

NOTES AND CORRESPONDENCE

Internal Boundary Layer Scaling in “Two Layer” Solutions of the Thermocline Equations

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ABSTRACT

The diffusivity dependence of internal boundary layers in solutions of the continuously stratified, diffusive thermocline equations is revisited. If a solution exists that approaches a two-layer solution of the ideal thermocline equations in the limit of small vertical diffusivity κ_v , it must contain an internal boundary layer that collapses to a discontinuity as $\kappa_v \rightarrow 0$. An asymptotic internal boundary layer equation is derived for this case, and the associated boundary layer thickness is proportional to $\kappa_v^{1/2}$. In general, the boundary layer remains three-dimensional and the thermodynamic equation does not reduce to a vertical advective–diffusive balance even as the boundary layer thickness becomes arbitrarily small. If the vertical convergence varies sufficiently slowly with horizontal position, a one-dimensional boundary layer equation does arise, and an explicit example is given for this case. The same one-dimensional equation arose previously in a related analysis of a similarity solution that does not itself approach a two-layer solution in the limit $\kappa_v \rightarrow 0$.

1. Internal boundary layer scaling

Stommel and Webster (1962) discovered a similarity solution of the thermocline equations with an internal boundary layer that could be interpreted as a model of the subtropical main thermocline. The internal boundary layer marks the base of the wind-driven motion, as the deeper circulation is driven by vertical diffusion of heat through the internal boundary layer. The characteristic thickness of the Stommel–Webster internal boundary layer is $\kappa_v^{1/2}$, where κ_v is a constant vertical diffusivity. Originally obtained by a linearized analysis, this scaling was confirmed by Young and Ierley (1986) using matched asymptotic expansions. It contrasts with the $\kappa_v^{1/3}$ thickness dependence that follows from the traditional advective–diffusive scaling of the thermocline equations (Welander 1971).

The $\kappa_v^{1/2}$ boundary layer is evidently not peculiar to the particular similarity form studied by Stommel and Webster (1962) and Young and Ierley (1986). Salmon (1990) showed that a $\kappa_v^{1/2}$ internal boundary layer should generally arise in subtropical-gyre solutions of the thermocline equations, although this result depended on a Taylor-series argument that itself relied on an assumption

that the vertical convergence in the boundary layer was constant and nonzero. A related discussion was given by Pedlosky (1979, p. 422). (As a reviewer has pointed out, these same arguments establish that a $\kappa_v^{1/2}$ scaling will generally arise in a stationary boundary layer when the advected scalar is passive rather than active, since constant vertical convergence is then generic.) Samelson and Vallis (1997) suggested that the $\kappa_v^{1/2}$ scaling will arise whenever isotherm slopes in the thermocline are fixed as $\kappa_v \rightarrow 0$ and reported evidence for a $\kappa_v^{1/2}$ scaling in numerical solutions of a closed-basin planetary geostrophic circulation model, despite clear differences between the horizontal structure of the numerical solutions and the Stommel–Webster similarity solution.

The present contribution should be read as a footnote to the articles cited above. In essence, it is a modest extension of the argument of Salmon (1990), cast in a different form. The starting point is a two-layer solution of the ideal ($\kappa_v = 0$) equations, in which temperature is discontinuous across the interface between the layers and the detailed structure of the wind forcing is not specified. A general internal boundary layer equation is then derived that must be satisfied asymptotically by any smooth solution of the diffusive ($\kappa_v > 0$) thermocline equations that approaches the two-layer ideal solution as $\kappa_v \rightarrow 0$. This point of view resembles that of Young and Ierley (1986), who interpret the ideal limit of the Stommel–Webster solution as a weak (discontinuous) solution of the continuously stratified ideal thermocline equations. Here, it is generally assumed that

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the relevant smooth solutions exist, but one explicit example is given.

2. The two-layer limit

Following Welander (1971), the dimensionless continuously stratified steady diffusive thermocline equations may be written

$$-f^{-1}M_{zy}M_{zzx} + f^{-1}M_{zx}M_{zzy} + \beta f^{-2}M_x M_{zzz} = \kappa_v M_{zzzz}, \tag{2.1}$$

where $M(x, y, z)$ satisfies

$$M_z = p. \tag{2.2}$$

The temperature T and velocities (u, v, w) may be written as derivatives of M ,

$$\begin{aligned} T &= M_{zz}, & u &= -f^{-1}M_{zy}, & v &= f^{-1}M_{zx}, \\ w &= \beta f^{-2}M_x, \end{aligned} \tag{2.3}$$

expressing the hydrostatic, geostrophic, and Sverdrup vorticity balances, where the additional boundary conditions $w, M \rightarrow 0$ as $z \rightarrow -\infty$ (or $w = M = 0$ at the bottom $z = -H_B$) have been enforced. The Sverdrup transport relation takes the form

$$\beta f^{-2}M_x(x, y, 0) = w_E, \tag{2.4}$$

where w_E is the Ekman vertical velocity at the base of the surface boundary layer.

For general $w_E < 0$, the ideal ($\kappa_v = 0$) thermocline equations have a two-layer (or ‘‘one-and-a-half-layer’’) solution, with an upper, moving layer of thickness $h(x, y)$ and uniform temperature $T = T_0$ overlying a deep motionless layer of temperature $T = 0$. The thickness of the moving layer is

$$h(x, y) = \left(H_0^2 + \frac{2f^2}{\beta T_0} \int_{x_e}^x w_E(x, y) dx \right)^{1/2}, \tag{2.5}$$

where H_0 is the depth of the upper layer at the eastern boundary. For example, such solutions have been considered by Parsons (1969) and Veronis (1973) and are equivalent to a ventilated thermocline (Luyten et al. 1983) with a single moving layer.

Now, for $0 < \kappa_v \ll 1$, suppose that there exists a solution of the diffusive thermocline equations (2.1) that matches the two-layer solution (2.5) except near $z = -h(x, y)$ where a smooth transition across an internal boundary layer of finite thickness replaces the discontinuity. The analysis below shows that any such solution must asymptotically satisfy an internal boundary layer equation following from (2.1).

To derive this general internal boundary layer equation, it is convenient to introduce the stretched boundary layer coordinate ζ , where

$$\zeta = \delta^{-1}(z + h(x, y)) \tag{2.6}$$

is the vertical distance from $z = -h$ scaled by the un-

known boundary layer thickness δ . The appropriate matching conditions on T outside the internal boundary layer are then $T \rightarrow T_0$ as $\zeta \rightarrow +\infty$ and $T \rightarrow 0$ as $\zeta \rightarrow -\infty$. In order to write the corresponding boundary conditions for $M_{\zeta\zeta}$ in a form that is independent of δ , it is necessary to rescale M in the boundary layer by the substitution

$$M(x, y, z) = \delta^2 T_0 A(x, y, \zeta), \tag{2.7}$$

where the absence of order-1 and order- δ contributions to M is consistent with the requirement that M vanish for $z < -h$ when $\kappa_v = 0$, and with the matching conditions on T . Then $M_{zz} = T_0 A_{\zeta\zeta}$, and the boundary conditions for A are

$$A_{\zeta\zeta} \rightarrow \begin{cases} 1 & \text{as } \zeta \rightarrow +\infty \\ 0 & \text{as } \zeta \rightarrow -\infty. \end{cases} \tag{2.8}$$

The matching conditions on w are $w \rightarrow (z/h + 1)w_E = \delta(\zeta/h)w_E$ as $\zeta \rightarrow +\infty$ and $w \rightarrow \text{const}$ as $\zeta \rightarrow -\infty$. The first of these matches the wind-driven vertical velocity above the internal boundary layer, while the second will give the diffusively driven abyssal upwelling velocity beneath the boundary layer, which must vanish along with the abyssal M as $\kappa_v \rightarrow 0$. Since $M_x = \delta T_0 (A_\zeta h_x + \delta A_x)$, to first order in δ these are

$$A_\zeta \rightarrow \begin{cases} \zeta & \text{as } \zeta \rightarrow +\infty \\ c & \text{as } \zeta \rightarrow -\infty, \end{cases} \tag{2.10}$$

$$\tag{2.11}$$

consistent with (2.8) and (2.9), where c is a constant and use has been made of (2.5).

With the substitution (2.7), the resulting equation for A is

$$\begin{aligned} f^{-1}[h_x(A_{\zeta\zeta y}A_{\zeta\zeta} - A_{\zeta\zeta\zeta}A_{\zeta y}) - h_y(A_{\zeta\zeta x}A_{\zeta\zeta} - A_{\zeta\zeta\zeta}A_{\zeta x})] \\ + \beta f^{-2}h_x A_\zeta A_{\zeta\zeta\zeta} + \delta[f^{-1}(A_{\zeta x}A_{\zeta\zeta y} - A_{\zeta y}A_{\zeta\zeta x}) + \beta f^{-2}A_x A_{\zeta\zeta\zeta}] \\ = \delta^{-2}T_0^{-1}\kappa_v A_{\zeta\zeta\zeta\zeta}. \end{aligned} \tag{2.12}$$

Balancing the diffusive term against the leading-order advective terms in (2.12) leads to $\delta^2 \propto \kappa_v$. Thus, the internal boundary layer in the two-layer limit should generally have thickness proportional to $\kappa_v^{1/2}$. Presumably, an appropriate solution of (2.12) exists. In general, it is not immediately clear how to solve (2.12). The leading horizontal advective terms contain first-order horizontal derivatives, and appropriate lateral boundary conditions must be determined. Note that it is the need to match to constant temperatures above and below the boundary layer that determines the rescaling (2.7) and ultimately requires $\delta \propto \kappa_v^{1/2}$.

Since the form (2.7) fixes the isotherm slopes independently of δ ($T_x/T_z = h_x, T_y/T_z = h_y$, to leading order), this result is consistent with the scaling argument of Samelson and Vallis (1997), who presented numerical evidence for a $\kappa_v^{1/2}$ thickness in the central subtropical gyre of a planetary geostrophic circulation model and suggested that the $\kappa_v^{1/2}$ thickness should arise whenever

isotherm slopes are fixed as $\kappa_v \rightarrow 0$. Because of the relative horizontal uniformity of the fluid immediately above (“subtropical mode water” analog) and below (abyssal fluid) the internal boundary layer in the numerical solutions of Samelson and Vallis (1997), the two-layer model can reasonably serve as an approximation to the numerical solutions near the internal boundary layer, despite the existence of a strongly stratified portion of the ventilated thermocline near the surface.

3. A one-dimensional equation

One might expect that the substitution (2.7) would lead to an asymptotic boundary layer equation involving only ζ derivatives of A in the limit $\delta \rightarrow 0$, corresponding to the thermodynamic balance $wT_z \approx \kappa_v T_{zz}$. However, this does not happen. The horizontal advective terms of order δ^{-1} that arise from the substitution (2.7) vanish identically from (2.12), but the leading-order horizontal advective terms that remain in (2.12) are still of order 1, the same order as the leading-order vertical advective term. Consequently, the thermodynamic balance does not, in general, reduce to $wT_z \approx \kappa_v T_{zz}$ as $\kappa_v \rightarrow 0$. This might be anticipated from the observation that the vertical velocity itself vanishes in this limit.

Special solutions of (2.12) may still be sought in which horizontal advection does vanish, either identically or to leading order. The matching conditions for A are themselves independent of x and y . In the case of (2.10), this reduction is possible because the Sverdrup relation enforces a proportionality between h_x and w_E/h at each point. This suggests the substitution $A(x, y, \zeta) = B(\zeta)$ in (2.12), or $M(x, y, z) = \delta^2 T_0 B(\zeta)$. If the resulting equation for $B(\zeta)$ were independent of x and y , then a one-dimensional boundary layer theory would exist in the two-layer limit. This substitution gives

$$\beta f^{-2} h_x T_0 \frac{dB}{d\zeta} \frac{d^3 B}{d\zeta^3} = \kappa_v \delta^{-2} \frac{d^4 B}{d\zeta^4}. \tag{3.1}$$

Comparison with (2.1) shows that this is a vertical advective–diffusive balance in which the term M_x is replaced by $M_\zeta h_x$, a consequence of fixing the isotherm slopes to first order. The equality in (3.1) can in general be satisfied only if the quantity $\kappa_v f^2 / (\beta h_x T_0)$ is constant since the latter is clearly independent of ζ while by assumption B depends only on ζ . From (2.5),

$$\frac{\kappa_v f^2}{\beta h_x T_0} = \frac{\kappa_v h}{w_E}, \tag{3.2}$$

but inspection shows that h/w_E is not generally constant. Thus, an asymptotic one-dimensional internal boundary layer theory for the two-layer limit of (2.1) does not exist in general. A solution may still exist that approaches the two-layer solution in the limit $\kappa_v \rightarrow 0$, and, if it exists, it must have $\delta \propto \kappa_v^{1/2}$, but the corresponding internal boundary layer will in general not be indepen-

dent of horizontal position; that is, it will remain intrinsically three-dimensional.

If the expression (3.2) were constant, the equation (3.1) would reduce to the equation ultimately solved by Young and Ierley (1986) in their asymptotic analysis of the similarity solution discovered by Stommel and Webster (1962). This can be seen as follows. Set

$$\delta = \left(\frac{\kappa_v f^2}{\beta h_x T_0} \right)^{1/2} = \left(\frac{\kappa_v h}{w_E} \right)^{1/2} \tag{3.3}$$

and let

$$F(\zeta) = \frac{dB}{d\zeta}. \tag{3.4}$$

Then (3.1) is

$$FF'' = F''' \tag{3.5}$$

and the boundary conditions (2.10) and (2.11) are

$$F(\zeta \rightarrow +\infty) \rightarrow \zeta, \quad F'(\zeta \rightarrow -\infty) \rightarrow 0, \tag{3.6}$$

where prime denotes derivative with respect to the argument. The two conditions (3.6) are sufficient for the third-order equation (3.5) since the first also implies $F'(\zeta \rightarrow +\infty) \rightarrow 1$. The abyssal upwelling velocity is determined by the value c , where $F(\zeta \rightarrow -\infty) \rightarrow c$, since $w(\zeta \rightarrow -\infty) \rightarrow c \delta w_E / h$.

The equation (3.5) with the boundary conditions (3.6) is solved by Young and Ierley (1986) in their analysis of the Stommel–Webster similarity solution (with the sign of ζ reversed). Their solution yields $c = 0.87574$. Note that the Stommel–Webster similarity solution is not of “two-layer” type: it retains zonal temperature gradients above and below the internal boundary layer even in the limit $\kappa_v \rightarrow 0$. Near the internal boundary layer, the two-layer model is a more accurate representation of the numerical solutions of Samelson and Vallis (1997) than is the Stommel–Webster similarity solution because of the large zonal temperature gradients in the latter and the dependence of the thermocline depth in the numerical solutions on horizontal position, which roughly follows the two-layer solution.

As an explicit example of a two-layer solution with this type of internal boundary layer, consider (2.5) with $H_0 = 0$ and $w_E = af^2(x - x_E)$, where a is a constant. In this case, w_E/h is constant, and (3.1) is independent of x and y and has the form (3.5), with boundary conditions (3.6). This is a two-layer solution with a one-dimensional internal boundary layer that has thickness $\delta \propto \kappa_v^{1/2}$ and is independent of horizontal position. In this case, the thermodynamic balance in the boundary layer does reduce to $wT_z \approx \kappa_v T_{zz}$.

Although (3.5) is formally valid only if δ from (3.3) is constant, as in the preceding example, it may still provide a useful approximation to M if δ is only approximately constant. If

$$(\delta_x, \delta_y) / \delta \sim \delta, \tag{3.7}$$

then (3.5) will be the correct leading-order approximation, and (3.7) will in turn be satisfied if

$$|\nabla(w_E/h)/(w_E/h)| \sim \delta. \quad (3.8)$$

That is, the accuracy of the approximation will be controlled by the extent to which the vertical convergence w_z in the two-layer solution is horizontally uniform. This requires that the fractional variations in w_E and h either be separately small or cancel to first order. The condition (3.7) can be recast in terms of w_E and H_0 using (2.5). If w_E is independent of x and y , and H_0 is sufficiently large, then the approximation will be accurate. If w_E is independent of x , and $h(x) - H_0 \approx H_0$, then the approximation may be inaccurate.

4. Related examples

Salmon and Hollerbach (1991) obtained some special solutions of the thermocline equations that are relevant to the present discussion. In one class of solutions (their “ S_{12} ”), the temperature changed abruptly across an internal boundary layer, as in the solution discussed above. In a second class of solutions (their “ S_{13} ”), the potential vorticity changed abruptly across an internal boundary layer, while the temperature field remained smooth as $\kappa_v \rightarrow 0$. For each of these classes, they presented specific examples for the special case in which the boundary layer is located at a constant depth, independent of horizontal position [their Eqs. (8.1)–(8.9) and (8.10)–(8.12), respectively]. In both of these specific examples, the thickness of the internal boundary layer was $\kappa_v^{1/2}$, as in the solution discussed above.

It is especially interesting that a $\kappa_v^{1/2}$ boundary layer appears also in their second example, for which the jump is in potential vorticity rather than temperature. An extrapolation of the argument given above would seem to suggest that the boundary layer thickness should be pro-

portional to $\kappa_v^{1/3}$, as in the traditional advective–diffusive scaling, since the potential vorticity $fT_z = fM_{zz}$ (rather than temperature $T = M_{zz}$) must be matched outside the boundary layer, and this would appear to lead to a factor δ^3 in the scaling (2.7). In this case, however, M in the boundary layer may have contributions of zero, first, and second order in δ , along with the third-order term associated with the potential vorticity matching. Thus, the simple extrapolation is misleading, and the boundary layer again scales with $\kappa_v^{1/2}$.

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REFERENCES

- Luyten, J., J. Pedlosky, and H. Stommel, 1983: The ventilated thermocline. *J. Phys. Oceanogr.*, **13**, 292–309.
- Parsons, A. T., 1969: A two-layer model of Gulf Stream separation. *J. Fluid Mech.*, **39**, 511–528.
- Pedlosky, J., 1979: *Geophysical Fluid Dynamics*. Springer-Verlag, 624 pp.
- Salmon, R., 1990: The thermocline as an “internal boundary layer.” *J. Mar. Res.*, **48**, 437–469.
- , and R. Hollerbach, 1991: Similarity solutions of the thermocline equations. *J. Mar. Res.*, **49**, 249–280.
- Samelson, R., and G. K. Vallis, 1997: Large-scale circulation with small diapycnal diffusion: The two-thermocline limit. *J. Mar. Res.*, **55**, 223–275.
- Stommel, H., and J. Webster, 1962: Some properties of the thermocline equations in a subtropical gyre. *J. Mar. Res.*, **44**, 695–711.
- Veronis, G., 1973: Model of the world ocean circulation, I: Wind-driven, two-layer. *J. Mar. Res.*, **31**, 228–288.
- Welander, P., 1971: The thermocline problem. *Philos. Trans. Roy. Soc. London, Series A*, **270**, 415–421.
- Young, W. R., and G. Ierley, 1986: Eastern boundary conditions and weak solutions of the ideal thermocline equations. *J. Phys. Oceanogr.*, **16**, 1884–1900.