Asymptotic Stability of the Viscous–Plastic Sea Ice Rheology

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ABSTRACT

The stability of the viscous–plastic rheology used extensively to model the dynamics of large-scale sea ice covers is studied. Using energy methods, it is shown that on a bounded domain, the viscous–plastic rheology is asymptotically stable when ocean drag terms are included.

1. Introduction

The viscous–plastic sea ice model introduced by Hibler (1977) forms the basis of a large fraction of computer codes that aim to model the dynamic behavior of the ice cover in the polar regions; see, for example, Walsh et al. (1985) and Stössel et al. (1990). The implementation of this model as described by Hibler (1979) requires the inclusion of artificial diffusive terms in order to maintain stability during long simulation runs. There is no physical justification for the inclusion of these diffusive terms, and as a consequence, questions naturally arise as to what the cause is of the instabilities on long timescales. There is some evidence to believe that the unstable behavior on long timescales is an artifact of the numerical implementation rather than an inherent property of the viscous–plastic rheology. Namely, Ip et al. (1991) did not appear to require artificial diffusive terms to maintain stability in their simulation runs. The numerical implementation of Ip et al. (1991) is fundamentally different from that of Hibler (1979).

Recently Gray and Killworth (1995) presented a linear stability analysis of the evolution equations for the area fraction, the ice thickness, and the momentum equations. Assuming a viscous–plastic rheology, they showed that an ice cover, at rest and with uniform thickness and area fraction, is unstable to infinitessimal perturbations, which give rise to a divergent flow field. Gray and Killworth (1995) argue that this instability may be the same instability as that which necessitates the inclusion of the diffusive terms in the viscous–plastic rheology as implemented by Hibler (1979). However, the argument of Gray and Killworth is not conclusive. Namely, the analysis of Gray and Killworth concerns the stability of the state of equilibrium of an ice cover. While the question of existence of an equilibrium configuration of an ice cover is important (presumably a physically realistic rheology should allow the existence of a static equilibrium configuration), absence of such a stable equilibrium configuration does not necessarily imply that instabilities occur in dynamic simulations. If a model is dynamically unstable, that is, if unbounded growth of the ice thickness or the velocity field occurs, an energy input is required which feeds the instabilities. If one can show that the total energy of the system is bounded, and decreasing with time, then there is no mechanism to sustain unbounded growth of either the velocity or the ice thickness. Thus, in order to determine whether or not the viscous–plastic rheology is an admissible rheology (in the sense that the energy remains bounded), one has to study the time evolution of the total energy of the ice–water system. A study of the stability of an equilibrium configuration is not sufficient.

Using energy methods, it will be shown in this paper that the viscous–plastic rheology is, in fact, asymptotically stable. This result still does not resolves the question as to why artificial diffusive terms are required to maintain stability in long simulation runs of an ice model based on the viscous–plastic rheology. As pointed out above, it is likely that observed instabilities should be attributed to the numerical implementation of the ice model since the model is not inherently unstable.

2. Stability analysis

It is assumed that the ice cover moves only in the horizontal plane (x_1, x_2) so that the area fraction A (defined to be the fractional ice cover per unit area), the

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mean ice thickness h, and the velocity components v_1, v_2 only depend on the coordinates x_i, $i = 1, 2$, and time t. The evolution equations to be studied are then as follows (cf. Hibler 1979). For the area fraction and the mean ice thickness obey the evolution equations

$$\frac{DA}{Dt} + A\frac{\partial v_i}{\partial x_i} = 0, \tag{1}$$

$$\frac{Dh}{Dt} + h\frac{\partial v_i}{\partial x_i} = 0, \tag{2}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i}$$

denotes the Lagrangian time derivative. The momentum equations are given by

$$\rho h\frac{Dv_i}{Dt} = \tau_i + \frac{\partial N_{ij}}{\partial x_j}, \tag{3}$$

in which ρ is the ice density, τ_i denote external stresses (wind stress or ocean drag), and N_{ij} denotes the stress tensor. The summation convention is used throughout this paper. Since this paper is primarily concerned with the stability of mechanical deformations, thermodynamic effects, Coriolis force, and the effects of sea surface tilt are neglected.

The kinetic energy per unit area is equal to $\frac{1}{2}\rho h v_i^2$. In order to calculate the time rate of change of the kinetic energy, Eq. (2) is multiplied by $\frac{1}{2}\rho v_i^2$ and Eq. (3) by v_i. Adding the resulting equations yields

$$\frac{1}{2}\rho\frac{Dhv_i^2}{Dt} = v_i\tau_i + v_i\frac{\partial N_{ij}}{\partial x_j} - \frac{1}{2}\rho v_i^2 h\frac{\partial v_i}{\partial x_i}. \tag{4}$$

Assume now that the region Ω in which Eqs. (1)–(3) are valid is finite. Integrating Eq. (4) over this domain one finds, after rearranging some terms, that the time rate of change of the total energy $\mathcal{E} = \frac{1}{2}\rho\int_\Omega hv_i^2 d\mathbf{x}$ is given by

$$\frac{\partial\mathcal{E}}{\partial t} = -\frac{1}{2}\rho\int_\Omega\frac{\partial}{\partial x_j}(hv_i^2 v_j)d\mathbf{x} + \int_\Omega v_i\frac{\partial N_{ij}}{\partial x_j}d\mathbf{x} + \int_\Omega \tau_i v_i d\mathbf{x}. \tag{5}$$

Using the divergence theorem, the first integral on the right-hand side in (5) is equal to

$$\int_\Omega\frac{\partial}{\partial x_j}(hv_i^2 v_j)d\mathbf{x} = \int_{\partial\Omega} hv_i^2 v_j n_j ds \tag{6}$$

in which $\partial\Omega$ denotes the boundary of Ω and $\mathbf{n} = (n_1, n_2)$ denotes the outward unit normal to $\partial\Omega$. If the domain Ω is surrounded by solid boundaries on which the velocity vanishes, then $v_{i|\partial\Omega} = 0$ and hence the right-hand side of (6) is identically zero.

Using Green's theorem, the second integral in (5) can be written as

$$\int_\Omega v_i\frac{\partial N_{ij}}{\partial x_j}d\mathbf{x} = \int_{\partial\Omega} v_i N_{ij} n_j ds - \int_\Omega D_{ij}N_{ij}d\mathbf{x}, \tag{7}$$

in which

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

No-slip boundary conditions eliminate the boundary integral in (7) and hence Eq. (5) becomes

$$\frac{\partial\mathcal{E}}{\partial t} = -\int_\Omega D_{ij}N_{ij}d\mathbf{x} + \int_\Omega \tau_i v_i d\mathbf{x}. \tag{8}$$

Clearly, the right-hand side of Eq. (8) determines whether or not the energy $\mathcal{E}$ will remain bounded. Of crucial importance in this respect is the relation between the stress and the strain rate. For the viscous–plastic model the relation between the stress N_{ij} and the strain rate D_{ij} is given by

$$N_{ij} = 2\mu D_{ij} + (\xi - \mu)D_{kk}\delta_{ij} - \frac{1}{2}P\delta_{ij}, \tag{9}$$

in which $\xi = P/2\Delta$ and $\mu = \xi/e^2$ are functions of the principal invariants

$$\eta = D_{ii}, \quad \gamma^2 = D_{12}^2 + \frac{1}{4}(D_{11} - D_{22})^2,$$

and

$$\Delta = (\eta^2 + 4\gamma^2/e^2)^{1/2}.$$

In the above equations e denotes the ratio of the principle axes of the elliptic yield curve and P is a pressure term postulated to depend on the ice thickness and the area fraction. With the stress–strain rate relation as in (9) it is not hard to show that

$$N_{ij}D_{ij} = \frac{1}{2}PD_{ii}\left[\left(1 + \frac{4\gamma^2}{e^2D_{ii}^2}\right)^{1/2}\operatorname{sgn}(D_{ii}) - 1\right], \tag{10}$$

in which $\operatorname{sgn}(D_{ii}) = D_{ii}/|D_{ii}|$. Substituting (10) into the integral in Eq. (8) yields finally

$$\frac{\partial\mathcal{E}}{\partial t} = -\int_\Omega P\left[\left(1 + \frac{4\gamma^2}{e^2D_{ii}^2}\right)^{1/2}\operatorname{sgn}(D_{ii}) - 1\right]D_{ii}d\mathbf{x} + \int_\Omega \tau_i v_i d\mathbf{x}. \tag{11}$$

Neglecting external forcing for the moment (i.e., taking $\tau_i = 0$ in the above equation), it is readily shown that the viscous–plastic model is asymptotically stable provided the pressure P is strictly positive. Namely, the term under the square root in the integrand in (11) is strictly larger than unity. Hence, during convergence (i.e., $D_{ii} < 0$) the term in large square brackets is negative so that the integrand is positive and the rate of change of energy is strictly negative. During divergence ($D_{ii} > 0$) the term in square brackets in (11) is positive and hence the integrand is positive, which implies that, once again, the rate of change of energy is negative. The pressure term is postulated to be of the form $P(h, A) = hF(A)$ in which $F(A)$ is often taken to be an exponential function of A. With this form of the pressure it is easy to see that the energy $\mathcal{E}$ is also bounded below. Namely, $\mathcal{E} \to 0$ only when h or v_i tend to zero. But when either h or v_i tend to zero, the right-hand side in (11) also approaches zero so that the rate of change of the energy with time also vanishes. Hence, $\mathcal{E}$ is bounded below by zero.

For very small strain rates the parameter Δ can become arbitrarily large. In order to avoid the occurance of arbitrarily large viscosities, the parameter Δ is assumed to be bounded below by some constant value $\Delta_{\min}$ (see Hibler 1979). For sufficiently small strain rates this procedure implies that the viscosity is rate independent. It is not hard to show that in this regime

$$N_{ij}D_{ij} = \frac{P}{2\Delta_{\min}}\left[(D_{11} + D_{22})^2 + \frac{1}{e^2}(D_{11} - D_{22})^2 + \frac{4}{e^2}D_{12}^2 - \Delta_{\min}D_{ii}\right]. \quad (12)$$

In convergence $D_{ii} < 0$ and hence (12) shows that $N_{ij}D_{ij} > 0$, implying stability since the rate of change of $\mathcal{E}$ is negative. On the other hand, in a diverging flow field ($D_{ii} > 0$) the right-hand side of Eq. (12) may be negative when D_{ij} is sufficiently small (note that the term containing $\Delta_{\min}$ inside the square brackets is linear in D_{ij} while the remaining terms are quadratic). This implies that sufficiently small divergence rates are unstable in the sense that the energy of the system increases. The energy can, however, not increase indefinitely when the domain of interest is bounded. Namely, an increase in the energy during divergence must lead to an increase in the velocity since the thickness decreases [as follows from Eq. (2)]. The velocity cannot, however, increase uniformly owing to the boundary conditions on $\partial\Omega$ and hence the small divergence rate cannot be sustained. It should be pointed out that while the energy cannot increase indefinitely due to the boundedness of the domain, it is somewhat disconcerting that the rheology requires the domain to be bounded for stability. In general, one would prefer the stability of the system to be independent of whether or not the domain of interest is bounded.

Next, the effect of external forcing on the rate of change of the energy is investigated. Assume that external forcing consists of a "positive" forcing term due to the wind stress and a "negative" forcing term due to ocean drag. Hence, the ocean drag depending linearly on the relative velocity of the ice cover and the ocean gives $\tau_i = \tau_i^w - C_d v_i$ in which τ_i^w denotes the wind stress and C_d is a drag coefficient. Equation (11) becomes

$$\frac{\partial \mathcal{E}}{\partial t} = -\int_\Omega \hat{P} D_{ii} d\mathbf{x} + \int_\Omega (\tau_i^w - C_d v_i) v_i d\mathbf{x} \quad (13)$$

in which

$$\hat{P} = P\left[\left(1 + \frac{4\gamma^2}{e^2 D_{ii}^2}\right)^{1/2} \operatorname{sgn}(D_{ii}) - 1\right].$$

Note that if the angle between the wind stress vector and the velocity vector is such that the inner product of these two vectors is positive, then energy is supplied by the action of the wind stress. Using the identity $\hat{P}\partial v_i/\partial x_i = \partial(\hat{P} v_i)/\partial x_i - v_i \partial\hat{P}/\partial x_i$ together with the divergence theorem and the assumption that the velocity vanishes on the boundary of Ω, it is found that (13) can be expressed in the form

$$\frac{\partial \mathcal{E}}{\partial t} = \int_\Omega v_i\left[\tau_i^w - C_d v_i + \frac{\partial \hat{P}}{\partial x_i}\right] d\mathbf{x}. \quad (14)$$

This equation shows that the energy may increase as a result of the action of wind stresses when τ_i^w is sufficiently large and when $\tau_i^w v_i > 0$. However, eventually the ocean drag term becomes dominant when the velocity has increased sufficiently [the ocean drag term in (14) depends on the square of the velocity, while the other terms are only linearly dependent on the velocity]. Hence, there is no possibility for unbounded growth of the energy when ocean drag is included. This conclusion is, of course, not effected when the ocean drag is a quadratic function of the relative velocity. Note that when $P(h, A)$ is a monotonically increasing function of both h and A, the pressure gradient term in (14) is negative, both during convergent and divergent flows as follows from the definition of $\hat{P}$ and the evolution equations (1) and (2). While this implies that the gradient of $\hat{P}$ acts in a stabilizing manner, sufficiently large wind stresses may render the system unstable if ocean drag terms were not taken into account.

3. Short-time behavior of the viscous–plastic model

In the previous section it was shown that on a finite domain, the viscous–plastic rheology cannot lead to unbounded growth of the energy, which implies that this rheology is asymptotically stable. This result does not give any information concerning the short time behavior of a model in which the viscous–plastic rheology is adopted. To study the short time behavior, consider Eqs. (1)–(3) together with the rheology described in (9). Assume that there is no external forcing so that $\tau_i = 0$ in (1). Small perturbations relative to the state of rest of the ice cover will be considered. Any perturbation will lead to small strain rates, and it is assumed that these strain rates are sufficiently small for the model to be in the linear viscous regime mentioned earlier. Assume next that at some time $t = 0$ the ice cover is at rest and initial values of the thickness and area fraction $h_0(\mathbf{x})$, $A_0(\mathbf{x})$ are given. Furthermore, assume that for small times the evolution equations are well behaved so that the velocity, area fraction, and thickness can be expanded in terms of the time variable t. In particular, consider the expansion

$$v_i(\mathbf{x}, t) = t v_i^{(1)}(\mathbf{x}) + O(t^2),$$
$$A(\mathbf{x}, t) = A_0(\mathbf{x}) + O(t^2),$$
$$h(\mathbf{x}, t) = h_0(\mathbf{x}) + O(t^2). \qquad (15)$$

Substituting the above expansions into Eqs. (1)–(3) and equating the coefficients of like powers of t to zero, it is found that the only zeroth-order contribution comes from the momentum equations, giving

$$\rho h_0 v_i^{(1)}(\mathbf{x}) = -\frac{1}{2}\frac{\partial P_0}{\partial x_i}. \qquad (16)$$

In the above equation $P_0 = P(h_0, A_0)$. Now, if P is a monotonically increasing function of h and A, then Eq. (16) states that $v_i^{(1)}$ is positive in the direction of decreasing thickness and decreasing area fraction. This implies that even without any external forcing, an ice cover that is initially nonhomogeneous diffuses toward a homogeneous state. The diffusive property of the viscous–plastic rheology was earlier remarked upon by Smith (1983). The presence of an equilibrium pressure that leads to continued dilation is not in accordance with experimental observations, and it would seem appropriate to aim for constitutive laws that do not contain equilibrium pressure terms.

Equation (16) shows that if $\partial P_0/\partial x_i = 0$, then $v_i^{(1)} = 0$. In fact, it is not hard to show that if P_0 is constant, that is, $A_0(\mathbf{x}) = \text{const}$ and $h_0(\mathbf{x}) = \text{const}$, then $v_i(\mathbf{x}, t) \equiv 0$. Hence, the equilibrium pressure vanishes for a homogeneous ice cover and the ice cover remains at rest. Clearly, when $P_0 = \text{const}$, the system is in an equilibrium configuration in which the ice thickness and the area fraction are constant throughout the domain. At this point it is important to relate the present work to the stability analysis of Gray and Killworth (1995). As mentioned earlier, Gray and Killworth have shown that the equilibrium configuration of a homogeneous ice cover is not stable when the viscous–plastic rheology is applied. Hence, the configuration in which $P_0 = \text{const}$ is not a stable equilibrium configuration.

It is instructive to estimate the velocity scale, which may be induced by the pressure term. To do so consider the pressure function that is commonly used, namely, $P = P^* h \exp(-K(1 - A))$ with $P^* = 5 \times 10^3$ N m^{-2}. Now consider the case in which characteristic length and time scales of the problem under consideration are L and T so that a characteristic velocity scale is given by L/T. Hence, (15) together with (16) yields

$$\rho[h]\frac{L}{T^2} \sim \frac{[h]}{L} P^* e^{-K(1-A)}, \qquad (17)$$

in which $[h]$ denotes a typical variation of the thickness over the lengthscale L. Since $P^*/\rho \sim O(1)$, it follows from (17) that

$$\frac{L}{T} \sim e^{-K(1-A)/2}. \qquad (18)$$

Thus, for large area fractions (that is, $A \sim 1$), the right-hand side of (18) is of order unity, which implies that the pressure term by itself can lead to velocities of the order of 1 m s^{-1}. For small area fractions ($A \ll 1$) Eq. (18) yields $L/T \sim e^{-K/2}$. In many calculations the choice $K = 20$ is made, which yields a velocity scale $L/T \sim 5 \times 10^{-5}$ m s^{-1}. These estimates show that in a system that consists of a nonhomogeneous ice cover, large velocities may occur solely as the result of the action of the pressure force. However, the pressure force will lead to a diverging flow field, which in turn, will lead to a decrease in the area fraction. Hence, owing to the exponential dependence of the velocity scale on the area fraction [as given by (18)] the pressure-induced velocity will decrease rapidly.

4. Conclusions

Gray and Killworth (1995) showed recently that the viscous–plastic rheology does not possess a stable equilibrium configuration and they suggested that the viscous–plastic rheology is inherently unstable. In this paper it has been shown that the viscous–plastic rheology is, in fact, asymptotically stable. This means that unbounded growth in the ice thickness or the ice velocity is not possible. In addition, it is shown that the inclusion of external forcing such as wind stress does not lead to unbounded increase in the energy of the system when ocean drag terms have been included. These results suggest that the artificial diffusive terms, inserted to maintain stability in long simulation runs by Hibler (1979), are required to smooth unwanted numerical effects rather than instabilities inherent in the viscous–plastic rheology. While the observed instabilities can be (and have been) surpressed by using artificial diffusion terms, it would seem more constructive to use stable numerical schemes, see, for example, Ip et al. (1991).

It has been shown how an ice cover, initially at rest, will move under the action of the equilibrium pressure. Significant velocities may occur as a result of this equilibrium pressure. Since there is no experimental evidence to suggest that the equilibrium pressure is a genuine feature of the dynamic behavior of ice covers, it would seem appropriate to search for models that do not contain the equilibrium pressure term. In view of this, it is important to point out that the equilibrium pressure is inherent to any rheology constructed using the normal flow rule on a given yield curve (e.g., see Smith 1983). Hence, eliminating the equilibrium pressure is only possible when the normal-flow rule in conjunction with a yield curve is abandoned.

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