The Primitive Equations in the Stochastic Theory of Adiabatic Stratified Turbulence

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ABSTRACT

The stochastic theory of compressible turbulent fluid transport recently developed by Dukowicz and Smith is applied to the ensemble-mean primitive equations (PEs) for adiabatic stratified flow. The theory predicts a generalized Gent–McWilliams form for the bolus velocity and a single symmetric positive-definite diffusivity tensor for along-isopycnal Fickian diffusion of layer thickness and tracer distributions. When the theory is applied to the active tracer potential vorticity it provides constraints on the form of the Reynolds correlation in the momentum equation, and the turbulence closure problem is reduced to the determination of one $2 \times 2$ symmetric diffusivity tensor and one scalar field related to the eddy kinetic energy. The role of the rotational eddy fluxes of thickness, tracers, and potential vorticity is investigated, and a key feature of the closure is that the mean PEs do not depend on the gauge field associated with the rotational component of thickness flux, thereby eliminating the need to parameterize it. The relationship between this closure and closure schemes proposed by others in the quasigeostrophic regime is discussed. It is shown that the eddy-induced transport velocity can be parameterized as diffusion of either thickness or potential vorticity, and the resulting closure schemes are equivalent in the quasigeostrophic regime. The implications of the theory for energy and enstrophy balances are also discussed.

1. Introduction

Mesoscale eddies play an important role in the mean momentum, mass, and potential vorticity balances in the ocean. By altering the mean circulation they affect the transport of heat and salt and, thereby, influence the thermohaline circulation and hence long-term variations in climate. However, eddy-resolving climate simulations with full three-dimensional ocean general circulation models (OGCMs) will not be computationally feasible for the foreseeable future. Even at eddy-resolving scales it is possible that subgrid-scale processes affect the resolved circulation and hence require modeling.

Recent parameterizations of eddy effects have received considerable attention because they appear to dramatically improve the fidelity of coarse-resolution level-coordinate OGCMs. The Gent–McWilliams parameterization of eddy-induced mean tracer transport (Gent and McWilliams 1990, hereafter GM) is fundamentally based on downgradient diffusion of thickness along isopycnal layers to mimic the effects of eddies generated through baroclinic instability. This type of parameterization has a theoretical basis in the stochastic theory of compressible-type flow, and in this paper we discuss the implications of this theory in turbulence closure schemes for the full set of primitive equations for adiabatic stratified flow.

The simplest approach to a turbulence closure is to start with the governing equations and perform a Reynolds decomposition of the prognostic variables into ensemble-mean and fluctuating components. In the standard approach the ensemble averages are taken at fixed points in space and time, but this is not an appropriate choice for ocean mesoscale turbulence. A key feature of flow in the interior ocean that distinguishes it from other types of three-dimensional turbulence is that it is stratified and adiabatic, meaning there is no mixing of water mass properties across isopycnal surfaces. Because of this, the most natural approach is to employ ensemble averages in isopycnal coordinates. Closure schemes based on ensemble means in level coordinates, such as the approaches based on the three-dimensional version of the transformed Eulerian mean equations (Andrews and McIntire 1976) are fundamentally unable to describe baroclinically unstable flow because the transport equations do not conserve mass between the ensemble-mean isopycnals in level coordinates.

The adiabatic closure scheme proposed here is based on the thickness-weighted ensemble-mean transport equations in isopycnal coordinates, as advocated by de Szoeke and Bennett (1993, hereafter DB), although we use the unweighted mean momentum equation to avoid introducing a nonlocal eddy thickness–pressure corre-
lation. The advantage of the thickness-weighted transport equations is that they are much simpler and conserve mass and tracers exactly. Furthermore, thickness-weighted mean tracer distributions appear naturally in the stochastic theory.

The fluid transport in three-dimensional adiabatic stratified flow is mathematically identical to two-dimensional compressible flow, where isopycnal layer thickness plays the role of density. The stochastic theory for turbulent transport in compressible flow has recently been developed by Dukowicz and Smith (1997, hereafter DS) and is applied here to mesoscale turbulence in the ocean. This theory assumes that the turbulence is random [i.e., it can be described by a probability density function (pdf)] and Markovian (meaning the pdf at time \( t + \Delta t \) only depends on its value at time \( t \) but not at earlier times \( t' < t \)). In the incompressible limit, this theory forms the microscopic basis for the standard semiempirical diffusion equation for a passive tracer and is the main theoretical justification for the fundamental assumption of downgradient or Fickian diffusion. In compressible flow (and hence in adiabatic stratified flow) it predicts the form of the bolus advection and compressible flow (and hence in adiabatic stratiﬁed is the main theoretical justiﬁcation for the fundamental semiempirical diffusion equation for a passive tracer and this theory forms the microscopic basis for the standard semiempirical diffusion equation for a passive tracer and is the main theoretical justification for the fundamental assumption of downgradient or Fickian diffusion.

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In section 2 we derive the ensemble-averaged adiabatic primitive equations (PEs). In section 3 various approaches to closure schemes in the quasigeostrophic limit are discussed. By redefining the mean velocity, the bolus component of the transport velocity can be expressed as turbulent mixing of either layer thickness or potential vorticity, and the two formulations are equivalent up to second or third order in the ageostrophic component of the flow. In section 4 the stochastic theory of adiabatic stratified turbulence is reviewed, based on the corresponding theory of two-dimensional compressible turbulent fluid transport, and it is applied to the full set of PEs. Section 5 discusses the stochastic theory in the quasigeostrophic and planetary geostrophic limits, and the parameterization recently proposed by Dukowicz and Greathatch (1999). In section 6 the transport equations for energy and entrophy are derived in the stochastic theory, and balances of energy and enstrophy in a statistical steady state are discussed. Section 7 is a summary and discussion.

2. The ensemble-mean adiabatic primitive equations

To understand the dynamics of turbulent flow in the adiabatic interior ocean, we follow GM and consider a simplified system with no boundary fluxes of mass or tracers and no instantaneous diapycnal mixing. This should be a good approximation for turbulence in the interior adiabatic ocean. Matching this description with diabatic turbulent flow in the surface and bottom boundary layers is an important problem that is beyond the scope of this paper. See DB for inclusion of diapycnal fluxes due to microstructure turbulence, Treguier et al. (1997) for a discussion of boundary conditions at the surface and bottom boundary layers, and Large et al. (1997) for a method for treating the boundary layers in an OGCM with the GM parameterization. We also ignore differences between in situ and potential density, so neutral surfaces are just constant density layers. For implementation of the closure schemes discussed here in comprehensive ocean models, the effect of the nonlinear equation of state for seawater can be incorporated by replacing isopycnal slopes with the slopes of local neutral surfaces computed from the neutral density gradients (McDougall 1987). The PEs for this simple adiabatic system are given in isopycnal coordinates \((\xi, \eta, \rho, t)\) by

\[
D_t \mathbf{u} + f \mathbf{k} \times \mathbf{u} + \nabla M = \partial_x h + \mathbf{u} \cdot \nabla M = 0 \quad (1)
\]

\[
D_t h = 0 \quad (2)
\]

\[
D_t \phi = 0, \quad (\phi = \tau, q) \quad (3)
\]

\[
\rho_\tau \partial_\tau M = g \eta \quad (4)
\]

\[
w = D_t z \quad (5)
\]

\[
D_t = \partial_t + \mathbf{u} \cdot \nabla, \quad (6)
\]

where \( \mathbf{u} \) is the horizontal velocity, \( w \) is the vertical velocity, \( \nabla = (\partial_x, \partial_y) \) is the horizontal gradient on a constant density layer, \( M = (p + \rho g z) / \rho_\tau \) is the Montgomery potential, and \( h = -\rho_\tau \partial_\tau z > 0 \) is a measure of layer thickness. Here \( \phi \) is either a tracer \( \tau \) or the potential vorticity \( q = (f + \zeta) h \) where \( \zeta = \mathbf{k} \times \nabla \mathbf{u} \) is the relative vorticity. In (1) we have made use of the identity \( \mathbf{u} \cdot \nabla \mathbf{u} = \nabla \mathbf{u} \cdot \mathbf{u} / 2 + \mathbf{k} \times \mathbf{u} \mathbf{z} \). The potential vorticity is an active tracer that obeys the same transport equation as passive tracers; however, the potential vorticity field will not in general obey the same no-flux boundary conditions as the passive tracers.

Performing a Reynolds decomposition of the mean and fluctuating components of the flow, ensemble-averages of (1)–(5) yield the PEs for the mean flow in terms of the mean variables and eddy statistical correlations. The equations are closed by parameterizing these correlations in terms of the mean prognostic variables subject to certain physical constraints. As stated in GM, parameterizations of the mass and tracer transport equations should maintain two essential properties of the instantaneous adiabatic flow in the mean equations: i) conservation of mass (or volume in incompressible or Boussinesq flow) between any two isopycnal layers and ii) conservation of tracers between layers,
with positive-definite diffusion of tracers along isopycnals. The GM parameterization, which satisfies these constraints, is given in level coordinates by

\[ \partial_t \langle u \rangle + \langle u \rangle \cdot \nabla \langle u \rangle + \nabla \cdot (\mu \nabla \langle \rho \rangle) / \rho = F(\langle u \rangle) \]  
(7)

\[ \nabla \cdot \langle u \rangle = \nabla \cdot U^s = 0 \]  
(8)

\[ \partial_t \langle \phi \rangle + (\langle u \rangle + U^s) \cdot \nabla \langle \phi \rangle = \nabla \cdot (\mu \mathbf{R}_i \cdot \nabla \langle \phi \rangle) \]  
(9)

\[ U^s = U^{GM}_c = \frac{\rho}{\langle h \rangle} \partial_z \kappa \nabla \langle z \rangle. \]  
(10)

where angle brackets denote the ensemble-mean variables; \( U^s \) is referred to as the “eddy-induced transport velocity,” or the “bolus” velocity (Gent et al. 1995); \( \langle u \rangle, \ U^s \) are the horizontal components of the full three-dimensional divergence-free velocities \( \langle u_x, u_y, u_z \rangle \); and \( \nabla \) is the horizontal gradient operator at constant \( z \). The thickness diffusion coefficient \( \kappa \) is usually taken to be constant. The rhs of (9) is a diffusion operator that mixes \( \langle \phi \rangle \) along isopycnals with diffusion coefficient \( \mu \), and \( \mathbf{R}_i \) is the small-slope version of the Redi diffusion tensor (Redi 1982). In (7) \( F \) is a viscous term, usually taken as Laplacian mixing of \( \langle u \rangle \) with different viscosities in the horizontal and vertical directions.

While this parameterization was designed only for the tracer transport equations (Gent et al. 1995), it involves an implied momentum equation that determines the mean velocity \( \langle u \rangle \), and in Bryan–Cox models this typically has the form of Eq. (7), although Gent and McWilliams (1996) have recently proposed a different form of the mean momentum equation based on the \( z \)-averaged equations and a parameterization of the generalized Eliassen–Palm fluxes. In isopycnal coordinates Eqs. (8)–(10) transform to

\[ \partial_t \langle h \rangle + \nabla \cdot \langle h \rangle (\langle u \rangle + U^s) = 0 \]  
(11)

\[ \partial_t \langle \phi \rangle + (\langle u \rangle + U^s) \cdot \nabla \langle \phi \rangle = \frac{1}{\langle h \rangle} \nabla \cdot \langle h \rangle \mu \nabla \langle \phi \rangle \]  
(12)

\[ U^s = U^{GM}_c = \rho \frac{\partial_z \kappa}{\langle h \rangle} \nabla \langle z \rangle. \]  
(13)

How does this parameterization arise from the ensemble-mean PEs? The answer depends on the type of averaging used, or equivalently, on how the mean variables are defined. The standard Reynolds decomposition involves averages at fixed depth in level coordinates \( (x, y, z, \tau) \), while GM arrived at their parameterization based on averaging at fixed density in isopycnal coordinates \( (\xi, \eta, \rho, \tau) \). We refer to these as “\( z \) averaging” and “\( \rho \) averaging,” respectively. De Szoeke and Bennett suggest that the best approach is thickness-weighted averaging (“\( h \) averaging”). The \( h \)-averaged transport equations are in conservative form, so they automatically satisfy the adiabatic constraints (i) and (ii). However, the other two types of averaging violate one or both of these constraints unless further approximations are made: the \( z \)-averaged transport equations do not conserve mass between mean isopycnals, and the \( \rho \)-averaged equations conserve mass but not tracers between isopycnals. The most serious problem is with the \( z \)-averaged statistics. To make the \( z \)-averaged transport equations conserve mass it is usually assumed that the turbulent mass flux lies along isopycnal surfaces and has a negligible normal component (Treguier et al. 1997; Gent and McWilliams 1996). This component is small in some parts of the ocean but can be large in regions of baroclinically unstable flow; that is, the turbulent mass flux has a large “diapycnal” component across the \( z \)-averaged isopycnal surfaces (Gille and Davis 1999). Because of this, \( z \)-averaged statistics are poorly suited for describing mesoscale turbulence. The \( \rho \)-averaged transport equations only conserve tracers if the eddy tracer-thickness correlation is assumed to be negligible, or at least constant in time. Thus \( h \) averaging of the transport equations is the most natural approach to an adiabatic closure scheme.

The mean variables for \( \rho \) and \( h \) averaging are denoted by \( \psi \) and \( \psi^* \), respectively, (here \( \psi = u, \tau, \) or \( q \)). With an ergodic hypothesis these ensemble averages may be evaluated either as time averages or horizontal spatial averages or both, but since the system has no instantaneous diapycnal mixing, the averaging should not include a vertical average over \( \rho \). The mean and fluctuating components of \( \psi \) and \( h \) satisfy

\[ h = \hat{h} + h^* \]  
(14)

\[ \psi = \hat{\psi} + \psi^* = \hat{\psi} + \psi^* \]  
(15)

\[ \hat{\psi} = \frac{\hat{\psi}}{\langle h \rangle} / \hat{h} \]  
(16)

\[ \psi^* = \frac{\psi^*}{\langle h \rangle} / \hat{h} \]  
(17)

\[ \hat{\psi} = \hat{\psi} = 0. \]  
(18)

The ensemble-mean adiabatic PEs, excluding the momentum equation, have the form

\[ \partial_t \hat{h} + \nabla \cdot \hat{h} \hat{u} = 0 \]  
(19)

\[ \hat{D}_t \hat{\psi} = \langle h \rangle \cdot \hat{h} \hat{\psi} \]  
(20)

\[ \rho \hat{\sigma} \hat{M} = g \hat{z} \]  
(21)

\[ \hat{v} = \hat{D}_t \hat{z} \]  
(22)

\[ \hat{D}_t = \partial_z + \hat{u} \cdot \nabla, \]  
(23)

where \( \hat{u} = \hat{u} + u^* \) and \( u^* \) is the thickness-weighted vertical velocity but instead is determined by (22). [For simplicity the tilde denoting isopycnal coordinates \( (\xi, \eta, \rho, \tau) \) has been dropped; henceforth \( (x, y, \rho, \tau) \) will denote isopycnal coordinates unless otherwise noted.] We have the freedom to associate the large-scale velocity field with any type of ensemble-mean velocity (e.g., \( z \), \( \rho \), or \( h \) averaged) as long as it satisfies the same boundary conditions as the instantaneous velocity field.
Consider the $\rho$-averaged and $h$-averaged momentum equations, which solve for $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$, respectively:

$$
\partial_t \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} + fk \times \tilde{\mathbf{u}} + \nabla \tilde{M} = \mathbf{u}^* \cdot \nabla \mathbf{u}^* \nonumber$$

or

$$
\partial_t \hat{\mathbf{u}} + (f + \tilde{\zeta})k \times \hat{\mathbf{u}} + \nabla (\tilde{M} + \hat{\mathbf{u}} \cdot \hat{\mathbf{u}}/2) = -\nabla \hat{\mathbf{u}}^* \cdot \hat{\mathbf{u}}^*/2 - k \times \hat{\mathbf{u}}^* \zeta \tag{24}$$

$$
\nabla \hat{\mathbf{u}} + f k \times \hat{\mathbf{u}} + \nabla \tilde{M} + (\nabla \mathbf{M}^*)^* = -\frac{1}{h} \nabla \cdot h \hat{\mathbf{u}} \hat{\mathbf{u}}^* \tau \hat{\mathbf{u}}^* \zeta^*, \tag{25}
$$

where the overbar also denotes an isopycnal average: $\overline{()} = \langle () \rangle$.

Both (24) and (25) are valid choices for the mean momentum equation but, since they involve different eddy correlations, they lead to different closure schemes. One advantage of (25) is that its solution yields the total tracer transport velocity $\tilde{\mathbf{u}}$, so it is not necessary to parameterize the eddy thickness-flux velocity $\mathbf{u}^*$ in (19) and (20). [We will use the term “bolus velocity” to denote the difference between the tracer-momentum velocity and the mean velocity, and denote it by $\mathbf{u}^*$]. The bolus velocity is not a uniquely defined quantity like $\mathbf{u}^*$, its definition depends on how the mean velocity, determined by the momentum equation, is defined. In general, $\mathbf{U}^*$ and $\mathbf{u}^*$ will differ even if the mean momentum is chosen to be $\mathbf{u}$, because the eddy tracer-flux divergence in (20) may contain an advective component that will contribute to $\mathbf{U}^*$. To avoid confusing these we will refer to $\mathbf{u}^*$ as the “eddy thickness-flux,” $\mathbf{u}^*$ as the “eddy thickness-flux velocity,” and reserve the terms “eddy-induced transport velocity” and “bolus velocity” for $\mathbf{U}^*$]. A disadvantage of (25) is the presence of the nonlocal eddy pressure-velocity correlation $\langle \nabla \mathbf{M} \rangle^*$ that could be difficult to parameterize. DB advocated (25) and argued that $\langle \nabla \mathbf{M} \rangle^*$ can be neglected if $|h| \ll \hat{h}$. Then the mean PEs take a form very similar to the standard $z$-averaged equations, where the only correlations that appear are the divergence of the Reynolds stress in (25) and the eddy tracer flux in (20). However, $h'$ and $\hat{h}$ can be the same order in regions of strong turbulence (Lee and Leach 1996), so the pressure correlation cannot in general be ignored. One advantage of the unweighted-mean-momentum equation (24) is that it has no pressure correlation. Another advantage is that the eddy flux of relative vorticity $\mathbf{u}^* \zeta^*$ on the rhs can be expressed in terms of the eddy fluxes of thickness and potential vorticity. These correlations are not independent; they satisfy the exact identity

$$
\frac{\mathbf{u}^* \zeta^*}{\hat{h}} = \hat{\mathbf{u}}^* \hat{h} + h \mathbf{u}^* \phi^* \nonumber$$

where the mean potential vorticity has the simple form $\hat{\mathbf{u}}^* = (f + \tilde{\zeta})/h$. This important identity simplifies the closure problem if the eddy $\phi$ flux can be parameterized in the same way as the eddy tracer fluxes. Substituting (26) into (24), taking the curl $(k \cdot \nabla \times)$, and using the continuity equation (19) yields the transport equation for $\hat{\phi}$:

$$
\hat{D}_{\hat{\phi}} = -\frac{1}{h} \nabla \cdot h \hat{\mathbf{u}} \hat{\mathbf{u}}^* \phi^* \zeta^*, \tag{27}
$$

which is the same as (20) with $\phi = \phi^*$. Any closure scheme based on (24) will automatically conserve potential vorticity in the sense of (27) as long as the parameterized correlations satisfy the identity (26).

We should emphasize the distinction between the isopycnal coordinates, which are held fixed during the ensemble-averaging, and the model coordinates. It is not necessary to use the $z$-averaged equations in order to derive a turbulence closure for level models. Any of the ensemble-mean equations above, with parameterized forms of the eddy correlations, can easily be transformed to $z$ coordinates for use in level models (see appendix B).

3. Closure schemes in the quasigeostrophic regime

In this section various forms of the ensemble-mean momentum equation in the quasigeostrophic (QG) scale regime are discussed and related to closure schemes that have been proposed by others. The ageostrophic terms in the momentum equation are order $\epsilon_e = \max(\epsilon_o, \epsilon_s, \epsilon_r)$, where $\epsilon_o = U/L$ is the Rossby number, $\epsilon_s = BL/\eta$, and $\epsilon_r = 1/\tau f$; $U$ and $\tau$ are typical velocity and timescales, and $L$ is a length scale comparable to the first baroclinic Rossby radius $R_f$ (see Gill 1982, p. 498). For mesoscale turbulence at midlatitudes (e.g., in the Gulf Stream) $\epsilon_s \sim 0.01$ and $\epsilon_r \sim R_o \sim 0.1$.

We introduce another parameter $\epsilon^*$, which is a measure of the scale of $\mathbf{u}^*$ relative to the mean velocity; that is, $[\mathbf{u}^*] = \epsilon^* [\mathbf{u}]$. We can also think of $\epsilon^*$ as the scale of $h^*$ relative to $h$, assuming $\mathbf{u}^*$ is the same order as $\mathbf{u}$, which is generally the case in baroclinically unstable flow. In reality $h^*$ and $\hat{h}$ can be the same order, but it is nevertheless true that $|h^*| \ll \hat{h}$ over most of the ocean, as was found in the simulations by Lee and Leach (1996), and we found similar results in our eddy-resolving simulations (Maltrud et al. 1998). The limit $h^* \rightarrow$ constant where $h^* \rightarrow 0$, $\epsilon^* \rightarrow 0$, and $\psi^* \rightarrow \psi$ corresponds to incompressible-type flow as in the rigid-lid barotropic equations.

For finite $\epsilon^*$ consider the scaling of the momentum equations when $\epsilon^*, \epsilon_s \ll 1$. Since the turbulent flow is in near-geostrophic balance $\nabla \mathbf{M} = -f k \times \mathbf{u} + O(\epsilon_s)$, and it therefore follows that $\langle \nabla \mathbf{M} \rangle^* = -f k \times \mathbf{u}^* + O(\epsilon_s \epsilon^*)$. Thus Eq. (25) can be written

$$
\hat{D}_{\hat{\phi}} + f k \times (\hat{\mathbf{u}} - \mathbf{u}^*) + \nabla \tilde{M} = -\frac{1}{h} \nabla \cdot h \hat{\mathbf{u}} \hat{\mathbf{u}}^* + O(\epsilon_s \epsilon^*), \nonumber$$

where $O(\epsilon_s \epsilon^*)$ denotes terms of this order relative to the dominant geostrophic terms. In this form $\mathbf{u}^*$ appears only in the momentum equation, and the closure prob-
lem is reduced to determining $u^*$ and the eddy tracer-flux and Reynolds-stress divergences.

Greatbatch (1998) suggests a somewhat different closure that can be derived as follows. Substituting (26) into (24), the momentum equation can be written

$$
\tilde{D}_{t} \tilde{u} + [f + \tilde{\xi}] k \times (\tilde{u} + u^*) + \nabla (\tilde{M} + \tilde{u} \cdot \tilde{u}/2) = -\nabla u \tilde{u} \cdot u'/2 - k \times \tilde{h} \tilde{u} q^*.
$$

(29)

Redefining the mean velocity as $\tilde{u}$, this can be written

$$
\tilde{D}_{t} \tilde{u} + f k \times \tilde{u} + \nabla \tilde{M} = \tilde{\nabla} \cdot \tilde{E}.
$$

(30)

Greatbatch proposes dropping the $O(\epsilon_0 e^*)$ terms and parameterizing the $q$ flux in the second term on the rhs as downgradient diffusion of $\tilde{q}$.

Lee and Leach (1996) formulate the ensemble-mean momentum equation such that the rhs is the divergence of a generalized isopycnal Eliassen–Palm (E–P) flux. A form similar to theirs, but given in terms of thickness-weighted averages, can be derived from (25) as follows. Using the hydrostatic equation (4) the pressure–height correlation can be written

$$
\tilde{h} (\nabla M)^* = h \nabla M^* = -\rho \tilde{p} \tilde{p} \cdot \nabla M^* + g \nabla \tilde{\epsilon}^2/2.
$$

(31)

Then (25) becomes

$$
\tilde{D}_{t} \tilde{u} + f k \times \tilde{u} + \nabla \tilde{M} = \tilde{\nabla} \cdot \tilde{E}.
$$

(32)

$$
\tilde{v} = \frac{1}{h} (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \quad \text{for} \quad \tilde{E} = (\tilde{E}_x, \tilde{E}_y, \tilde{E}_z),
$$

$$
\tilde{E}_x = (-\tilde{h} \tilde{u} u'' - g \tilde{\epsilon}^2/2, -\tilde{h} \tilde{u} v''), \quad \tilde{E}_y = (-\tilde{h} \tilde{v} v'' - g \tilde{\epsilon}^2/2, -\tilde{h} \tilde{v} w''), \quad \tilde{E}_z = (-\tilde{h} \tilde{w} w'' - g \tilde{\epsilon}^2/2, -\tilde{h} \tilde{w} u''),
$$

where $\tilde{E}$ is the generalized E–P flux in isopycnal coordinates. Comparing (30) and (32), it is clear that, relative to the dominant geostrophic terms,

$$
\tilde{\nabla} \cdot \tilde{E} = -\nabla u \tilde{u} \cdot u'/2 - k \times \tilde{h} \tilde{u} q^* + O(\epsilon_0 e^*).
$$

(33)

In this sense the closure schemes advocated by Greatbatch and by Lee and Leach are equivalent. Lee and Leach found in their simulations that the dominant balance was between the divergence of the E–P flux and the eddy $q$ flux. They found that the term $\nabla u \tilde{u} \cdot u'/2$ was small compared to the $q$-flux term, although not completely negligible, and suggested dropping it and approximating the E–P flux divergence by the $q$-flux term. Note, however, that $\nabla u \tilde{u} \cdot u'/2$ is $O(\epsilon_0)$ and therefore cannot be neglected in the QG regime based on a scaling argument alone.

Is the assumption $\epsilon^* \ll 1$ valid? This same assumption is needed to transform the GM parameterization into the vertical mixing parameterization proposed by Greatbatch and Lamb (1990). Gent et al. (1995) showed that this parameterization gives similar results to GM in a wind-driven double-gyre solution using a coarse-resolution model with a rectangular domain (the results differed by $\sim 5\%$). We have found in our eddy resolving simulations (Maltrud et al. 1998) that $e^* \sim 0.1–0.2$ in global average; however, it is larger (0.3–0.6 or more in zonal average) in the Tropics and in the midlatitude jets. This is expected since $|h'| \sim \tilde{h}$ in regions of strong turbulence or strong stratification. Because this approximation is only marginally adequate, we think the best approach is to use the momentum equation (24) with (26), which solves for $\tilde{u}$. The price to be paid for this, relative to using (30) or (32), is that $u^*$ has to be parameterized.

In the QG regime it is possible to transform the PEs (19)–(24) to a form in which $u^*$ does not appear and the assumption $\epsilon^* \ll 1$ is not needed. Furthermore, it is not only necessary to drop terms that are $O(\epsilon_0)$ relative to the $O(1)$ geostrophic terms. If $\tilde{a} \neq 0$, then (26) can be written

$$
u^* = u^* + u^v,
$$

(36)

$$
u^* = \tilde{u} h'/\tilde{h},
$$

(37)

$$
u^* = \tilde{u} q'/\tilde{q},
$$

(38)

$$
u^* = \tilde{u} \tilde{z}/\tilde{h}.
$$

(39)

In appendix A it is shown that $u^v$ is $O(\epsilon_0)$ relative to the mean velocity $\tilde{u}$, provided $u^v_{\tilde{z} \tilde{z}} = 0$ (where the subscript $g$ denotes the geostrophic component of the flow), which is valid in the limit of large spatial averaging (see §5). If we redefine the mean momentum as

$$
u^* = \tilde{u} + u^v = \tilde{u} + O(\epsilon_0),
$$

(40)

then $\tilde{u}$ can be written

$$\tilde{u} = \tilde{u} - u^v,
$$

(41)

and the momentum equation becomes

$$
\tilde{D}_{t} \tilde{u} + [f + \tilde{\xi}] k \times \tilde{u} + \nabla (\tilde{M} + \tilde{u} \cdot \tilde{u}/2) + \tilde{u} \tilde{u} \cdot u'/2 = -\nabla u \tilde{u} \cdot u'/2 + O(\epsilon_0),
$$

(42)

where again the rhs is $O(\epsilon_0)$ relative to the dominant geostrophic terms. Thus the ensemble-mean PEs (19), (20), and (24) become

$$
\tilde{D}_{t} \tilde{u} + [f + \tilde{\xi}] k \times \tilde{u} + \nabla (\tilde{M} + \tilde{u} \cdot \tilde{u}/2) = -\nabla u \tilde{u} \cdot u'/2 + O(\epsilon_0),
$$

(43)

$$
\tilde{D}_{t} \tilde{h} + \nabla \cdot \tilde{h} (\tilde{u} - u^v) = 0,
$$

(44)

$$
\tilde{D}_{t} \tilde{\phi} + (\tilde{u} - u^v) \cdot \nabla \tilde{\phi} = -\frac{1}{\tilde{h}} \nabla \cdot \tilde{h} \tilde{u} \phi^*.
$$

(45)

[If the condition $u^v_{\tilde{z} \tilde{z}} = 0$ is not satisfied, then $u^v$ is only $O(\epsilon_0)$ relative to $\tilde{u}$, and the rhs of (43) and (44) is $O(\epsilon_0)$.] Therefore, except for a possible advective con-
tions will apply to fluid turbulence provided the time third, a hypothesis relating the ensemble-mean Eulerian particles forget all but their most recent past history; and first, that the turbulence is fundamental assumptions underlying the stochastic models: second, that the turbulence is described by a probability distribution function (pdf); and third, a hypothesis relating the ensemble-mean Eulerian and Lagrangian velocity fields. The first two assumptions will apply to fluid turbulence provided the time averages are taken over times much longer than the Lagrangian integral timescale $T_L$. They permit the derivation of a Fokker–Planck equation (FPE) for the time evolution of the pdf. The third assumption leads from the FPE to the semiempirical diffusion equation for the ensemble-mean tracer distribution.

These assumptions are not always valid in fluid turbulence, and undoubtedly they do not apply everywhere to mesoscale turbulence in the ocean. Random-walk models do not account for subgrid-scale dynamics, which may violate any of the above assumptions. Nevertheless, the theory of geostrophic turbulence is not fully developed, and there is potentially much to be gained by applying to it the basic ideas from random-walk theory. However, until recently it was not possible to derive a semiempirical diffusion equation for adiabatic stratified turbulence in this theory because it had only been developed for compressible or divergence-free flow. In ocean mesoscale turbulence, mixing is confined to isopycnal layers where the horizontal velocity field is divergent, and hence a compressible-type theory is needed. Such a theory was recently developed by DS and applied to tracer transport in mesoscale turbulence. In the incompressible case the assumptions outlined above, with a specific hypothesis relating the mean Eulerian and Lagrangian transport velocities [Eq. (70) below], lead to the standard semiempirical diffusion equation (Monin and Yaglom 1971, p. 606, hereafter MY). In that simple theory the mean tracer distributions are advected with the Eulerian-mean velocity and diffused downgradient (Fickian diffusion). For engineering purposes the semiempirical diffusion equation is widely used and quite successful, in spite of the fact that turbulence does not always satisfy the underlying assumptions. In DS it was shown that by making the same assumptions as in the standard incompressible case, a semiempirical diffusion equation can be derived for compressible flow that also involves Fickian diffusion, but in addition predicts tracer advection with the Eulerian-mean velocity plus an eddy-induced transport velocity involving downgradient diffusion of the mean density field.

In adiabatic stratified flow the layer thickness plays the role of density. The continuity and tracer transport equations written in conservative form are

$$\partial_t h + \nabla \cdot \mathbf{u} h = 0 \quad (47)$$

$$\partial_t h \phi + \nabla \cdot u h \phi = 0. \quad (48)$$

These are mathematically identical to the two-dimensional compressible flow equations with $p \rightarrow h$. In the stochastic theory random displacements of fluid parcels are described by a probability density function, denoted $p(x, t|y, t_o)$, which represents the conditional probability of finding a particle in volume $dx$ at time $t$ given that it was in volume $dy$ at time $t_o \leq t$. We will briefly outline the derivation of the Fokker–Planck equation for the pdf and then show that the standard incompressible semiempirical diffusion equation and its compressible generalization (DS) are both derived from the same FPE using identical assumptions. For Markovian stochastic processes the pdf satisfies the Chapman–Kolmogorov integral equation (Gardiner 1985):

$$p(x, t|y, t_o) = \int dx' p(x|y, t_o) p(x', t|y, t_o) \quad (49)$$

for any $t_i$ satisfying $t \geq t_i \geq t_o$. Then assuming that the random particle paths are continuous (but not differentiable) in time (i.e., no instantaneous “jumps”), the equivalent partial differential equation for the pdf is the FPE

$$\partial_t p(x, t|y, t_o) + \nabla \cdot p(x, t|y, t_o) = \nabla \cdot K \nabla p(x, t|y, t_o) \quad (50)$$

$$U = v - \nabla \cdot K \quad (51)$$

$$v(x, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int dx' (x' - x)p(x', t + \Delta t|x, t) \quad (52)$$

$$K(x, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int dx' \frac{1}{2} (x' - x)(x' - x)p(x', t + \Delta t|x, t) \quad (53)$$

Since the pdf is positive, it is clear from (53) that $K$ is a $2 \times 2$ symmetric positive-definite tensor. We will refer to $v$ as the Lagrangian mean velocity, although this identification is not exact (see Bennett 1996, p. 7). In (52) and (53) the limit is understood to mean that $\Delta t$ remains
greater than the Lagrangian integral timescale as $\Delta t \to 0$. A word of caution: if the turbulence is such that the fluid parcels remain correlated over very long times, then these integrals will not converge, even if the particle displacements are Markovian. In the real ocean eddies can sometimes transport water parcels over very large distances, even from one ocean basin to another, and in these situations the transport equations derived from the FPE will not apply.

At this point we have not specified whether the flow is compressible or incompressible, and Eqs. (50)–(53) apply to both. The semiempirical diffusion equations are derived by integrating the pdf over an initial tracer distribution. As discussed by Bennett (1996) and DS, the appropriate ensemble average for incompressible flow is

$$\bar{\phi}(x, t) = \int dy \phi(y, t_c)p(x, t | y, t_c)$$

and for compressible type flow the appropriate average is

$$\tilde{\phi}(x, t) = \int dy \phi(y, t_c)p(x, t | y, t_c)$$

Equations (55) and (57) follow from (54) and (56), respectively, in the limit $\phi \to 1$. Note, however, that (55) does not apply to compressible flow. The pdf is normalized to unity when integrated over arrival points $x$, but not when integrated over departure points $y$ unless the flow is incompressible. Also note that the expressions (54) and (56) for the ensemble-averages only apply to conserved quantities that satisfy $D_t \phi = 0$. For example, with $\phi = q$ Eq. (56) can be written

$$f(x) + \tilde{\zeta}(x, t) = \int dy[f(y) + \tilde{\zeta}(y, t_c)]p(x, t | y, t_c).$$

This effectively defines the ensemble-mean relative vorticity $\tilde{\zeta}(x, t)$.

Applying (56) and (57) to the FPE yields the transport equations for compressible flow:

$$\partial_t \tilde{\phi} + \nabla \cdot \tilde{U} \tilde{\phi} = \nabla \cdot \mathbf{K} \cdot \nabla \tilde{\phi}$$

$$\partial_t \tilde{\phi} + \nabla \cdot \tilde{U} = 0.$$

Using (60), (59) can be rewritten

$$\partial_t \tilde{\phi} + \left( U - \frac{1}{\hat{h}} \mathbf{K} \cdot \nabla \tilde{h} \right) \cdot \nabla \tilde{\phi} = \mathcal{R} (\tilde{\phi})$$

$$\mathcal{R} (\tilde{\phi}) = \frac{1}{\hat{h}} \nabla \cdot \mathbf{h} \mathbf{K} \cdot \nabla \tilde{\phi}.$$

where $\mathbf{U} - \tilde{h}^{-1} \mathbf{K} \cdot \nabla \tilde{h}$ is the total tracer transport velocity, and $\mathcal{R}$ is the along-isopycnal diffusion operator, as in Eq. (12). In the incompressible case, an analogous derivation starting from (50) and (54)–(55) leads to

$$\partial_t \tilde{\phi} + \mathbf{U} \cdot \nabla \tilde{\phi} = \nabla \cdot \mathbf{K} \cdot \nabla \tilde{\phi}\)$$

$$\mathbf{U} \cdot \nabla = 0.$$

To relate $\mathbf{U}$ and $\mathbf{K}$ to the correlations in the ensemble-mean PEs, write the transport equation (20) in conservative form:

$$\partial_t \tilde{\phi} + \nabla \cdot (\mathbf{U} \tilde{\phi}) = \nabla \cdot \mathbf{U} \tilde{\phi} = -\nabla \cdot \tilde{h} \mathbf{u}^* \tilde{\phi}^*.$$

Equating the divergence terms in (19), (60), and in (59), (65), we identify

$$\mathbf{u}^* \tilde{h}^* = \tilde{h} (\mathbf{U} - \tilde{\mathbf{u}}) - \mathbf{K} \cdot \nabla \tilde{h} + \mathbf{k} \times \nabla \psi_h$$

$$\mathbf{U} = \mathbf{v} - \nabla \cdot \mathbf{K}$$

Note that the bolus velocity and the thickness-flux velocity $\mathbf{u}^*$ differ by the gauge term $\tilde{h}^{-1} \mathbf{k} \times \nabla \psi_h$. In the transport equation (65) this term is canceled by an advective contribution from the eddy tracer-flux divergence that arises from the second term on the rhs of (67).

To proceed further we need to relate the Lagrangian-mean velocity $\mathbf{v}$ to the Eulerian-mean velocity $\tilde{\mathbf{u}}$. Using (66), $\mathbf{U} = \mathbf{v} - \nabla \cdot \mathbf{K}$ can be expressed as

$$\mathbf{U} = \tilde{\mathbf{u}} + \frac{1}{\hat{h}} [\mathbf{u}^* \tilde{h}^* + \mathbf{K} \cdot \nabla \tilde{h} - \mathbf{k} \times \nabla \psi_h].$$

Dukowicz and Smith (1997) show that the simplest plausible hypothesis for $\mathbf{v}$ is to set $\mathbf{U} = \tilde{\mathbf{u}}$, or

$$\mathbf{v} = \tilde{\mathbf{u}} + \nabla \cdot \mathbf{K}.$$

With this hypothesis (63) reduces to the standard semiempirical diffusion equation for incompressible flow (MY), and (60)–(61) reduce to the compressible generalization of DS. The term $\nabla \cdot \mathbf{K}$ is the difference between the Lagrangian and Eulerian-mean velocities. It is analogous to the Stokes drift velocity, which is associated with the displacement of fluid parcels in linear waves rather than in turbulent flow. As discussed by MY, this term is associated with the tendency of a cloud of particles in inhomogeneous turbulence to move or be dispersed even in the absence of mean flow. Equation (52) suggests that $\mathbf{v}$ is closely related to the unweighted
mean velocity rather than the weighted mean (56) appropriate for tracers. Hence (70) is also the simplest hypothesis in the compressible case, which includes this effect of inhomogeneous turbulence (see DS for a more detailed discussion of this point).

It should be emphasized, however, that the hypothesis (70) is not the only possible choice for $v$. For example, in incompressible flow ($h \to \text{const}$) Eq. (69) becomes

$$U = \tilde{u} - k \times \nabla (\psi_h/h),$$

(71)

where the gauge term represents an arbitrary rotational velocity field. Bennett (1996) presents a more general derivation of the incompressible FPE that does not start from the Chapman–Kolmogorov equation (49) but instead is based on the Reynolds averaged kinematics of a particle. It does not require a hypothesis like (70), but it involves a diffusion tensor that is not in general symmetric. This leads to an advection term with a rotational component of $q$, which will also apply to the potential vorticity $q^{*}$.

**Inserting (67), (72), and (74) into the PEs (19)–(24), we obtain the stochastic theory prediction**

$$\partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + f k \times \tilde{u} + \nabla \tilde{U}^* = 0$$

(75)

$$\partial_t \tilde{h} + \nabla \cdot \tilde{h} (\tilde{u} + \tilde{U}^*) = 0$$

(76)

$$\tilde{D}_i \partial_i \phi = \frac{1}{h} \tilde{u} \cdot \tilde{h} k \cdot \nabla \phi = R(\phi)$$

(77)

$$\rho \partial_t \tilde{M} = g z$$

(78)

$$W = \tilde{D} \tilde{z}$$

(79)

$$\tilde{D}_i = \partial_i + (\tilde{u} + \tilde{U}^*) \cdot \nabla,$$

(80)

where $W$ is the vertical velocity associated with the horizontal tracer transport velocity $\tilde{u} + \tilde{U}^*$ and satisfies $\nabla_q (\tilde{u} + \tilde{U}^*) + \partial_i W = 0$. The scalar field $\psi$ and bolus velocity $U^*$ are given by

$$\psi = \tilde{u} \cdot \tilde{u} / 2 - \psi_q$$

(81)

$$U^* = -\frac{1}{h} \tilde{K} \cdot \nabla \tilde{h}.$$  

(82)

Equations (75)–(82) constitute the principle result of this paper. They are transformed to $z$ coordinates for use in level models in appendix B.

Thus the stochastic theory predicts a single positive-
definite diffusion tensor for \( \hat{h}, \hat{\phi}, \) and \( \hat{q} \), and the PEs are close to within one scalar field \( \psi \) and one \( 2 \times 2 \) symmetric tensor \( K \). Furthermore, it is clear from (53) that this tensor is not constant in space, but increases with the intensity of the turbulence. With the hypothesis (70) the PEs are independent of all gauge fields \( \psi_h \) and \( \psi_0 \) associated with the rotational components of eddy fluxes of thickness and passive tracers. This is a crucial result, since it eliminates the need to parameterize these rotational fluxes, which are generally much larger than the corresponding divergent components. The momentum equation still depends on \( \psi_h \), although this gauge can be combined with the unspecified correlation \( \vec{u}' \cdot \vec{u}''/2 \) into a single scalar field \( \psi \). The term \( \nabla \psi_o \) may be difficult to parameterize, but if (70) is valid, and if \( \vec{u} \cdot \vec{z} \) is \( O(\varepsilon) \), then it can be neglected away from the equation since by (74) \( \nabla \psi_o \) contributes an \( O(\varepsilon) \) term to the momentum equation. By the same argument the term \( K \cdot \nabla (f + \zeta) \) is also \( O(\varepsilon) \), but it must be retained in order to ensure exact conservation of \( q \) in the sense that the curl of the momentum equation yields the transport equation (77) with \( \phi = q \). It is also worth noting that in the homogeneous isotropic limit, \( \nabla \psi \) and \( -K \times K \cdot \nabla (f + \zeta) \) correspond, respectively, to the divergent and rotational components of the Reynolds correlation \( \vec{u}' \cdot \nabla \vec{u}'' = \vec{u}' \cdot \vec{u}''/2 + K \times \vec{u}' \cdot \vec{z} \).

Some insight into the meaning of the parameterized Reynolds correlation can be gained by considering flow in a periodic zonal channel and replacing the ensemble averages with zonal averages. Then the mean variables as well as \( K \) and \( \psi \) are independent of \( \psi \), and the turbulent fluxes take the form

\[
\begin{align*}
\vec{u} \cdot \nabla \vec{u}'' &= -\partial \psi_h \\
\vec{v} \cdot \nabla \vec{v}'' &= -\nu \partial \psi_h \\
\vec{u} \cdot \nabla \vec{z}'' &= -\partial \psi_q \\
\vec{v} \cdot \nabla \vec{z}'' &= -\nu \zeta + \nu \zeta \vec{u} \\
h\vec{u} \cdot \nabla \vec{q}'' &= \hat{q} \partial \psi_h - \partial \psi_q \\
h\vec{v} \cdot \nabla \vec{q}'' &= -\kappa \partial \psi, 
\end{align*}
\]

(83)–(88)

where we have assumed an isotropic diffusivity \( \kappa \). Note that \( \kappa \) appears only in the meridional fluxes and the gauge fields \( \psi_h \) and \( \psi_q \) appear only in the zonal fluxes. Thus the channel model provides a unique setting for investigating parameterizations of the diffusivity and the gauge fields since these can be separately extracted from the turbulent fluxes. The zonal and meridional momentum equations are

\[
\begin{align*}
\partial \vec{u} + \vec{v} \cdot \nabla \vec{u} - f \vec{v} &= -\kappa \beta + \nu \partial \psi, \\
\partial \vec{v} + \vec{u} \cdot \nabla \vec{v} + f \vec{u} + \partial \vec{u} &= \partial \psi.
\end{align*}
\]

(89)–(90)

Here \( \kappa \) appears only in the zonal equation, and \( \psi \) only in the meridional equation. In the zonal equation the first term on the right, \(-\kappa \beta\), is a small eddy-induced westward acceleration of order \( 10^{-8} \) m s\(^{-2}\). The presence of this term was first pointed out by Welander (1973), associated with a downgradient parameterization of the potential vorticity flux. The second term on the right represents diffusion of the zonal momentum associated with a downgradient flux of relative vorticity. If \( \kappa \) is uniform across the channel, this term will laterally diffuse an eastward zonal jet. Numerical simulations with QG models have shown that the relative vorticity flux actually accelerates the eastward flow into a narrow concentrated jet rather than diffusing it (McWilliams and Chow 1981). If \( \kappa \) is uniform this implies a negative diffusivity. However, it has recently been shown by Ivchenko et al. (1997) that the jet acceleration can be accomplished with a positive but spatially varying diffusivity. Their form of \( \kappa \) is proportional to the magnitude of the vertical velocity shear and inversely proportional to the square of the meridional gradient of potential vorticity so that \( \kappa \) has a minimum at the center of the jet and maxima on each side. The net effect is that momentum is laterally diffused toward the center, resulting in a narrow concentrated jet. Therefore the downgradient parameterization of the relative vorticity flux in (75) and (89) does not preclude this effect.

The transport equations (76), (77), with (82) are equivalent to the GM parameterization (11)–(13) in the limit of horizontally isotropic turbulence with a depth-independent diffusion coefficient \( (K \rightarrow \kappa) \) and \( \partial \psi, \kappa \rightarrow 0 \). However, as seen in the last section, with \( \nu = \kappa \). To date, essentially all simulations with the GM parameterization have used a spatially constant \( \kappa \). Danabasoglu and McWilliams (1995) tried using a depth-dependent form of \( \kappa \), but found only modest effects on the solution relative to a constant value. So there is, as yet, no numerical evidence to suggest which form is preferable.

5. The planetary geostrophic and quasigeostrophic regimes in the stochastic theory

Recently it has been argued by several authors (Treguier et al. 1997; Killworth 1997; Lee et al. 1997) that the bolus transport should be expressed as diffusion of \( \hat{q} \) rather than diffusion of \( \hat{h} \) (as in the GM parameterization) because \( q \) is conserved along trajectories of fluid parcels, whereas \( h \) is not. The basis for this argument is that the standard stochastic theory of turbulent mixing, which constitutes the theoretical basis for Fickian diffusion, only applies to such conserved quantities. That is, it may be justified to parameterize \( \langle \vec{u}' \cdot \vec{v} \rangle \rightarrow -K \cdot \nabla \langle \phi \rangle \) because \( \phi \) satisfies the continuity equation \( D \phi = 0 \), but not \( \langle \vec{u}' h' \rangle \rightarrow -K \cdot \nabla \langle h \rangle \) because \( h \) satisfies the continuity equation \( D h = -\delta \vec{v} \cdot \vec{u} \neq 0 \). However, as seen in the last section, with the hypothesis (70) the correspondence \( \vec{u} h' \rightarrow -K \cdot \nabla h \) arises in compressible flow from the mean tracer transport equation in the limit of a constant tracer distribution \( \vec{v} = 1 \).

There does exist at least one regime of the flow, the
planetary geostrophic (PG) limit, where this correspondence does not hold, and (70) must be modified. Dukowicz and Greatbatch (1999) apply the stochastic theory in this regime with a different hypothesis, which leads to a bolus velocity given by \( \dot{q} \) diffusion. In the PG system where \( q = q_\ast = f\bar{h} \) the turbulent fluxes of \( \dot{q} \) and \( h \) exactly satisfy

\[
\mathbf{u}^\prime \dot{q}^\prime = -\mathbf{u} \dot{q} \dot{q},
\]

which is the limit \( \mathbf{u}^\prime \xi \to 0 \) of the correlation identity (26). In the QG system \( \mathbf{u}^\prime \xi \) is given in terms of the geostrophic velocity fluctuations alone: \( \mathbf{u}^\prime \xi \to \mathbf{u}^\prime \zeta^\prime \).

DG show that this can be written as the divergence of a tensor, so if the ensemble mean is replaced with a horizontal spatial average, the integral over area can be transformed to a line integral along the boundary using the Gauss theorem. Therefore the correlation can be expressed as the ratio of a line integral to an area integral, and so should vanish if the averaging area is sufficiently large. Thus DG consider the limit \( \mathbf{u}^\prime \zeta = 0 \) in their theoretical analysis. In this limit the stochastic theory prediction (73) becomes

\[
\mathbf{U} = \mathbf{u} + \frac{1}{\bar{h}q} \left( \mathbf{K} \cdot \nabla \bar{h} \mathbf{q} - \mathbf{k} \times \nabla \psi_e \right).
\]  

This is inconsistent with the hypothesis (70): the two terms in brackets cannot cancel since the first has a divergent component (it is a pure gradient in the homogeneous isotropic limit) while the second term is purely rotational. Therefore DG conclude that (70) must be modified for the QG and PG systems. Is this also true for the PE system? If \( \mathbf{u}^\prime \zeta = 0 \), then using only (67), the ensemble mean PEs (20)–(24) become

\[
\partial_t \mathbf{u} + (f + \zeta) \mathbf{k} \times \mathbf{u} + \nabla (\dot{M} + \mathbf{u} \cdot \mathbf{u}/2) = -\nabla \mathbf{u}^\prime \cdot \mathbf{u}^\prime/2
\]

(93)

\[
\partial_t \mathbf{h} + \nabla \cdot \dot{h} (\mathbf{u} + \mathbf{U}^\ast) = 0
\]

(94)

\[
\partial_t \dot{\phi} + (\mathbf{u} + \mathbf{U}^\ast) \cdot \nabla \dot{\phi} = \mathcal{R}(\dot{\phi})
\]

(95)

\[
\mathbf{U}^\ast = -\frac{1}{\dot{q}^\ast} \mathbf{K} \cdot \nabla \dot{q}^\ast - \frac{1}{\bar{h}q} \mathbf{k} \times \nabla \psi_e,
\]

(96)

and \( \mathbf{U}^\ast \) is again related to \( \mathbf{u}^\ast \) by (68). This is the same form obtained by DG for the continuity and tracer transport equations. Note that the same results would be obtained by applying \( \mathbf{u}^\prime \zeta = 0 \) and (67) directly to the transformed PEs (44)–(46). Consider the PG limit of (93)–(96), where \( \dot{q} = f\bar{h} \) and the momentum equation is simply \( \mathbf{j} \times \mathbf{u} + \nabla \mathcal{M} = 0 \). Thus the bolus velocity can be written

\[
\mathbf{U}^\ast = -\frac{1}{\bar{h}q} \mathbf{K} \cdot \nabla \dot{h} + \frac{1}{f} (\mathbf{K} \cdot \nabla f - \mathbf{k} \times \nabla \psi_e).
\]  

(97)

Redefining the mean velocity as

\[
\mathbf{\bar{v}} = \mathbf{u} + \frac{1}{f} (\mathbf{K} \cdot \nabla f - \mathbf{k} \times \nabla \psi_e),
\]

(98)

the momentum equation becomes

\[
\mathbf{j} \mathbf{k} \times \mathbf{\bar{v}} + \nabla \mathcal{M} = \mathbf{k} \times \mathbf{K} \cdot \nabla f + \nabla \psi_e,
\]

(99)

which is the appropriate form of (75) in the PG limit. The curl of (99) combined with the continuity equation \( [(76) \ with \ \mathbf{u} \ replaced \ by \ \mathbf{v}] \) yields the transport equation (77) for \( \dot{\phi} = \dot{q} = f\bar{h} \). This demonstrates the equivalence of the two closure schemes (75)–(82) and (93)–(96) in the PG regime.

In order to close their transport equations, DG hypothesize that the gauge term \( \nabla \psi_e \) vanishes, so the bolus velocity (96) is given simply by

\[
\mathbf{U}^\ast = -\frac{1}{\dot{q}^\ast} \mathbf{K} \cdot \nabla \dot{q}^\ast.
\]  

(100)

Thus the bolus velocity involves diffusion of \( \dot{q} \) rather than \( \bar{h} \). Earlier, Killworth (1997) deduced the same result based on analysis of the linear solutions of the QG equations.

Dukowicz and Greatbatch (1999) avoided making the hypothesis (70) but, instead, assumed that \( \nabla \psi_e \) vanishes in the PG and QG systems. This is equivalent to assuming \( \nabla \psi_e \) is \( O(\varepsilon^3) \) or smaller. As shown in appendix A, \( \mathbf{u}^\prime \zeta \) does not vanish in the PE system; it is an \( O(\varepsilon) \) term in the momentum equation. Furthermore, the gauge term \( \nabla \psi_e \) has a contribution from the rotational component of the eddy \( q \) flux (67), which might not be small. So, in general, neither \( \mathbf{u}^\prime \zeta \) nor \( \nabla \psi_e \) vanish in the PE system.

However, if the hypothesis (70) is made, then it follows from (74) that \( \nabla \psi_e \) is \( O(\varepsilon^3) \) since it is then associated with rotational component of \( \mathbf{u}^\prime \zeta \) and therefore the transport equations proposed by DG, (94)–(95) with (100), are recovered exactly in the QG limit \( \varepsilon \to 0 \) of the PEs (75)–(82). Alternatively, if \( \nabla \psi_e \) is not set to zero but instead assumed to be \( O(\varepsilon^3) \), then, applying the transformation (98) with \( f = f^\ast \) to the full equations (93)–(96), these can be shown to be equivalent to the PEs (75)–(82) to \( O(\varepsilon^3) \), and hence the hypothesis (70) is valid to \( O(\varepsilon^3) \). Since the two forms of the PEs are equivalent to this order, there is no practical reason for preferring one over the other in the PG or QG scale regimes.

The two forms of the bolus velocity (82) and (100) differ by the term

\[
-\frac{1}{\dot{q}^\ast} \mathbf{K} \cdot \nabla \dot{q}^\ast + \frac{1}{f} \mathbf{K} \cdot \nabla \dot{h} = \frac{1}{\bar{h}q} \mathbf{K} \cdot \nabla \dot{h} \dot{q} \to \frac{1}{f} \mathbf{K} \cdot \nabla f,
\]

(101)

where the arrow indicates the PG limit, which in horizontally isotropic turbulence is a poleward component of the bolus velocity with magnitude \( \kappa \beta f \). It could be argued that such a term should also be present in the PEs as a component of the thickness flux since these reduce to the PG system in the limit \( \varepsilon \to 0 \) and \( \nabla f \gg \nabla \xi \). The latter is equivalent to \( \varepsilon \gg \varepsilon_0 \), \( \varepsilon_0 \) or \( L \gg \sqrt{U/\beta} \), so this limit of the PEs only describes turbulence at scales much larger than the Rhines length \( \sqrt{U/\beta} \), which is order 100 km in the ocean. Nevertheless, the PG system exhibits turbulence at all length scales, so it
could be argued that this term should be included to accurately parameterize turbulence in the PEs at all scales. On the other hand, there are other regimes of the flow where this term cannot be present. Consider again the correlation identity (26). In the PG limit $\mathbf{u} \cdot \mathbf{q} \to 0$ and if the $q$ flux is downgradient, then by (82) and (96) $\mathbf{u} \cdot \mathbf{h}^\prime$ must contain the beta term (101), but in the incompressible limit $h \to \text{const}$ (e.g., rigid-lid barotropic flow) $\mathbf{u} \cdot \mathbf{h}^\prime \to 0$, hence it has no beta term. Another example is the regime of homogeneous turbulence, where the ensemble-mean Eulerian and Lagrangian velocities should be equal ($\mathbf{v} = \mathbf{u} = \mathbf{U}$), consistent with (70). Furthermore, the term (101) will be singular in any flow regime where $\hat{q}$ can vanish (e.g., the Tropics where $|f| \sim |\hat{q}|$, or nonrotating flow), unless $K \to 0$. In the PEs (75)–(82) there is no $\hat{q}^{-1}$ singularity, but the effect of the beta term (101) is still present, as it must be since the two forms of the PEs are equivalent to $O(\epsilon)$. There it appears as a component of the relative-vorticity flux that produces westward acceleration of the zonal momentum, rather than a component of the thickness flux that produces poleward bolus transport of tracers.

A nice series of eddy-resolving simulations designed to test whether $\mathbf{u}^*$ is associated with $\hat{q}$ diffusion or $\hat{h}$ diffusion have been carried out by Lee et al. (1997) and Marshall et al. (1999). In these experiments a three-layer isopycnal model was configured as a reentrant zonal channel. Eddy fluxes of potential vorticity were computed as unweighted isopycnal averages: $\mathbf{u} \cdot \mathbf{q}^\prime \to \mathbf{u} \cdot \mathbf{q}^\prime$. This is only valid in the limit $|h^\prime| \ll \hat{h}$, which does not apply everywhere in the real ocean, but it should be adequate for these experiments, where $\hat{h}$ is very large ($\geq 500 \text{ m}$) since there are only three layers. To maintain a mean zonal flow in near-geostrophic balance, the thickness in each layer was strongly restored to its initial value at the meridional boundaries. This allowed the mean gradient of thickness across the channel to be controlled. Ensemble averages such as $\mathbf{u} \cdot \mathbf{h}^\prime$ were computed by taking zonal averages as well as a time averages. As discussed in the last section, the advantage of this procedure is that the zonal averaging eliminates the unknown gauge terms in the vertical components of the turbulent fluxes. In the PEs (75)–(82) these fluxes are given by (83)–(88), which can be used to directly extract the isotropic diffusivity $\kappa$. Or, if the QG form of the PEs (93)–(96) is correct, then (87)–(88) should still hold, but instead of (83)–(86), one should find that $\mathbf{u} \cdot \mathbf{q} = 0$ and the thickness fluxes should satisfy

$$
\mathbf{u} \cdot \mathbf{h}^\prime = -\partial_y \psi_h + \frac{1}{\hat{q}} \partial_z \psi_q \tag{102}
$$

$$
\mathbf{v} \cdot \mathbf{h}^\prime = \hat{h} \left( \mathbf{\hat{\kappa}} \partial_z \mathbf{h} \right) = -\mathbf{\hat{\kappa}} \mathbf{h} + \frac{1}{\hat{q}} (\mathbf{\hat{\kappa}} \beta - \mathbf{\hat{\kappa}} \mathbf{\hat{\kappa}} \mathbf{h}). \tag{103}
$$

In the Lee et al. (1997) experiments it was found that when the boundary forcing was chosen to maintain a constant mean thickness in the intermediate layer, a non-zero poleward thickness flux $\mathbf{\hat{v}} \cdot \mathbf{h}^\prime$ resulted, so they concluded that (103) rather than (84) is the correct parameterization of the thickness flux. This result is perhaps not surprising with zonal averaging because the statistics contain a contribution from fluctuations on scales much larger than the Rhines length, which are governed by PG dynamics; hence the beta term should appear in $\mathbf{v} \cdot \mathbf{h}^\prime$. Marshall et al. (1999) performed a similar set of experiments in which the gradient of $\hat{q} = \partial h$ was controlled across the channel, and the results all supported the downgradient hypothesis (88) for the $q$ flux. In one experiment $\partial h$ was held constant across the channel, and they found $\mathbf{v} \cdot \mathbf{q}^\prime \approx 0$, again in agreement with (88). However, they also found $\mathbf{v} \cdot \mathbf{h}^\prime = \mathbf{v} \cdot \mathbf{q}^\prime / \hat{h} \neq 0$, and while both these terms were small, this nevertheless contradicts the PG parameterization (103). [Both terms were order $10^{-3} \text{ m s}^{-1}$, which is consistent with the $O(\epsilon^2)$ scaling: the typical zonal velocities were order 5 cm s$^{-1}$; using $f = 0.73 \times 10^{-4} \text{ s}^{-1}$, $\beta = 2 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$, and an eddy-length scale $L = R_\text{h} \sim 37 \text{ km}$ at 30$^\circ$lat, then $\rho = U/fL \sim 0.02$ and $\epsilon_\text{h} = \beta L f \sim 0.01$ so $\epsilon_\text{h} = R_\text{h}$; thus with $\hat{h} = 500 \text{ m}$ for their middle layer we find $\mathbf{v} \cdot \mathbf{h}^\prime \sim \epsilon_\text{h}^2 \hat{h} = 10^{-5} \text{ m s}^{-1}$ in agreement with their results.] So the two sets of experiments would appear to contradict both forms of the parameterization for $\mathbf{v} \cdot \mathbf{h}^\prime$. Further analysis of these and similar experiments is needed to settle this question, but we emphasize again that there is no practical difference between the two forms of the PEs in the QG regime and that the effect of the beta term is present in either case. The only issue is whether $\mathbf{v} \cdot \mathbf{h}^\prime$, diagnosed from a simulation or measured in an experiment, will contain the beta term. We expect that it will appear only when the spatial averaging is over scales much larger than the Rhines length. Since these scales are well resolved even in coarse-resolution simulations, this suggests that a parameterization for mesoscale turbulence need not correspond exactly with the PG limit, which is the only regime of the flow where the beta term must necessarily appear in $\mathbf{u}^*$.

6. Energy and enstrophy balances in the stochastic theory

a. Transport of enstrophy and tracer variance

The transport equations for the mean and eddy enstrophy $\hat{q}^2/2$ and $\hat{q}^2/2$ are given by

$$
\partial_t \hat{q}^2/2 + \mathbf{u} \cdot \nabla \hat{q}^2/2 = -\mathbf{u} \cdot \mathbf{q}^\prime \cdot \nabla \hat{q} - \frac{1}{\hat{h}} \nabla \cdot \mathbf{h} \mathbf{u} \cdot \mathbf{q}^\prime
$$

$$
= -\frac{\hat{q}}{\hat{h}} \nabla \cdot \mathbf{h} \mathbf{u} \cdot \mathbf{q}^\prime \tag{104}
$$

$$
\partial_t \hat{q}^2/2 + \mathbf{u} \cdot \nabla \hat{q}^2/2 = -\mathbf{u} \cdot \mathbf{q}^\prime \cdot \nabla \hat{q} - \frac{1}{\hat{h}} \nabla \cdot \mathbf{h} \mathbf{u} \cdot \mathbf{q}^\prime
$$

$$
= \frac{\hat{q}}{\hat{h}} \nabla \cdot \mathbf{h} \mathbf{u} \cdot \mathbf{q}^\prime - \frac{1}{\hat{h}} \nabla \cdot \mathbf{h} \mathbf{u} \cdot (\mathbf{q}^\prime)^2/2, \tag{105}
$$

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where the last line follows by using the identity \( q^2 = (q^2)^* - 2 \overline{q} q^* + \overline{q}^2 \) to express the triple correlation as the difference of two double correlations. In the stochastic theory the transport equations become

\[
\begin{align*}
\partial_t q^2/2 + (\dot{u} + \mathbf{U}^*) \cdot \nabla \overline{q}^2/2 &= -\nabla \overline{q} \cdot \mathbf{K} \cdot \nabla \overline{q} + \mathcal{R}(\overline{q}^2/2) \\
(106) \\
\partial_t \overline{q}^2/2 + (\dot{u} + \mathbf{U}^*) \cdot \nabla \overline{q}^2/2 &= +\nabla \overline{q} \cdot \mathbf{K} \cdot \nabla \overline{q} + \mathcal{R}(\overline{q}^2/2). \\
(107)
\end{align*}
\]

These are exact results derived using the fact that enstrophy is conserved to evaluate the last term in (105) in the stochastic theory using (67) with \( q^* \). The diffusion terms \( \mathcal{R}(\overline{q}^2/2) \) and \( \mathcal{R}(\overline{q}^2/2) \) represent positive-definite diffusion of mean and eddy enstrophy. The term \( \nabla \overline{q} \cdot \mathbf{K} \cdot \nabla \overline{q} \) is a conversion from mean to eddy enstrophy that is positive at each point in space since \( \mathbf{K} \) is positive definite, so in domain average there is a sign-definite sink from mean to eddy enstrophy. In a statistical steady state the last integral must vanish, and, if \( \mathbf{K} \neq 0 \), this implies \( \nabla \overline{q} = 0 \). Therefore, as in the theory of Rhines and Young (1982), the turbulence drives the system toward a state of horizontally uniform potential vorticity. Note this does not imply that an unforced system must relax toward a state of uniform \( \overline{q} \). If the turbulent flow decreases as the system spins down, then \( \mathbf{K} \) will decrease as well. If \( \mathbf{K} \) vanishes in the final state, then \( \nabla \overline{q} \) need not vanish, and the system can approach a final state of minimum available potential energy with \( \nabla \varepsilon = 0 \). In realistic flow the boundary fluxes should be considered in the overall balances. In general, the boundary conditions are such that \( \nabla \overline{q} \) cannot vanish everywhere, and if \( \mathbf{K} \neq 0 \) in a statistical steady state, then something else must balance the source term \( \nabla \overline{q} \cdot \mathbf{K} \cdot \nabla \overline{q} \) in domain average: either it is balanced by boundary fluxes of \( \overline{q} \) or there is a friction term on the rhs of (107) that dissipates eddy enstrophy in the interior. The need for dissipation of eddy enstrophy when the \( \overline{q} \) flux has a component down the mean gradient was originally pointed out by Rhines and Holland (1979).

### b. Energy conversions

In isopycnal coordinates the kinetic energy per unit volume is \( \rho u^2/2 \) (here \( u^2 \) is shorthand notation for \( \mathbf{u} \cdot \mathbf{u} \)), and the available potential energy is \( g(z - z_o)^2/2 \), where \( z_o = z_o(\rho) \) is the depth of the isopycnal with density \( \rho \) in the minimum potential energy state obtained by adiabatically redistributing the fluid to a state with flat isopycnals. The transport equations for the mean and eddy available potential energy and kinetic energy are derived in appendix C, which closely follows the derivation by Hallberg (1995) for the discrete-layer case. The mean kinetic energy (MKE), eddy kinetic energy (EKE), mean available potential energy (MPE), and eddy available potential energy (EPE) are defined here as

\[
\begin{align*}
\text{MKE:} & \quad \rho \overline{u^2}/2 \\
\text{EKE:} & \quad \rho \overline{\mathbf{u} \cdot \mathbf{u}} - \overline{\rho \overline{u^2}} \\
\text{MPE:} & \quad g(z - z_o)^2/2 \\
\text{EPE:} & \quad g\overline{z_o^2}/2.
\end{align*}
\]

The conversion rates between these components of the energy are identified by integrating the energy transport equations over the domain and assuming no fluxes of energy through the boundaries. Figure 1a shows a schematic diagram of the conversion rates. They are given by

![Fig. 1. Conversion rates between mean and eddy kinetic and available potential energy in isopycnal coordinates with two different choices for the individual conversion rates. Only the net conversion of each component of the energy is uniquely defined.](image)
The theory predicts downgradient diffusion of tracers and potential vorticity with a single symmetric positive-definite diffusivity tensor, and transport of tracers with a bolus velocity involving downgradient thickness-diffusion with the same diffusivity tensor. The closure problem is reduced to the determination of this $2 \times 2$ symmetric tensor $\mathbf{K}$ and one scalar field $\psi$ related to the eddy kinetic energy.

The rotational components of the eddy fluxes have traditionally been either overlooked or ignored in deriving eddy parameterizations, in spite of the fact that they are generally larger than the corresponding divergent components. In the closure scheme proposed here, the only unknown rotational eddy flux that appears in the mean PEs is the gauge term $\mathbf{k} \cdot \nabla \psi_\alpha$ associated with the eddy flux of relative vorticity. The equations do not depend on other rotational gauge fields associated with the eddy fluxes of thickness and passive tracers, and hence these need not be parameterized.

The question remains how to fully close the equations by parameterizing $\psi$ and $\mathbf{K}$ in terms of the mean prognostic variables. Larichev and Held (1995) and Visbeck et al. (1997) have suggested forms for the spatial dependence of an isotropic scalar diffusivity $\kappa$ based on the earlier work of Green (1970) and Stone (1972). It may be possible to extend these ideas to parameterize the kinetic energy term $\mathbf{u} \cdot \mathbf{u}^{\prime}/2$ as well. The gauge field $\psi_\alpha$ could pose a more difficult problem, but as discussed in section 4, it is $O(\xi_2^3)$ and may be negligible away from the equator if $\mathbf{u}^{\prime} \cdot \mathbf{u}^{\prime}/2$ is well and if the hypothesis (70) is valid. Parameterizations of these correlations should be guided by isopycnally averaged statistics from realistic eddy-resolving simulations and from idealized simulations such as those of Lee et al. (1997). In particular, the role of the gauge fields and the validity of (70) are key questions that need to be investigated in this context.

An important issue that has not been discussed here is the nature of the boundary conditions near the surface mixed layer and the ocean floor. In principle the effect of topographic stress in adiabatic stratified turbulence can be included in the stochastic theory through the gradient of the ocean depth $H$, which modifies the thickness of isopycnal layers where they meet topography. Random turbulent motion in an adiabatic layer overlying sloping topography should produce an upslope motion that would generate rectified alongslope mean geostrophic flow with both barotropic and baroclinic components. Greatbatch and Li (1997) have shown that including this effect in a barotropic model produces flow similar to the barotropic Neptune parameterization proposed by Holloway (1997). The problem with this approach is that near a sloping boundary the gradient of layer thickness remains constant while the thickness itself approaches zero. If $\mathbf{K}$ is roughly constant near topography, this leads to a singularity in the bolus velocity or, equivalently, an unphysical thickness flux that does not vary with layer thickness. The stochastic theory as
developed here cannot be strictly applied in this context because the boundary conditions were not properly applied to the Fokker–Planck equation; it was derived assuming random turbulent motion in an isopycnal layer without considering the pressure forces that would tend to oppose upwelling near topography. A more careful consideration of the boundary conditions in this situation could lead to a nonsingular parameterization of the upwelling bolus velocity.

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APPENDIX A

\( \mathbf{\tilde{u}} \mathbf{\xi}' \) in the Geostrophic-Scale Regime

Here we show that \( \mathbf{u}_\xi = \mathbf{\tilde{u}} \mathbf{\xi}' / (\mathbf{f} + \mathbf{\tilde{\xi}}) \) is \( O(\varepsilon) \) relative to the mean velocity \( \mathbf{\tilde{u}} \). Decomposing the velocity into its geostrophic and ageostrophic components on a \( \beta \) plane \( (f \to f_o + \beta y) \), we have

\[
\begin{align*}
\mathbf{u} &= \mathbf{u}_g + \mathbf{u}_a, \\
\mathbf{u}_g &= k \times \mathbf{\nabla} (M/f_o), \\
\mathbf{u}_a &= -\frac{\beta y}{f_o} \mathbf{u}_g + \frac{1}{f_o} k \times [\partial \mathbf{u}_g + \mathbf{u}_a \cdot \mathbf{\nabla} \mathbf{u}_g],
\end{align*}
\]

where \( \mathbf{u}_a \) is the leading \( O(\varepsilon) \) ageostrophic contribution to the velocity (Gill 1982, 498). Thus \( \mathbf{u}_\xi \) can be expressed to leading order as

\[
\mathbf{u}_\xi = \frac{1}{f_o} \left( \mathbf{u}_g \mathbf{\tilde{\xi}} + \mathbf{u}_a \mathbf{\tilde{\xi}} + \mathbf{u}_j \mathbf{\tilde{\xi}} \right).
\]

For any rotational velocity field the velocity–vorticity correlation vanishes for sufficiently large spatial averaging (DG). Since the geostrophic velocity \( \mathbf{u}_g \) is rotational, this implies

\[
\mathbf{u}_g \mathbf{\tilde{\xi}} \to 0.
\]

So the first term on the right in (A4) vanishes. The second term on the right is

\[
\frac{1}{f_o} \mathbf{\tilde{u}} \mathbf{\tilde{\xi}} = \frac{\beta y}{f_o} \mathbf{\tilde{u}} \mathbf{\tilde{\xi}} \mathbf{\tilde{\xi}} + \frac{1}{f_o} \mathbf{\tilde{u}} \mathbf{\tilde{\xi}} \mathbf{\nabla} \cdot (\mathbf{u}_a \mathbf{\nabla} \mathbf{u}_g),
\]

where the long overbars also denote isopycnal averages \( \langle \cdot \rangle = \mathbf{\tilde{\cdot}} \). With geostrophic scaling, \( \mathbf{\tilde{u}} \sim U, \mathbf{u}_g \sim U', \mathbf{\nabla} \mathbf{\tilde{u}} \sim \mathbf{\tilde{\xi}} \sim U' \mathbf{L}', \) and \( \partial \mathbf{\tilde{u}} \mathbf{\tilde{\xi}} \sim U' \mathbf{L}' \mathbf{\tilde{\xi}} \); here primed (unprimed) quantities indicate the scales of turbulent (mean) flow. For baroclinically unstable flow at mid-latitudes (e.g., the Gulf Stream), we take \( U' \sim U, \) and \( L' \sim R_1, \) the first baroclinic Rossby radius. The first term on the rhs of (A6) scales like \( \beta U'^2 / f_o^2 = \varepsilon_f R_1 U \), and the second term scales like \( U'^3 / f_o^2 R_1^2 = R_1 U \). Therefore \( \mathbf{\tilde{u}} \mathbf{\tilde{\xi}} / f_o \mathbf{\tilde{\xi}} = O(\varepsilon_f) \) relative to the mean velocity \( \mathbf{\tilde{u}} \). A similar analysis shows that the last term in (A4), \( \mathbf{u}_j \mathbf{\tilde{\xi}} / f_o \mathbf{\tilde{\xi}} \), is the same order, and therefore \( \mathbf{u}_j \mathbf{\tilde{\xi}} = O(\varepsilon_f) \), as stated in (40).

APPENDIX B

Transformation to Level Coordinates

The transformation of the ensemble-mean thickness-weighted PEs (1)–(6) from isopycnal coordinates \( (\mathbf{x}, \rho, t) \) to \( \mathbf{z} \) coordinates \( (\mathbf{x}, \mathbf{z}, \mathbf{t}) \) is performed in DB (1993). (In this appendix we retain the tilde to denote isopycnal coordinates.) The transformation of the stochastic theory form of the PEs (75)–(80) is similar. Here \( \mathbf{\tilde{x}} = \mathbf{x}, \mathbf{\tilde{t}} = \mathbf{t}, \) and \( \rho = \tilde{\rho}(\mathbf{x}, \mathbf{z}, \mathbf{t}) \), where the function \( \tilde{\rho} \) at fixed \( \mathbf{x} \), \( \mathbf{t} \) is the inverse of the function \( \tilde{z}(\mathbf{x}, \rho, \mathbf{t}) \), the mean height of isopycnal \( \rho \). The derivatives transform according to

\[
\begin{align*}
\partial_\rho &= \frac{1}{\tilde{\rho}_z}, \\
\mathbf{\nabla} &= \mathbf{\nabla}_z + \mathbf{L} \partial_\mathbf{z}, \quad \mathbf{L} = -\mathbf{\nabla} \tilde{\rho} / \tilde{\rho}_z \\
\partial_\mathbf{z} &= \partial_\mathbf{z} - \tilde{\rho}_z \partial_\mathbf{z},
\end{align*}
\]

where \( \mathbf{L} \) is the slope vector. Applying these transformations to Eqs. (76)–(80), we find

\[
\begin{align*}
\mathbf{\nabla} \cdot \mathbf{V}_3 &= 0, \quad \mathbf{V}_3 = (\mathbf{V}, \mathbf{W}), \quad \mathbf{V} = \mathbf{\tilde{u}} + \mathbf{U}_\xi \\
\widehat{D}_z \phi &= \mathbf{\nabla}_z \cdot \mathbf{K}_3 \mathbf{\nabla}_z \phi = \mathbf{\tilde{K}}_3 \phi \\
\partial_\mathbf{t} \tilde{\rho} &= -\tilde{\rho}_z \\
\widehat{D}_z \rho &= 0 \\
\widehat{D}_z \mathbf{v} &= \partial_\mathbf{z} + \mathbf{V} \cdot \mathbf{\nabla} = \partial_\mathbf{z} + \mathbf{V}_3 \cdot \mathbf{\nabla}_z, \quad \mathbf{W} = \mathbf{\tilde{V}}_3.
\end{align*}
\]

where \( W \) is the vertical component of the three-dimensional tracer transport velocity. The horizontal component of the bolus velocity (82) transforms to

\[
\mathbf{U}_\xi = -\mathbf{K} \cdot \partial_\mathbf{z} \mathbf{L}.
\]

This is similar to the GM form (10) except that the diffusion tensor has not been assumed isotropic and is outside the \( \mathbf{z} \) derivative. The \( 3 \times 3 \) tensor \( \mathbf{K}_3 \) is given by

\[
\mathbf{K}_3 = \begin{pmatrix} \mathbf{K} & \mathbf{K} \cdot \mathbf{L} \\ \mathbf{K} \cdot \mathbf{L} & \mathbf{L} \cdot \mathbf{K} \cdot \mathbf{L} \end{pmatrix}.
\]

In the limit \( \mathbf{K} \to \kappa \mathbf{I} \), \( \mathbf{K}_3 \) reduces to the small-slope version of the Redi diffusion tensor [Eq. (13) of Gent et al. 1995] as in (9) with \( \mu = \kappa \). It reduces to the full
Redi tensor in the limit $\mathbf{K} \rightarrow \kappa \mathbf{J}$, where $\mathbf{J}$ is a $2 \times 2$ matrix that arises from transforming to a local orthogonal coordinate system with horizontal axes tangent to the local isopycnal surface (DS).

The momentum equation (75) can be rewritten using (B8) as

$$\tilde{D} \tilde{u} - \mathbf{U}^* \cdot \nabla \tilde{u} + f \mathbf{k} \times \tilde{u} + \nabla \tilde{M} = -\nabla \tilde{\psi} + \mathbf{k} \times \mathbf{K} \cdot \nabla (f + \tilde{\zeta}),$$  \hspace{1cm} (B11)

which transforms to

$$\begin{align*}
\partial_t \tilde{u} + (V^*_h - V^*_z) \cdot \nabla \tilde{u} + f \mathbf{k} \times \tilde{u} + \nabla \tilde{M} \tilde{\rho} = -\nabla \tilde{\psi} + \mathbf{k} \times \mathbf{K} \cdot (\nabla \tilde{\psi} + \nabla \partial_z(f + \tilde{\zeta})) \hspace{1cm} (B12)
\end{align*}$$

Therefore, in the z-coordinate PEs (B4)–(B8) and (B12) the only vertical velocity that appears is the vertical component of the full transport velocity $W$. It is not necessary to separately define vertical components for $\tilde{u}$ and $\mathbf{U}^*$ as in (8).

The condition of no normal flow through the ocean floor leads to the boundary condition

$$W = -\mathbf{V} \cdot \nabla_H z \hspace{1cm} \text{at} \hspace{0.5cm} z = -H. \hspace{1cm} (B16)$$

To determine the vertical velocity, the continuity equation (B4) should be integrated up from the bottom with this boundary condition.

**APPENDIX C

Energy Balances in Isopycnal Coordinates**

The energy conversion rates (113)–(116) for the adiabatic system can be obtained directly from the transport equations for mean and eddy kinetic and available potential energy. The (unaveraged) kinetic energy transport equation is derived by taking the dot product of $\tilde{u}^H$ with the momentum equation (1) and subtracting $u^H/2$ times the continuity equation (2) to obtain

$$\partial_t (\tilde{h}u^H/2) + \nabla \cdot [\tilde{h} \mathbf{u} (\tilde{h}u^H/2)] + \tilde{h} \mathbf{u} \cdot \nabla M = 0. \hspace{1cm} (C1)$$

The transport equation for the mean kinetic energy (109) is similarly obtained using the mean momentum and continuity equations (24) and (19):

$$\begin{align*}
\partial_t (\tilde{h}u^H/2) + \nabla \cdot [(\tilde{h} \mathbf{u} + \tilde{u}^H \tilde{h}^H/2) - \tilde{u}^H \tilde{h}^H \mathbf{u} \cdot \nabla M = 0. \hspace{1cm} (C2)
\end{align*}$$

The condition of no normal flow through the ocean floor leads to the boundary condition

$$W = -\mathbf{V} \cdot \nabla_H z \hspace{1cm} \text{at} \hspace{0.5cm} z = -H. \hspace{1cm} (B16)$$

To determine the vertical velocity, the continuity equation (B4) should be integrated up from the bottom with this boundary condition.

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$$\partial_t (\tilde{h}u^H/2) + \nabla \cdot [\tilde{h} \mathbf{u} (\tilde{h}u^H/2)] + \tilde{h} \mathbf{u} \cdot \nabla M = 0. \hspace{1cm} (C1)$$

The transport equation for the mean kinetic energy (109) is similarly obtained using the mean momentum and continuity equations (24) and (19):

$$\begin{align*}
\partial_t (\tilde{h}u^H/2) + \nabla \cdot [(\tilde{h} \mathbf{u} + \tilde{u}^H \tilde{h}^H/2) - \tilde{u}^H \tilde{h}^H \mathbf{u} \cdot \nabla M = 0. \hspace{1cm} (C2)
\end{align*}$$

The transport equation for the eddy kinetic energy (110) is obtained by subtracting (C2) from the mean of (C1), resulting in

$$\begin{align*}
\partial_t (\tilde{h}u^H/2) + \nabla \cdot [\tilde{h} \mathbf{u} (\tilde{h}u^H/2)] + \tilde{h} \mathbf{u} \cdot \nabla M = 0. \hspace{1cm} (C3)
\end{align*}$$

The (unaveraged) available potential energy transport equation is derived as follows. Define

$$\pi = [p + \rho g (z - z_o)]/\rho_o, \hspace{1cm} (C4)$$

and note that $\nabla \pi = \nabla M$. Then take the $\rho$ derivative of $\pi$ times the continuity equation (2) to obtain

$$\partial_t g(z - z_o)^2/2 + \nabla \cdot h (\tilde{h} \mathbf{u} \pi - \partial_z (\tilde{h} \mathbf{u} \pi) = \rho g h \partial_z \pi - \partial_\rho \rho g \mathbf{u} \cdot \nabla M = 0. \hspace{1cm} (C5)$$

Finally, the transport equation for the eddy available potential energy (112) is obtained by subtracting (C6) from the mean of (C5), resulting in

$$\begin{align*}
\partial_t g(z - z_o)^2/2 + \nabla \cdot h (\tilde{h} \mathbf{u} \pi - \partial_z (\tilde{h} \mathbf{u} \pi) - \mathbf{u} \cdot \nabla M = 0. \hspace{1cm} (C7)
\end{align*}$$

The energy balances in a statistical steady state result from dropping all terms in (C2), (C3), (C6), and (C7) that involve time derivatives of mean variables. The energy conversion rates (113)–(116) are then obtained by integrating the remaining terms over volume and dropping surface terms. These results are essentially equivalent to the discrete-layer balances and conversion rates derived by Hallberg (1995); he also includes diabatic forcing terms in the momentum and continuity equations.

It should be noted that the energy transport equations depend on how the kinetic energy is defined. If we had chosen to define the mean momentum as $\tilde{u}$ and used the $h$-averaged momentum equation (25) rather than (24), then the MKE and EKE would be more naturally defined as

$$\begin{align*}
\text{MKE:} & \hspace{0.5cm} \rho \tilde{h}u^2/2 \hspace{1cm} (C8) \\
\text{EKE:} & \hspace{0.5cm} \rho \tilde{h}u^2/2. \hspace{1cm} (C9)
\end{align*}$$

For completeness, we give the conversion rates for this case:
MKE → MPE: \( \mathbf{T}_1 = \int dV \mathbf{u} \cdot \nabla \mathbf{M} \) \hspace{1cm} (C10)

MKE → EPE: \( \mathbf{T}_2 = \int dV \mathbf{u} \cdot (\nabla \mathbf{M})^* \) \hspace{1cm} (C11)

EKE → EPE: \( \mathbf{T}_3 = \int dV \mathbf{u} \cdot \nabla \mathbf{M}^* \) \hspace{1cm} (C12)

MKE → EKE: \( \mathbf{T}_4 = \int dV [\mathbf{u} \cdot \nabla \mathbf{M}]^* + \mathbf{\hat{u}} \cdot (\nabla \mathbf{M})^* : \mathbf{u}' \mathbf{u}''] \). \hspace{1cm} (C13)

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