Comments on "Stability of the Viscous–Plastic Sea Ice Rheology"

JOHN K. DUKOWICZ

Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico

14 August 1995 and 12 June 1996

Recently, Gray and Killworth (1995, henceforth GK) appeared to show that the viscous–plastic sea ice rheology based on an elliptical yield curve is linearly unstable in divergent flow. They recommended modifying the yield curve to avoid this apparent instability. However, as shown below, energy considerations demonstrate that this rheology is necessarily dissipative and therefore always stabilizing.

For simplicity, following GK, we restrict the analysis to the one-dimensional case; a similar analysis holds in the two-dimensional case. The momentum equation considered by GK for the elliptical yield curve is

$$m^i\frac{dv}{dt} = -\alpha v + \frac{\partial}{\partial x}\left[\frac{P}{2}(1 + e^{-2})^{1/2}\frac{D}{|D|} - \frac{P}{2}\right], \quad (1)$$

where $m^i = r^i h \geq 0$ is the ice mass per unit area, r^i is the density of ice, $h \geq 0$ is the ice thickness, $\alpha \geq 0$ is a frictional drag coefficient, $P \geq 0$ is the ice strength, e is the eccentricity of the elliptical yield curve (typically, $e \sim 2$), v is the ice velocity, $D = \partial v/\partial x$ is the divergence, and $d/dt = \partial/\partial t + v(\partial/\partial x)$ is the total time derivative. The continuity equation, rewritten in terms of m^i, instead of h, and neglecting source terms, is

$$\frac{\partial m^i}{\partial t} + \frac{\partial m^i v}{\partial x} = 0. \quad (2)$$

Multiplying (1) by v, and combining with (2), we obtain an equation for the rate of change of the ice kinetic energy:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}m^i v^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}m^i v^2 v\right) = -\alpha v^2 + v\frac{\partial}{\partial x}\left[\frac{P}{2}(1 + e^{-2})^{1/2}\frac{D}{|D|} - \frac{P}{2}\right]. \quad (3)$$

We now integrate this equation over a fixed region of ocean sufficiently large to contain all the ice. The second term on the left produces only boundary terms proportional to vm^i, and the rheology term on the rhs may be integrated by parts such that it contributes boundary terms proportional to vP and a term from the interior. The region of integration is bounded either by land boundaries, where $v = 0$, or by ice-free ocean, where $h = 0$, and therefore $m^i = 0$ and $P = 0$, because P is proportional to h. Should the region of integration be bounded by the ice edge, the correct boundary condition is that the ice stress is zero. In any case, all boundary terms vanish, and the result is

$$\frac{\partial}{\partial t}\int \frac{1}{2}m^i v^2\, dx = -\int \alpha v^2\, dx - \int \frac{1}{2}P[(1 + e^{-2})^{1/2}|D| - D]\, dx \leq 0, \quad (4)$$

where we have moved the time derivative outside the integral because the region of integration is fixed. This result is sufficient to demonstrate the stability of the dynamics by Liapunov's direct method (Lefschetz 1957) or the energy method (Straughan 1992). Each of the terms on the rhs is negative definite; in particular, the rheology term is negative definite, and so the viscous–plastic rheology is always dissipative. The rheology term on the rhs of (1) will therefore always tend to drive the total kinetic energy (a positive norm of the solution, i.e., a Liapunov functional) to zero, *no matter what the velocity field v is.* That is, the rheology term will always drive the dynamics toward an equilibrium such that v and/or D are zero (absent forcing). This consequence implies that this term will always contribute to stability, rather than detract from it, in the sense of Liapunov. The conclusion, in contradiction to the impression given in GK, is that viscous–plastic rheology with an elliptic yield curve is always stabilizing.

The preceding, as in GK, is a continuum analysis. It demonstrates the usefulness of the dissipative nature of the viscous–plastic ice rheology. This property should be preserved as much as possible in the discretization. A discretization typically involves a timewise lineari-

Corresponding author address: Dr. John K. Dukowicz, Theoretical Division, Los Alamos National Laboratory, Group T-3, Mail Stop B216, Los Alamos, NM 87545.

zation of the equations. The above considerations imply that the eigenvalues associated with the linear operator derived from the rheology term would *all* be real and negative, provided care is taken in the discretization.

Acknowledgments. This work was performed with support from the DOE CHAMMP program.

REFERENCES

Gray, J. M. N. T., and P. D. Killworth, 1995: Stability of the viscous-plastic sea ice rheology. *J. Phys. Oceanogr.,* **25,** 971–978.

Lefschetz, S., 1957: *Differential Equations: Geometric Theory*. Interscience, 364 pp.

Straughan, B., 1992: *The Energy Method, Stability, and Nonlinear Convection.* Springer-Verlag, 242 pp.