to approximately 300 km offshore, as shown in Fig. 1. Two sections are modeled, primarily the more northerly section through the North Rankin (NR) location, but some runs are included for the southern section from Dampier through Goodwyn (G1) location to investigate the sensitivity of the model to varying topography. At approximately 1600-m depth the bathymetry flattens and a constant maximum depth of 1600 m is assumed.

Input temperature and salinity profiles are taken from the Levitus atlas (Levitus 1982) for the region 15°–20°S, 115°–120°E. The data are an average over the region and averages from 3-month periods in the top 250-m depth and annual means in deeper water. Southern Hemisphere summer (February to April) and winter (May to July) values are considered and the profiles plotted in Fig. 2 along with the corresponding buoyancy frequency. Significant seasonal changes in temperature, salinity, and buoyancy frequency are only seen in the upper 100–200 m.

4. The barotropic tide

The barotropic tide is defined by running the model with no stratification. The offshore \( M_2 \) tidal elevation is specified as 0.6 m with a phase of 293°. These values come from the global tidal atlas of Schwiderski (1980) for the region approximately 300 km offshore (also see Holloway 1983). However, the phase used is increased by 8° from that given by Schwiderski to give better agreement with the observed shelf values. The model is run for 4 days, with a ramp over the first day, and the last 2 days time series are harmonically analyzed using the tidal analysis package of Foreman (1978). The surface elevations and the two orthogonal current components from each cross shelf and vertical grid point are tidally analyzed giving \( M_2 \) elevations and phases and tidal ellipse semimajor and semiminor axis lengths and ellipse orientation and phases.

Figure 3 shows the cross-shelf distribution of tidal elevation and phase from the model along with observed constituent values from a number of locations,
as shown in Fig. 1 (from Holloway 1983). The cross-shelf amplification of tidal elevation is well reproduced by the model. There is little phase change across the shelf. In the upper part of the water column the barotropic tide is relatively uniform in amplitude and phase but changes significantly in the bottom few tens of meters where friction reduces the speed and the phase. The cross-shelf distribution of tidal ellipse properties is also plotted in Fig. 3 where values are taken from approximately middepth and represent the free-stream barotropic tidal currents. Observations from several locations (see Fig. 1) are also included where the data are from Holloway (1983, 1984). Around the shelf break there is good agreement in the semimajor axis lengths, although on the shelf the observed values are slightly higher than the model predictions, while the observed semiminor axis lengths are consistently smaller than the model values. The observed phase is consistently larger than the model by about 10° to 15°, although phase is very nearly constant across the shelf in both model and observations, and orientation of the ellipses is directly cross shelf in the model where observed values are generally about 30° anticlockwise from this plane. Most of the discrepancies between model and observations are probably a consequence of the two-dimensional approximation and the neglect of long-shore variations of tidal properties. However, the most important property in forcing the internal tide is the cross-isobath tidal velocity component, and this is accurately reproduced by the model.

Tidal ellipse properties are plotted against depth for two locations to indicate how the barotropic tide is predicted to vary with depth. Profiles from 100-m and 300-m depths are shown in Fig. 4. The profiles are relatively uniform through depth except for the lower 20–30 m where the velocity reduces rapidly, phase lag decreases by about 10°, but orientation is unchanged. These trends are consistent with analytical models of tidal boundary layer flow (e.g., Soulsby 1983). In deeper water the boundary layer is not as well resolved as in shallow water.

5. The baroclinic tide

When the model is run including vertical stratification, the currents generated contain both a barotropic
and a baroclinic component. As an aid to interpreting the model results, the baroclinic component is separated from the total signal and defined as the difference between the total time series and the barotropic time series (from the model run with no stratification). The resulting baroclinic time series of hourly values are then tidally analyzed in the same way as previously described for the barotropic tide.

In addition to the horizontal currents, the vertical velocity component (corrected from $\sigma$ coordinate to $z$ coordinate, see Blumberg and Mellor 1987) is also tidally analyzed and the corresponding values of vertical-displacement amplitude and phase are calculated. If the vertical-velocity amplitude is $w$ and the phase is $w_{\phi}$, then the vertical displacement amplitude and phases are $\zeta = w/\omega$ and $\zeta_{\phi} = w_{\phi} + 90^\circ$, respectively (i.e., $w = d\zeta/dt$).

Vertical sections, over the whole model domain (but excluding the flow relaxation zone), of semimajor axis length of the baroclinic currents and of vertical displacement amplitude are shown in Figs. 5 and 6. There is clearly a significant internal tide with strongest motion centered around the top of the continental slope and outer shelf. Little motion appears to propagate across the shelf or reach the coast and similarly there is only a weak signal offshore of the 500-m isobath. The vertical displacements show a series of cell-like structures over the upper slope with maxima in amplitude occurring near the seabed around 350-m depth. Figure 7 shows the internal tide characteristic paths.
placement amplitudes with maxima following the ray paths. This is particularly evident when the signals emanating from the seabed at around 350-m depth and moving both onshore and offshore.

Some comparisons are made between observations and the model output. Vertical profiles of observed M\textsubscript{2} vertical displacement amplitudes and phases and baroclinic tidal ellipse semimajor axis and phases from a number of mooring locations are presented in Figs. 8 and 9, respectively. The mooring locations are shown in Fig. 1 and the internal tide properties are previously reported values from Holloway (1994). The observations are all from the summer months but are from different years of observation and are from different durations of observation. The values are averages from data spanning a month or longer. The corresponding model profiles for locations at the same depth and distance along the cross section (bearing in mind that not all locations are exactly on the cross section) are also plotted in Figs. 8 and 9. The comparisons for G1 and G2, model locations 53 and 55, are from a model run using a bathymetric profile through the G1 location discussed in section 8. The model vertical displacements show considerable variability with depth and at some locations show an enhanced value near the seabed. This is also seen in the observations. The large vertical displacements in the lower part of the water column may also be influenced by a barotropic component

\[ \frac{u}{\omega} \frac{dh}{dx} . \]

At some locations, most noticeably M7 and NR, the observed displacements are about three times the amplitude of the modeled values. The agreement between the observed and modeled baroclinic currents is reasonable with the semimajor axis lengths of similar magnitude at each location and the large phase variations with depth are seen in both model and observations. The model shows intensified currents near the seabed at all locations but, while this is also seen to some extent in the observations, the observations are not detailed enough to show the strong variations in amplitude and phase near the seabed.

The baroclinic tidal properties are further considered by looking in more detail in the region around the upper slope and outer shelf. Sections of baroclinic tidal ellipse properties, semimajor axis length and phase, are plotted in Fig. 10 over the region 100 to 250 km from shore and to 500-m depth with corresponding sections of vertical displacement amplitude and phase given in Fig. 11. Very rapid variations of amplitude and phase with both horizontal position and depth are shown. Currents are intensified near the seabed over the whole region and are strongest near the seabed in about 120-m water depth but with a second maximum around 300–350 m. At these locations the characteristic slopes are nearly critical (Fig. 7), and it appears that these are regions
of generation of internal waves. This relationship was also observed in the modeling work of Craig (1988). The vertical displacement amplitudes also show regions of large elevation near the seabed for three levels at 100, 150, and 350 m. From these regions, and particularly from 350-m depth, the maxima in vertical displacement follow the internal wave characteristic paths emanating from the seabed and reflecting off the sea surface and seabed while moving both onshore and offshore. This pattern is also seen in the currents (Fig. 10), particularly in the signal moving offshore from 350-m depth. The results show that the internal waves are to a large extent propagating along characteristic paths, although the signal is significantly dissipated as it propagates.

Inshore of 120 km there is very little internal wave motion. The waves propagating onshore are dissipated by the time they have propagated onto the shelf. This is consistent with the observations of Holloway (1991), who found very strong dissipation of the internal tide between moorings at depths of 123 to 70 m.

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**Fig. 4.** Vertical profiles of modeled $M_2$ barotropic tidal ellipse properties at two locations. Model run is for North Rankin bathymetry and summer stratification. Values are semimajor and semiminor axis lengths, orientation, and phase. Phases are in local time, 8 h ahead of GMT, and orientation is in degrees anticlockwise from the onshore direction—that is, 135° east of north.

**Fig. 5.** Cross section of modeled $M_2$ baroclinic current semimajor axis length over the whole model domain. Model run is for North Rankin bathymetry and summer stratification. Contour interval is 2 cm s$^{-1}$.

**Fig. 6.** Cross section of modeled $M_2$ vertical displacement amplitude over the whole model domain. Model run is for North Rankin bathymetry and summer stratification. Contour interval is 1 m.
(M7 and M5 in Fig. 1). This also highlights the importance of modeling the dissipation, as with no dissipation the model would unrealistically predict very large amplitude internal waves on the shelf.

6. Internal wave energetics

In this section calculations are made of the energy flux of the modeled internal tide. The energy flux vector is the transport of energy past a point in space and, averaged over a tidal period \( T = 2\pi / \omega \), is given as (LeBlond and Mysak 1978)

\[
F = \int_0^T \mathbf{u}(z, t)p(z, t)\, dt,
\]

where \( \mathbf{u}(z, t) = (u, v) \) is the horizontal velocity vector, \( p(z, t) \) is the fluctuating component of pressure, \( z \) is the depth coordinate measured positive upward, and \( t \) is time. This is the energy flux per unit meter of wave crest and has SI units of joules per square meter. Alternatively, the average flux per unit time \( (F/T) \) has SI units of watts per square meter. For sinusoidal motion in time \( e^{-i \omega t} \), the linearized equations of motion relate pressure and vertical velocity by

\[
p_0(z) = -i \frac{\rho_0}{\omega} \int_z^0 w_0(z)((N^2(z) - \omega^2))\, dz,
\]

where \( p_0 \) and \( w_0 \) are the amplitude functions for pressure and vertical velocity.

If the velocity and pressure are each considered to have a real and imaginary component \((u_r + iu_i\) and \(p_r + ip_i\), respectively), the cross-shelf energy flux component can be written from (5) as

\[
F_x = \frac{T}{2} (u_r p_i + u_i p_r).
\]

Further, from tidal analysis the horizontal and vertical velocity amplitudes and phases \([(U, g_u)\) and \( (W, g_w)\), respectively] can be found and are related to the real and imaginary components by

\[
U^2 = u_r^2 + u_i^2, \quad g_u = \tan^{-1}(u_i / u_r),
\]

\[
W^2 = w_r^2 + w_i^2, \quad g_w = \tan^{-1}(w_i / w_r).
\]

Then (7) gives, using (6), (8), and (9), the cross-shelf energy flux as

\[
F_x = \frac{\rho_0 T U}{2 \omega} \left( \cos(g_u) \int_z^0 W \sin(g_u)(N^2 - \omega^2)\, dz \right.
\]

\[
- \sin(g_u) \int_z^0 W \cos(g_u)(N^2 - \omega^2)\, dz \right),
\]

and this allows the energy flux to be computed from the results of the tidal analysis of the model output. The depth-integrated energy flux is defined as \( \int_{-h}^{0} F_x\, dz \).

The cross-shelf component of energy flux and depth-integrated energy flux as computed from the model are plotted in Fig. 12 over part of the model domain. The largest values of energy flux are confined to the regions close to the seabed and at locations with depths of 100, 150, and 350 m, previously identified as regions of near-critical bottom slope and of internal wave generation. The energy flux has components going both onshore and offshore. The depth-integrated energy flux
7. Winter stratification

The model is rerun using the winter (May–July) temperature and salinity profiles from the Levitus atlas, as shown in Fig. 2, as a means of assessing the sensitivity of the model to changes in vertical stratification. The profiles only vary from the summer values in the top 250 m. The winter temperature profile is close to homogeneous in the top 75 m and has a slightly weaker thermocline between 100 and 300 m than during sum-

![Graph showing vertical profiles of observed M_s vertical displacement amplitudes and phases from a series of mooring locations, as shown in Fig. 1, along with modeled values. Modeled values are from summer stratification and the North Rankin bathymetry for locations M5, M7, NR, and G3, and Goodwyn bathymetry for locations G1 and G2. Phase lag (g) is with respect to local time, 8 h ahead of GMT. Observed values are from Holloway (1994).](image)

shows that, over most of the region, the flow of energy is offshore with a peak value of around 50 W m$^{-1}$. At the very top of the slope, at around 100-m depth, there is a net onshore energy flux of approximately 20 W m$^{-1}$ and this is rapidly dissipated with very little energy propagating far onto the shelf. The zero crossings in the depth-integrated energy flux indicate generation regions with subcritical slopes in the topography. From these regions energy propagates both onshore and offshore as seen at 140 km and again at 200 km from shore. Between these two regions the offshore energy flux components from 140-km and 180-km regions appear to add to give the maximum value observed.

![Graph showing vertical profiles of observed M_s tidal current ellipse properties from a series of mooring locations, as shown in Fig. 1, along with modeled values. Modeled values are from summer stratification and the North Rankin bathymetry for locations M5, M7, NR, and G3, and Goodwyn bathymetry for locations G1 and G2. Semimajor (a) and semiminor (b) axis lengths, phase lag (g) with respect to local time, 8 h ahead of GMT, and orientation (θ) measured anticlockwise from the onshore velocity component (135° east of north) are given. Observed values are from Holloway (1994).](image)
m depth, very little energy propagates onshore or offshore from the upper parts of the slope in winter.

The results show that the vertical structure of the horizontal velocity profiles is significantly changed from the summer to winter stratification, with strong bottom intensification of currents seen in winter. The energy flux is also significantly reduced by the weaker winter stratification.

8. Goodwyn bathymetry

Further sensitivity tests are undertaken by running the model for the more southern bathymetric cross section through the G1 (Goodwyn) location, 31 km to the southwest (Fig. 1). This represents a relatively small change in cross section from the original run through North Rankin when compared to the variations in bathymetric cross sections for the region evident in Fig. 1. However, the upper part of the slope, deeper than about 200 m, is slightly steeper than through the North Rankin cross section.
placements of 15 m are found at a depth of about 300 m. Vertical displacement phase is relatively uniform with depth. The results show strong bottom intensification of the motion over the region of approximately critical slope. Where the bottom slope changes to subcritical at about 200 m, the bottom intensification disappears, but the signal follows the path of a characteristic onshore.

The energy flux, plotted in Fig. 16, is much larger than for the North Rankin bathymetry and results from the strong bottom intensification. From about the 200-m mark there is energy flux both onshore and offshore, but as shown in the depth-averaged values, the offshore flux is reinforced over the critical slope region reaching a peak of 425 W m⁻¹. This value is an order of magnitude larger than found for the North Rankin section. Moving farther offshore, the value of the energy flux slowly decreases indicating some dissipation in the deeper water, whereas the onshore flux is quickly dissipated on the shelf.

These results show a very strong dependence of the internal tide strength on the slope of the bathymetry.
An example of the model-computed vertical profiles of vertical eddy viscosity ($K_n$) are shown in Fig. 17 at a water depth of 100 m for runs including and excluding stratification. In the absence of stratification, $K_n$ has an approximate parabolic distribution with depth, reaching a maximum value of $\sim 20 \times 10^{-3}$ m$^2$ s$^{-1}$. This shape is influenced by the mixing length imposed in the turbulence closure scheme. The effect of stratification on $K_n$ is quite dramatic with $K_n$ nonzero only in the bottom boundary layer where the current shear is strongest. The maximum value at this water depth and time is $\sim 2 \times 10^{-3}$ m$^2$ s$^{-1}$. The effect of stratification is to suppress the turbulence, and the modeled internal waves do not generate sufficient shear to overcome this stable density profile. Here $K_n$ increases slightly in shallower water and decreases in deeper water and varies slightly with time over a tidal cycle. Some runs were also made including vertical mixing of heat and salt, and it was found that over only 4-day simulations that there was little change in the temper-

relative to the characteristic slope. In this Goodwyn cross section, there appears to be a large section of the slope at depths between 200 and 500 m that is contributing to the generation compared to an approximate point source of generation at around 300 m on the North Rankin section. In particular, the energy flux is seen to be strongly dependent on relatively small changes in the bathymetric slope.

9. The model representation of mixing

The vertical and horizontal mixing of momentum are included in the model runs through parameterizations that compute an appropriate mixing coefficient. These are the Mellor–Yamada turbulence closure scheme for vertical mixing and Smagorinsky horizontal diffusivity, as discussed in section 2. While the intent of these parameterizations is to give an accurate representation of the relevant physical processes, it is instructive to look at the effects of these processes on the internal wave properties.

Fig. 14. Cross section of modeled $M_z$ baroclinic current semimajor axis length and corresponding phase over the region between 80 and 230 km from the coast and to a depth of 500 m. Model run is for Goodwyn bathymetry and summer stratification. Contour interval is 2 cm s$^{-1}$ for velocity and 90° for phase.

Fig. 15. Cross section of modeled $M_z$ vertical displacement amplitude and corresponding phase over the region between 80 and 230 km from the coast and to a depth of 500 m. Model run is for Goodwyn bathymetry and summer stratification. Contour interval is 1 m for amplitude and 90° for phase.
vertical mixing of momentum. Without vertical mixing there is a significant increase in the magnitude of the energy flux in both the onshore and offshore directions but particularly in the energy propagating onto the shelf.

Further, the model was run with horizontal eddy viscosity \( \nu_h \) set to zero as well as having zero vertical mixing. In this case, dissipation is achieved primarily through bottom friction, although some shear dispersion can also occur. Example model profiles of vertical displacements and baroclinic tidal currents are included in Fig. 18. The displacement amplitudes are further increased above the solutions that included and excluded vertical mixing. There is also an increase in the phase lag of the vertical displacements. The currents through much of the water column, and particularly near the seabed, are more than double the values obtained when vertical and horizontal mixing are included, and this creates a region of very high current shear close to the seabed. The removal of mixing can also be seen in Fig. 19 to significantly increase the depth-integrated energy flux where values of onshore propagating energy flux on the shelf are generally more than double those for the case with no vertical mixing. Results are clearly sensitive to specifications of vertical and horizontal mixing.

Fig. 16. Energy flux (100 W m\(^{-2}\)) and depth-integrated energy flux (W m\(^{-1}\)) over the region between 80 and 230 km from the coast and to a depth of 500 m. Model run is for Goodwyn bathymetry and summer stratification. Positive values (solid contours) are onshore and negative values (dashed contour) are offshore. Contour interval is 0.5 W m\(^{-2}\).

Fig. 17. Vertical profiles of vertical eddy viscosity at a location 100 m deep from the North Rankin section for (a) homogeneous water and (b) with summer stratification.
10. Possible three-dimensional effects

The bathymetric chart (Fig. 1) and the difference between the cross-shelf profiles used in the model runs through North Rankin and Goodwyn, show that there are alongshelf topographic gradients in the region being modeled. These gradients have been neglected in the model, but it is useful to have an estimate of how significant the alongshelf topographic gradients are in generating the internal tide.

The amplitude of the forcing function for the internal tide has been shown by Baines (1982) to be

$$F = \frac{\rho_0 Q N^2 z}{\omega h^2} \frac{dh}{dx}, \quad \text{(11)}$$

where

$$Q = \int_{-h}^{0} u(z) \, dz \quad \text{(12)}$$

is the cross-shelf volume flux due to the barotropic tide $u(z)$, $\omega$ is wave frequency, $h$ water depth, $\rho_0$ a reference density, and $dh/dx$ the cross-shelf slope of the seabed. Assuming a constant buoyancy frequency $N(z) = N_0$ and depth-uniform barotropic tidal current of amplitude $u(z) = u_0$, the depth-integrated forcing function is then, from (11) and (12),

$$\bar{F} = \frac{\rho_0 N_0^2 u_0 h}{2\omega} \frac{dh}{dx}. \quad \text{(13)}$$
Equation (13) is evaluated for the North Rankin and Goodwyn sections. At North Rankin \((h = 123 \text{ m})\), Holloway (1994) gives \(M_2\) tidal ellipse properties of semimajor axis \(0.17 \text{ m s}^{-1}\), semiminor axis \(0.028 \text{ m s}^{-1}\), and orientation 57°W of \(N\). The major axis is then only 12° off alignment with the model cross-shelf sections and so the semimajor axis is taken as the \(u_0\) value. Then \(Q = 20.9 \text{ m}^2 \text{ s}^{-1}\) and this is assumed to be a representative value over the upper slope and equal for both the North Rankin and Goodwyn sections. Average cross-shelf slopes over the region between 200 and 500 m deep are \(9.2 \times 10^{-3}\) for North Rankin and \(19 \times 10^{-3}\) for the Goodwyn section. With \(N_0 = 0.007 \text{ s}^{-1}\) (Fig. 2), and \(\rho_0 = 1025 \text{ kg m}^{-3}\), Eq. (13) gives \(\bar{F} = 34 \text{ N m}^{-2}\) for North Rankin and \(71 \text{ N m}^{-2}\) for Goodwyn—that is, the forcing is about double over the Goodwyn section, consistent with the larger internal tide modeled on this section.

Similarly, an estimate can be made of \(\bar{F}\) in the longshelf \((or y)\) direction. By comparing the differing slopes at depths between 200 and 500 m from the North Rankin and Goodwyn sections (separated by a distance of 31 km) the longshelf topographic slope is estimated as \(dh/dy = 5 \times 10^{-3}\). Using the semiminor axis length of the \(M_2\) tidal ellipse from North Rankin as the appropriate longshelf barotropic tidal velocity gives the longshelf volume flux as \(Q = 3.4 \text{ m}^2 \text{ s}^{-1}\). Then (13) gives the longshelf internal tide forcing as \(\bar{F} = 3 \text{ N m}^{-2}\).

This simple calculation estimates the longshelf forcing as approximately 10% of the cross-shelf forcing on the North Rankin section (although this could easily vary by a factor of 2 or 3). Hence the longshelf forcing would seem secondary to the cross-shelf forcing, but could still be of significance. Further longshelf forcing would result from the fact that there are longshelf variations in amplitude and phase of the generated internal tide.

11. Summary and discussion

A finite-difference, nonlinear, primitive equation model has been applied to a continental shelf/slope region on the NWS to model the \(M_2\) tidal field over a two-dimensional cross section. In particular, emphasis has been placed on resolving the internal tide that results when the water is vertically stratified. The model was run for different stratifications and bathym-
etrical cross sections. Results showed an energetic internal tide generated at water depths of around 300–400 m and possibly further generation at around 120-m depth. The bathymetry in both cross sections is subcritical with some regions of critical and marginally supercritical slopes, when compared to the slope of the internal wave characteristics. The magnitude of the resulting internal tide is strongly dependent on both the strength of the density stratification as well as the slope of the bathymetry. Changing the bathymetry from largely subcritical (North Rankin section) to large sections of critical slope (Goodwyn section) caused an order of magnitude increase in the energy flux and particularly the offshore energy flux. This is consistent with the findings of Craig (1987a), who showed that energy flux increases rapidly for subcritical slopes as the slope increases toward the critical value, whereas the rate of increase slows dramatically for supercritical slopes. Vertical and horizontal mixing have been shown to reduce the strength of the internal tide.

The energy flux of the internal tide in comparison to the energy flux of the barotropic tide is important in understanding the global dissipation of tidal energy. Baines (1982) made a series of calculations using his analytical model to estimate that approximately 0.3% of the global dissipation of tidal energy could be accounted for by transfer of energy from barotropic to internal tides. He also showed that the NWS is one of the most significant global regions for internal tides. His estimated summer average energy flux along the NWS of 174 W m$^{-1}$ is consistent with values found in this paper of approximately 50 W m$^{-1}$ for the North Rankin section and 400 W m$^{-1}$ for the Goodwyn section. However, this variability over a short distance demonstrates the uncertainty in extrapolating results to global estimates. The model used in this paper produced a barotropic energy flux of approximately 4000 W m$^{-1}$ so that the internal tide represents a dissipation of between 1% to 10%, much higher than the global average of 0.3% estimated by Baines.

The model produces a number of features in the internal tide that are consistent with observations from the region. Observations show that the internal tides are predominantly first mode and propagate onshore at the top of the slope and shelf and that little energy propagates far onto the shelf. The currents tend to be intensified near the seabed particularly near the generation regions. However, there is a tendency for the model to underestimate the amplitude of the internal tide, and hence the energy flux, around the top of the slope where 10-m amplitude waves are reported from the observations. This was also found by Craig (1988) using an analytical model. Unfortunately, there are no observations from deeper water to compare to the model predictions. It is likely that the discrepancy between model and observed amplitudes from the upper slope is, at least in part, due to three-dimensional effects where forcing of the internal tide occurs along the slope due to a component of the barotropic tidal current flowing over longshelf topographic gradients. Also, the model was run using average summer stratification, which may be significantly different from the times at which observations were made.

Results from the North Rankin summer run, forced by the $M_2$ tidal elevation, are analysed for the $M_4$ tidal constituent. This has a period (6.12 h) half that of the $M_2$ constituent and results from the nonlinear advection terms ($\partial u/\partial x$) in the momentum equations. The strength of the $M_4$ internal tide currents (Fig. 20) then provides a measure of the importance of the nonlinear terms in the model. It can be seen that the maximum $M_4$ currents (about 2 cm s$^{-1}$) occur near the seabed at about 100-m depth. This is about 25% of the magnitude of the $M_2$ values. Significant $M_4$ currents are seen close to the seabed across most of the upper slope but particularly near regions of generation. The maximum in the $M_4$ depth-integrated energy flux is 2 W m$^{-1}$ and occurs near the generation region at 100-m water depth. This is approximately 12% of the $M_2$ energy flux.

Earlier models (e.g., Rattray et al. 1969) showed the internal tide propagating beams of energy away from the generation region. However, such models did not incorporate dissipation. In the current study there is some evidence of internal tide beams. The current and vertical displacement plots (Figs. 10 and 11) are consistent with a beam propagating away from around 300–400 m depth, shown by maxima in the semi-major axis and vertical displacement amplitudes. These beams move both onshore and offshore, but are broad, or smooth, presumably resulting from the effects of diffusion of momentum and from generation over a finite section of slope rather than from a point. The beams only persist for short distances before being completely dissipated.

The model shows a very high degree of spatial variability in amplitude and phase of both currents and vertical displacements. Phase propagates both horizontally and vertically, and varies from regions of onshore to regions of offshore propagation. This complexity highlights the need for fine-resolution measurements in the ocean in order to properly detail the internal tide and also the value of using numerical models as an aid to interpreting ocean observations.

Acknowledgments. This modeling work has been based on the Princeton Ocean Model, most generously provided by George Mellor. I thank Peter Craig and Toby Sherwin for many helpful discussions on the work.

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