Coastal Upwelling on a β -Plane¹

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ABSTRACT

The β effect is found to produce a poleward undercurrent in a wind-driven model of an eastern ocean circulation. The model is on an x, z plane and longshore derivatives of the velocity field are neglected. The nonlinear two-layer model is time-dependent and is solved numerically. Beta is found to exhibit little effect on the vertical mass transport, but exerts a dominant influence on the longshore flow by inducing a north-south sea surface slope.

For an equatorward wind stress and flat bottom topography, the model predicts upwelling adjacent to the coast with a mean vertical velocity of 10^{-3} to 10^{-2} cm sec⁻¹ and an e -folding width of about 15 km. The longshore flow is characterized by an equatorward surface jet and a poleward undercurrent. Outside the upwelling zone the longshore flow is weak. The offshore flow in the upper layer is slightly weaker than that predicted by Ekman drift. The compensating onshore flow in the lower layer is balanced by the north-south sea surface slope.

Positive wind stress curl in the coastal upwelling zone tends to diminish the surface jet and to enhance the poleward undercurrent. Bottom topography is shown to modify the dynamics and a secondary upwelling zone is found over the continental slope.

1. Introduction

In this paper we discuss a wind-driven model of coastal upwelling on a beta-plane. Attention is focused on the longshore flow, especially a poleward undercurrent and an equatorward surface current found within the upwelling zone. Although O'Brien and Hurlburt (1972) examine the dynamics of the equatorward surface current in detail, they find no poleward undercurrent in their model. Theoretical explanations of the poleward undercurrent, a common feature of the eastern ocean boundary circulation, remain a matter of much controversy.

These undercurrents have been observed near the west coasts of South America (Gunther, 1936; Wyrtki, 1963, 1967; Wooster and Gilmartin, 1961), North America (Pavlova, 1966; Pillsbury *et al.*, 1970; Mooers *et al.*, 1972) and Africa (Defant, 1936; Hart and Currie, 1960). They may appear both seaward and shoreward of the continental shelf break at depths of 50–1000 m with speeds of 5–20 cm sec⁻¹. Below the pycnocline the undercurrent is often accompanied by weakening of the vertical density gradient and downwarping of the isopycnals toward the coast (Wooster and Reid, 1963). Seaward of the continental shelf break the undercurrent may persist in all seasons (Reid *et al.*, 1958; Brandhorst, 1959), with maximum intensity occurring during active coastal upwelling, according to Pavlova

(1966). Over the continental shelf the undercurrent is highly time-dependent and is strongly related to the occurrence of coastal upwelling. Collins *et al.* (1968), Pillsbury *et al.* (1970), and Mooers *et al.* (1972) have clearly shown that the inshore undercurrent intensifies with the onset of coastal upwelling, and may completely disappear in the absence of upwelling. The relationship between the undercurrent in the upwelling zone and that over the continental slope and offshore is not well known.

Recently Mooers *et al.* (1972) have presented evidence for a near-surface equatorward jet and a subsurface poleward jet inside the upwelling zone within 30 km of the Oregon coast. The maximum undercurrent speed is found to be 10 cm sec⁻¹, with a vertical extent of 200 m, and a maximum width of 50 km. This California-Oregon Coastal Upwelling Undercurrent is distinguished from the California Countercurrent by strong temporal variations in current intensity and closer proximity to the coastal boundary.

The eastern boundary circulation has not been adequately represented in general circulation models of the ocean. In particular, the nature of eastern boundary currents not directly related to an equatorward wind stress in that region is not well known (Gill, 1971). Several theories have been proposed to explain the observed undercurrents, both inside and outside the coastal upwelling zone. For example, on the basis of water mass identification, Wyrtki (1967) has theorized that near the northern Peruvian Coast the poleward

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flowing Peru-Chile Undercurrent is an extension of the Cromwell Current. Stevenson (1972) demonstrates that the characteristic features of the coastal undercurrent over the continental shelf off Peru are distinct from those of the Peru-Chile Undercurrent, although both exhibit characteristics of Cromwell Current source water.

Yoshida (1967) shows that a poleward flowing current due to thermohaline effects is possible. Cooler, denser water at higher latitudes causes the isopycnals to rise and the sea surface to slope downward toward the poles, producing a thermal circulation. In the eastern boundary region the flow normal to the coast becomes subgeostrophic in order to satisfy the kinematic boundary condition. Ignoring the effects of bottom topography, the N-S sea surface slope may produce a poleward current, while the rise of the isopycnals adds an equatorward baroclinic component to the flow. As a result the current must decrease in strength from the surface, and possibly reverse in direction at some depth. Presumably in the real ocean the surface flow would be overcome by the effect of wind stress during the upwelling season, and equatorward flow in the upper layer would result. Subsurface poleward flow would either vanish or appear as a weak undercurrent, contrary to the observation that the upwelling undercurrent is strongest during the upwelling season.

Durance and Johnson (1970) have shown that an upwelling undercurrent can occur in a three-dimensional homogeneous ocean model driven by equatorward wind stress, but it is confined to a non-hydrostatic boundary layer of width $(\delta E)^{1/3}$, where E is the horizontal Ekman number and δ the aspect ratio of the ocean basin. They find, as do others, that the $E^{1/3}$ "beta layer" vanishes to lowest order in the eastern ocean boundary region, in contrast to the strong $E^{1/3}$ layer found on the western side of the ocean. It should be noted that they obtain their upwelling undercurrent only for $E \ll \delta^2$. It is unlikely that this condition is met in the real ocean.

Mooers *et al.* (1972) have proposed that frontal mixing might play an important role in driving the upwelling undercurrent. The front is represented by an inclined pycnocline which typically breaks the surface about 10 km off shore in regions of active coastal upwelling. According to the theory, frontal mixing produces a water mass of intermediate density which flows offshore at the base of the pycnocline. As it sinks below the Ekman layer, it becomes dynamically unbalanced and acquires a poleward acceleration due to the Coriolis force. The dynamic balance for the compensating subsurface onshore flow is unclear. Our own work suggests that this branch of the circulation may be subgeostrophic in the upwelling zone, and also subject to poleward acceleration.

Garvine (1971) has shown that for a homogeneous steady-state model of coastal upwelling the longshore pressure gradient is an essential dynamic ingredient of

his model. Without this longshore pressure gradient the subsurface onshore flow required by mass continuity must occur in a strong bottom Ekman layer in his model, a feature not observed in the real ocean. Specification of this pressure gradient introduces the upwelling undercurrent in Garvine's model. No physical mechanism for the production of a longshore pressure gradient of just the magnitude required to balance the onshore flow is included in his model, but Garvine suggests that in a closed basin, for example, "wind 'piles up' the water in the direction of the stress."

Yoshida (1967) finds a poleward undercurrent in a wind-driven steady-state model of the eastern ocean circulation on a β -plane. His linear two-layer model has a flat bottom and ignores bottom stress and horizontal friction. The occurrence of an equatorward surface current and a poleward undercurrent in his model depends on a particular parameterization of thermohaline mixing and the existence of a Sverdrup balance within the upwelling zone, which does not occur in the real ocean.

O'Brien and Hurlburt (1972) consider a time-dependent, two-dimensional numerical model of coastal upwelling on a vertical section normal to a straight N-S coast. Their wind-driven two-layer model is on an f -plane and neglects all longshore variations. For an equatorward wind stress they find an equatorward surface jet, but no poleward undercurrent. They also obtain an unrealistically large barotropic longshore flow outside the upwelling zone.

Our work is an outgrowth of that by O'Brien and Hurlburt, and is formulated for a similar two-dimensional geometry. Two new features of our model are the inclusion of the β effect and the longshore pressure gradient. We find that β exerts a secondary influence on the vertical mass transport in the upwelling zone, but has a profound effect on the longshore flow. For an equatorward wind stress and a flat bottom we find small velocities outside the upwelling zone. Inside the zone we find a realistic equatorward surface jet and poleward undercurrent with the same width scale as the zone itself. The poleward undercurrent appears because the β effect requires a N-S sea surface slope of just the right magnitude to balance the onshore flow outside the upwelling zone. In order to satisfy the kinematic boundary conditions the flow must become subgeostrophic inside the upwelling zone, resulting in poleward acceleration in the lower layer.

We find that when the β effect is included, positive wind stress curl in the coastal upwelling zone significantly reduces the equatorward surface current and enhances the poleward undercurrent. We also find that the effect of bottom topography depends on the effective depth of the Sverdrup balance. As the ocean depth becomes shallower than this depth, the onshore flow becomes supergeostrophic. This results in a temporary equatorward acceleration, eventual frictional balance,

and the formation of a secondary upwelling zone over the continental slope. Very near shore the flow again becomes subgeostrophic and a poleward acceleration occurs.

Additional discussion of upwelling theory and observations can be found in the review paper by Smith (1968).

2. Formulation of the model

Consider the equations governing a stably stratified, rotating ocean in a right-handed Cartesian coordinate system as defined in the Appendix. The origin is centered at the coast and at a reference level below the sea surface. A schematic of the model geometry is shown in Fig. 1. Thermodynamic effects are neglected and the time scale for vertical transport is assumed to be much less than the diffusive time scale. Hence it is consistent to ignore thermohaline mixing. We also ignore the effects of horizontal atmospheric pressure gradients, and assume that the fluid is hydrostatic and Boussinesq. Under these assumptions the momentum equations and the equation of continuity form a closed set. Retaining only the first term in the perturbation expansion for the advective terms, the vertically integrated equations appropriate for a two-layer model are:

$$\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 + \hat{k} \times f \mathbf{v}_1 = -g \nabla (h_1 + h_2 + D) + \frac{\tau_s - \tau_I}{\rho h_1} + A \nabla^2 \mathbf{v}_1, \quad (1)$$

$$\frac{\partial h_1}{\partial t} + \nabla \cdot (h_1 \mathbf{v}_1) = 0, \quad (2)$$

$$\frac{\partial \mathbf{v}_2}{\partial t} + \mathbf{v}_2 \cdot \nabla \mathbf{v}_2 + \hat{k} \times f \mathbf{v}_2 = -g \nabla (h_1 + h_2 + D) + g' \nabla h_1 + \frac{\tau_I - \tau_B}{\rho h_2} + A \nabla^2 \mathbf{v}_2, \quad (3)$$

$$\frac{\partial h_2}{\partial t} + \nabla \cdot (h_2 \mathbf{v}_2) = 0, \quad (4)$$

FIG. 1. Typical geometry for a two-layer model with a free surface. $D(x)$ is the height of the bottom topography above some reference level.

where

$$\left. \begin{aligned} \nabla &= \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} \\ \mathbf{v}_i &= u_i \hat{i} + v_i \hat{j} \\ g' &= g(\rho_2 - \rho_1) / \rho_2 \\ \tau_I &= \rho C_I \bar{q} (\mathbf{v}_1 - \mathbf{v}_2) \\ \tau_B &= \rho C_B \bar{q} \mathbf{v}_2 \\ \tau_s &= \tau_{sx} \hat{i} + \tau_{sy} \hat{j} \\ q_i &= (u_i^2 + v_i^2)^{1/2} \\ \bar{q} &= (q_1 + q_2) / 2 \end{aligned} \right\}, \quad (5)$$

and $i=1, 2$. Symbols are defined in the Appendix, although the notation used is common in oceanography. The height of the bottom topography above a reference level is given by $D(x)$. Note that both barotropic and baroclinic modes are retained in the model.

Consider the case of an infinite meridional coast with wind-stress forcing and bottom topography which are independent of latitude. We will show that the problem can be simplified to two dimensions by applying a procedure similar to that used by Veronis and Stommel (1956). In our problem, scale analysis demonstrates that terms in the momentum equations containing N-S derivatives of the velocity field are small, but as was suggested in the Introduction, the N-S pressure gradient may be an important part of the momentum balance. In addition, all terms in the continuity equations containing longshore derivatives are small. Now consider the vorticity equations which can be derived from (1) and (3). Scale analysis is sufficient to show that, except in a thin boundary layer which contributes negligible transport, the meridional currents are highly geostrophic, and again, longshore derivatives of the velocity field are of secondary importance. One possible exception is the horizontal divergence term $f v_y$, which in the next section is also shown to be unimportant. Therefore, by neglecting longshore variations in the velocity field, we can write a closed system of equations on an x, z plane which retain the β effect and the N-S pressure gradient:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} - f v_1 = -g \frac{\partial}{\partial x} (h_1 + h_2 + D) + \frac{\tau_{sx} - \tau_{Ix}}{\rho h_1} + A \frac{\partial^2 u_1}{\partial x^2}, \quad (6)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial v_1}{\partial x} + \frac{\partial}{\partial x} \left(u_1 \frac{\partial v_1}{\partial x} \right) + f \frac{\partial u_1}{\partial x} + \beta v_1 \right) = -\frac{\partial}{\partial x} \left(\frac{\tau_{sy} - \tau_{Iy}}{\rho h_1} \right) + A \frac{\partial^3 v_1}{\partial x^3}, \quad (7)$$

$$\frac{\partial h_1}{\partial t} + \frac{\partial}{\partial x}(u_1 h_1) = 0, \quad (8)$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} - f v_2 = -g \frac{\partial}{\partial x}(h_1 + h_2 + D) + g' \frac{\partial h_1}{\partial x} + \frac{\tau_{Ix} - \tau_{Bx}}{\rho h_2} + A \frac{\partial^2 u_2}{\partial x^2}, \quad (9)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial v_2}{\partial x} \right) + \frac{\partial}{\partial x} \left(u_2 \frac{\partial v_2}{\partial x} \right) + f \frac{\partial u_2}{\partial x} + \beta v_2 = \frac{\partial}{\partial x} \left(\frac{\tau_{Iy} - \tau_{By}}{\rho h_2} \right) + A \frac{\partial^3 v_2}{\partial x^3}, \quad (10)$$

$$\frac{\partial h_2}{\partial t} + \frac{\partial}{\partial x}(u_2 h_2) = 0. \quad (11)$$

Eqs. (6)–(11) represent the governing equations for our model of the eastern ocean circulation in the six unknowns u_1 , u_2 , v_1 , v_2 , h_1 , h_2 . The model is started from rest with no horizontal pressure gradients. Application of the kinematic and no-slip conditions at the coast and the far boundary requires that the velocity field vanishes there. An additional boundary condition on the longshore flow is needed to close the problem and is obtained by noting that the y -momentum equations reduce to

$$\left. \begin{aligned} -g \frac{\partial}{\partial y}(h_1 + h_2 + D) + \frac{\tau_{sy}}{\rho h_1} + A \frac{\partial^2 v_1}{\partial x^2} &= 0 \\ -g \frac{\partial}{\partial y}(h_1 + h_2 + D) + g' \frac{\partial h_1}{\partial y} + A \frac{\partial^2 v_2}{\partial x^2} &= 0 \end{aligned} \right\} \quad (12)$$

at the boundaries. These yield the additional boundary conditions required, if two of the three terms in each are independently specified. We will apply (12) at the far boundary, requiring the wind stress and the N–S pressure gradients to vanish there. This formulation of the boundary condition precludes the possibility of westward intensification in our model. Additional discussion of this topic is found in Section 5c. It will become apparent from the discussion of the model solutions in Section 3 that the solutions in the upwelling zone are little affected by the width of the model region, the width of the forced region, and a variety of boundary conditions at the outer boundary.

At this point we need only specify the model parameters to complete the statement of the problem. Appropriate values are given in Table 1 and are those used in the model solutions discussed at length in Sections 3 and 4. The model is driven by a steady wind stress distributed as shown in Fig. 2. Note that

TABLE 1. Constants of the model and scaling parameters.

Parameter	Value	Parameter	Value
f	10^{-4} sec^{-1}	H_1	$5 \times 10^3 \text{ cm}$
g	10^3 cm sec^{-2}	H_2	$15 \times 10^3 \text{ cm}$
g'	2 cm sec^{-2}	L	1200 km
A	$10^6 \text{ cm}^2 \text{ sec}^{-1}$	β	$2 \times 10^{-13} \text{ cm}^{-1} \text{ sec}^{-1}$
ρ	1 gm cm^{-3}	Δx	2 km
C_I	10^{-5}	Δt	1 hr
C_B	10^{-3}		

the model region is 1200 km wide and $\tau_{sx} = 0$ everywhere. From the coast to 300 km off shore τ_{sy} is constant. From 300 to 1100 km, $\text{curl}_z \tau = 6.25 \times 10^{-9} \text{ dyn cm}^{-3}$. The outer region of the model is unforced. An interfacial stress coefficient C_I of 10^{-5} is specified after Thompson and O'Brien (1973) and the bottom stress coefficient C_B is taken to be 10^{-3} after Proudman (1953). Bottom topography is described in Sections 3 and 4.

The x -directed momentum equations and the continuity equations are solved using a semi-implicit numerical scheme patterned after Kwizak and Robert (1971) and extended to a two-layer baroclinic model. Both internal and external gravity wave modes are treated implicitly. By using this semi-implicit method, we obtain an 80-fold increase in computational efficiency over an explicit scheme with no significant change in the solutions. Details of the numerical procedure may be found in O'Brien and Hurlburt (1972).

The model equations are finite-differenced using leap-frog in time, quadratic averaging for the advective terms (Scheme F from Grammelvedt, 1969), and centered-in-time Coriolis terms. The vorticity equations are integrated from the outer boundary to the coast. Planetary vorticity advection, βv , is treated implicitly using the trapezoidal rule in both space and time.

FIG. 2. Wind stress components as a function of distance off shore: τ_{sy} is -0.5 dyn cm^{-2} near the coast; τ_{sx} is zero everywhere.

3. Flat bottom case

Because of the efficiency of the numerical computations, it was possible to run a large number of cases [O(100)]. Two of these have been singled out for detailed discussion in Sections 3 and 4. The first is a case with a flat bottom.

Consider the simple case of a uniform depth ocean driven from rest by the wind stress shown in Fig. 2. Initially the upper layer is 50 m thick, and the lower layer is 150 m thick. The interface between layers is a simple representation of the permanent pycnocline. The solutions near the coast are of primary interest. Therefore, although the model region is 1200 km wide, only the 200 km nearest the coast are shown in displays of the model solutions.

Fig. 3 depicts the velocity profiles at model day 15. The most striking features are the equatorward surface jet and the poleward undercurrent, which appear with maximum intensity about 8 km off shore. Note that the flow normal to the coast is offshore in the upper layer and onshore in the lower layer, as expected from Ekman dynamics (Ekman, 1905). Outside the upwelling zone the longshore flow is weak and is characterized by Rossby waves and inertial oscillations. The Rossby waves are excited by the impulsive starting of the wind stress curl. The inertial oscillations are excited by the impulsive starting of the wind stress and have a period of 17.44 hr for $f = 10^{-4} \text{ sec}^{-1}$. These are depicted clearly in Fig. 4. The winds are shut off at day 15 in this figure. In the downwelling zone (not shown) the longshore flow oscillates about a Sverdrup balance

$$\beta(v_1 h_1 + v_2 h_2) = -\frac{1}{\rho} \text{curl}_z \tau \quad (13)$$

of -1.6 cm sec^{-1} . The flow near the edge of the downwelling zone is weakly baroclinic.

FIG. 3. Velocity component profiles as a function of distance off shore after 15 days. The profiles for the lower layer are dashed. The longshore flow in the upwelling zone exhibits an equatorward jet in the upper layer and a poleward jet in the lower layer: $u_1 < 0$, $u_2 > 0$.

TABLE 2. The e -folding width (km) of the upwelling zone.

Day	f (sec^{-1})	Model result (km)	λ (km)
5	10^{-4}	13.2	8.7
10	10^{-4}	14.6	8.7
15	10^{-4}	14.9	8.7
10	2.5×10^{-5}	38.8	34.6

The upwelling is depicted by height anomalies of the pycnocline at 3-day intervals up to day 15, as shown in Fig. 5. The term "height anomaly" refers to departures of a surface from its initial position. In this case the interface surfaces after 19 days. This is equivalent to a mean vertical velocity at the coast of 80 m month^{-1} or $3 \times 10^{-3} \text{ cm sec}^{-1}$. The mean vertical velocity varies from 10^{-2} to $10^{-3} \text{ cm sec}^{-1}$ for the range of parameter values appropriate for coastal upwelling.

Fig. 5 indicates that the width of the region of positive height anomaly remains almost constant at about 32 km. Mass continuity yields a scaling relation,

$$L_u \sim H_1 U_1 t / \Delta H, \quad (14)$$

which explains this result. The scaling parameters are defined in the Appendix. Because u_1 averaged over an inertial oscillation is virtually independent of time and $\Delta H \propto t$, it is clear that L_u should vary little during the model integration.

Additional scaling relations are needed, if we are to estimate the ratio $t/\Delta H$ without using the model solutions. As an aid in understanding the model we might hope to estimate the width of the upwelling zone from a closed system of scaling relations. However, the complicated balance for the upwelling zone in the y -momentum equations precludes any but the crudest estimate. Likewise, it is not possible to obtain good estimates of the time scale for pycnocline surfacing using scale analysis. However, an estimate of the width of the upwelling zone can be obtained from conservation of potential vorticity, which predicts an interface having exponential shape and an e -folding width given by the baroclinic Rossby radius of deformation for a two-layer model:

$$\lambda = \left[\frac{g' H_1 H_2}{f^2 (H_1 + H_2)} \right]^{1/2}. \quad (15)$$

Table 2 compares the e -folding width predicted by (15) with that from the model solutions, obtained by determining the point at which the height anomaly decreases to e^{-1} of its value at the coast.

Eq. (15) shows that the width of the upwelling zone should vary directly with stratification and inversely with latitude. The differences between the baroclinic Rossby radius of deformation and the width of the upwelling zone predicted by the model are almost entirely due to frictional effects, as has been verified by numerical experiment. The agreement is particularly good at

FIG. 4. Contours v_1 (left) and v_2 (right) as a function of time and distance off shore. Contours are labeled in cm sec^{-1} . The wind is turned off at day 15.

moderately low latitudes, since as f decreases, λ increases faster than the width of the viscous boundary layer. Table 2 indicates that the width of the upwelling zone increases slowly with time, consistent with (14) and the observation that the rate of upwelling decreases slowly (see Fig. 5).

Outside the upwelling zone the height anomalies are < 5 cm in magnitude except in a downwelling zone where $\text{curl}_z \tau \neq 0$ and in another weak downwelling zone just outside the upwelling zone. The vertical velocity at the pycnocline predicted by Ekman pumping corrected for planetary vorticity advection using the Sverdrup relation (13) is given by

$$w_I = -\frac{dh_1}{dt} = \left[\frac{h_2}{h_1 + h_2} \right] \frac{1}{\rho f} \text{curl}_z \tau. \quad (16)$$

Since $\text{curl}_z \tau$ is constant from 300 to 1100 km, the model gives a downwelling zone (not shown) with an x -independent height anomaly of -0.6 m after 15 days, in close agreement with (16). It should be noted that the planetary vorticity advection reduces the magnitude of the Ekman vertical velocity by 25% in this case.

As an aid in revealing the dynamics of the model solutions, the value of each term in the model equations and the antecedent primitive equations has been calculated from the solutions at each grid point for several different times. We find that the longshore flow is quasi-geostrophic everywhere that it differs significantly from zero. Most of the ageostrophic component is the inertial oscillation. The baroclinic pressure gradient, $g'(h_1)_{x_1}$ is important only in the upwelling zone and at the edge of the downwelling.

It is possible to calculate the vertical velocity at the interface diagnostically from the continuity equation,

$$w_I = \frac{dh_2}{dt} = -h_2 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial D}{\partial x}. \quad (17)$$

This clearly indicates the role of the kinematic boundary condition and one-sided divergence in the Ekman theory. The divergence term is important in the vorticity equation as well. In fact, calculation of term magnitudes in the model vorticity equations indicates that all terms are large in the upwelling zone, except βv and the interfacial stress terms. This suggests that βv should have little effect on the vertical velocity. This is confirmed in Fig. 6 which compares the height anomalies of f -plane and β -plane solutions after 15 days. If the width of the upwelling zone were $O(100 \text{ km})$, βv would have an important effect on the vertical velocity, as shown by Arthur (1965) using scale analysis. Using (17), the vorticity equation suggests that in any case βv should enhance the upwelling; yet the opposite is observed in Fig. 6. Although Fig. 6 implies that along a straight coast the longshore currents have little effect on the upwelling, the large difference in the longshore flow between the f -plane and β -plane

FIG. 5. Pycnocline height anomaly as a function of distance off shore at 3-day intervals up to day 15.

FIG. 6. The f -plane versus the β -plane solutions at day 15: pycnocline and free surface height anomalies (left) and longshore velocity profiles (right) versus distance off shore for $\beta = 2 \times 10^{-10} \text{ cm}^{-1} \text{ sec}^{-1}$ (solid) and $\beta = 0$ (dashed).

solutions might be responsible for the small decrease in the upwelling. This is not the case as will be explained shortly.

So far, it has been shown that βv has little effect on the upwelling velocity in mid-latitudes, but how it exerts a dominating influence on the longshore flow has not been demonstrated. The first integral of (7) and the corresponding y -momentum equation show that

$$\int_{-L}^x \beta v_1 dx = g \frac{\partial}{\partial y} (h_1 + h_2 + D) \Big|_x - g \frac{\partial}{\partial y} (h_1 + h_2 + D) \Big|_{-L}. \quad (18)$$

This shows directly how N-S pressure gradients are induced by the β effect. Recall we specify the N-S pressure gradient to be zero at the far boundary. Since $\text{curl}_z \tau \leq 0$ everywhere, (13) and (18) show that the sea surface slopes downward toward the pole.

As an aside, we now reconsider the neglect of $f v_y$ in (7) and (10). Since $f v_y$ is a small term in the vorticity equations, we need only show that it also would be a small contribution to the N-S pressure gradient, i.e., that

$$\left| \int_{-L}^x f \frac{\partial v}{\partial y} dx \right| \ll \left| \int_{-L}^x \beta v dx \right|.$$

From the Sverdrup relation and the assumed wind stress distribution, $f v_y$ is negligible outside the upwelling zone, although inside the zone it may be comparable to βv . However, the integral of βv over the downwelling zone is roughly four times that over the remaining region; hence, $f v_y$ could provide only a secondary contribution to the N-S pressure gradient.

A picture of the circulation from a dynamical point of view including an explanation of the coastal jets can be completed by consideration of the y -momentum equations. Outside the upwelling zone the balance in

the upper layer is

$$f u_1 = -g \frac{\partial}{\partial y} (h_1 + h_2 + D) + \frac{\tau_{sy}}{\rho h_1}. \quad (19)$$

The geopotential gradient in (19) is about four times smaller in magnitude than the Ekman drift and of opposite sign. Thus, the offshore transport is weaker than the Ekman drift exhibited by the f -plane solutions. By mass continuity this results in a slight reduction in the upwelling for the β -plane solution.

As we will show, the return flow in the lower layer outside the upwelling zone must be balanced by the N-S sea surface slope. Integrating by parts, the barotropic Sverdrup balance in the downwelling zone yields

$$g \frac{\partial}{\partial y} (h_1 + h_2 + D) \Big|_{L_u} = \int_{-L}^{-L_u} \beta v_1 dx = \frac{\tau_{sy}}{\rho (h_1 + h_2)}. \quad (20)$$

Neglecting the tendency of the free surface, the sum of (8) and (11) plus the kinematic boundary condition demonstrate that

$$u_1 h_1 = -u_2 h_2. \quad (21)$$

Use of (19)–(21) verifies our assertion that

$$f u_2 = -g \frac{\partial}{\partial y} (h_1 + h_2 + D). \quad (22)$$

Inside the upwelling zone there is a breakdown in the combination of geostrophy and Ekman drift due to the kinematic boundary condition. Thus, at least initially, we expect the imposed surface stress to force longshore flow near the coast with

$$\frac{\partial v_1}{\partial t} < 0 \quad \text{and} \quad \frac{\partial v_2}{\partial t} > 0.$$

The bottom stress is unimportant in the flat bottom case during the 15-day period considered.

In the subsequent analysis we shall demonstrate that for the special case of flat topography the net longshore mass transport vanishes locally where $\text{curl}_z \tau = 0$. This result applies even in the upwelling zone where the Sverdrup relation is invalid. We also obtain diagnostic relations which allow direct calculation of the longshore flow from the baroclinic E-W pressure gradient.

As a first step in the analysis, consider the sum of the depth-weighted y -momentum equations:

$$\begin{aligned} & h_1 \frac{\partial v_1}{\partial t} + h_2 \frac{\partial v_2}{\partial t} - A \left[h_1 \frac{\partial^2 v_1}{\partial x^2} + h_2 \frac{\partial^2 v_2}{\partial x^2} \right] \\ &= - \left[h_1 u_1 \frac{\partial v_1}{\partial x} + h_2 u_2 \frac{\partial v_2}{\partial x} + f(h_1 u_1 + h_2 u_2) \right. \\ & \quad + g(h_1 + h_2) \frac{\partial}{\partial y} (h_1 + h_2 + D) - g' h_2 \frac{\partial h_1}{\partial y} \\ & \quad \left. - \frac{(\tau_{sy} - \tau_{Iy} + \tau_{Iy} - \tau_{By})}{\rho} \right]. \quad (23) \end{aligned}$$

The depth-weighted Coriolis terms cancel in (23) as do the interfacial stress terms. Using (20) and barotropy in the longshore flow outside the upwelling zone, we find

$$\begin{aligned} & g(h_1 + h_2) \frac{\partial}{\partial y} (h_1 + h_2 + D) - \frac{\tau_{sy}}{\rho} - g' h_2 \frac{\partial h_1}{\partial y} \\ &= h_1 \int_{-L_u}^x \beta v_1 dx + h_2 \int_{-L_u}^x \beta v_2 dx \quad (24) \end{aligned}$$

inside the upwelling zone. From the discussion of (18) and (19),

$$\begin{aligned} & \left| h_1 \int_{-L_u}^x \beta v_1 dx + h_2 \int_{-L_u}^x \beta v_2 dx \right| \\ & \ll \left| h_1 g \frac{\partial}{\partial y} (h_1 + h_2 + D) - \frac{\tau_{sy}}{\rho} \right| \\ & \approx \left| h_2 g \frac{\partial}{\partial y} (h_1 + h_2 + D) - h_2 g' \frac{\partial h_1}{\partial x} \right|. \quad (25) \end{aligned}$$

As we shall show shortly, the integrals on the right-hand side of (24) actually tend to cancel for the flow which develops when (25) and the other conditions in the derivation are valid. The advective terms add constructively in the upwelling zone, but are small even there. Also, the bottom stress has little effect on the solution in the flat bottom case for $C_B = 10^{-3}$. Thus (23) reduces to

$$h_1 \frac{\partial v_1}{\partial t} + h_2 \frac{\partial v_2}{\partial t} - A \left[h_1 \frac{\partial^2 v_1}{\partial x^2} + h_2 \frac{\partial^2 v_2}{\partial x^2} \right] \approx 0. \quad (26)$$

Linearizing (26), we obtain the simple initial value problem

$$\frac{\partial}{\partial t} (v_1 H_1 + v_2 H_2) - A \frac{\partial^2 (v_1 H_1 + v_2 H_2)}{\partial x^2} = 0. \quad (27)$$

With rest as the initial condition and for a variety of boundary conditions,

$$v_1 H_1 = -v_2 H_2. \quad (28)$$

Recall this derivation is valid only where $\text{curl}_z \tau = 0$ and requires the existence of a Sverdrup balance in the downwelling zone at the initial time. About two days are required for the establishment of a Sverdrup balance in the downwelling zone in the sense that (20) holds. However, due to the Rossby waves in the solutions and the short time scale required for the establishment of the Sverdrup balance, the conditions used in the derivation remain appropriate. Although (28) can be obtained by assuming a Sverdrup interior and conservation of potential vorticity, the analysis presented here provides a stronger result.

Outside the viscous boundary layer

$$h_1 \frac{\partial v_1}{\partial t} + h_2 \frac{\partial v_2}{\partial t} \approx 0. \quad (29)$$

Expanding h_1 and h_2 in a Taylor series and retaining the first two terms in each case, we obtain

$$\frac{\partial}{\partial t} [v_1 (H_1 - \bar{w}_I t) + v_2 (H_2 + \bar{w}_I t)] + \bar{w}_I (v_1 - v_2) = 0, \quad (30)$$

where $\bar{w}_I = \bar{w}_I(x)$. For $\bar{w}_I t \ll H_1$

$$v_1 h_1 \approx -v_2 h_2. \quad (31)$$

Otherwise, when $\bar{w}_I > 0$,

$$v_1 h_1 < -v_2 h_2. \quad (32)$$

From (31) and geostrophy in the baroclinic component of the longshore flow,

$$f(v_1 - v_2) = g' \frac{\partial h_1}{\partial x}, \quad (33)$$

we obtain useful diagnostic relations for the longshore velocity shoreward of the region where $\text{curl}_z \tau \neq 0$:

$$v_1 = -\frac{g'}{f} \left[\frac{h_2}{h_1 + h_2} \right] \frac{\partial h_1}{\partial x}, \quad (34)$$

$$v_2 = -\frac{g'}{f} \left[\frac{h_1}{h_1 + h_2} \right] \frac{\partial h_1}{\partial x}. \quad (35)$$

Recall that in deriving (31) we used the barotropic Sverdrup balance in the downwelling zone. Hence, (34)

FIG. 7. Model solution versus diagnostic calculation at day 15. Longshore velocity profiles versus distance off shore for the model solution (solid) and from (34) and (35) (dashed). In both cases the flow exhibits an equatorward jet in the upper layer and a poleward jet in the lower layer.

and (35) really represent baroclinic departures from a barotropic Sverdrup balance outside the upwelling zone. Use of (13) in (33) adds a Sverdrup component, $\text{curl}_z \tau / [\rho \beta (h_1 + h_2)]$, to (34) and (35). It is important to recognize that (34) and (35) are not based on the existence of a Sverdrup balance in the upwelling zone. Recall we find that βv is one of the smallest terms in our vorticity equations in that region.

Profiles of v_1 and v_2 have been calculated diagnostically using (34) and (35) and layer thickness data from the model solutions. These are compared with the model solutions in Figs. 7 and 8. The winds are turned off at day 15. The discrepancy is everywhere $< 2 \text{ cm sec}^{-1}$ and is nearly the same in both layers. When a similar comparison is performed at day 15 for a linear (except for the quadratic stresses) model and linearized diagnostic relations, the extreme difference is 0.9 cm sec^{-1} for $C_B = 0$ (no bottom friction) and 1.1 cm sec^{-1} for $C_B = 10^{-3}$. In both cases the discrepancy is primarily

FIG. 8. Model solution versus diagnostic calculation at day 50. The wind was turned off at day 15.

due to Rossby waves. In the nonlinear case the discrepancy in the upwelling zone is consistent with (32) outside the viscous boundary layer. At day 50, even though no wind stress has been applied for 35 days, longshore jets are still identifiable with maximum intensity shifted to 24 km off shore (see Fig. 8).

It should be noted that except for the secondary contribution due to the N-S pressure gradient, the upwelling circulation in the zonal plane is independent of the basin width if it is wide enough to separate the eastern and western boundary circulations. This requires a minimum width of about 100 km at mid-latitudes. The longshore flow is independent of the width of the basin as long as a Sverdrup interior flow can be established in the sense that (20) holds. For $\beta = 2 \times 10^{-13} \text{ cm}^{-1} \text{ sec}^{-1}$ this requires a width $> 1000 \text{ km}$. Thus, since our model region is 1200 km wide, the kinematic and no slip conditions are suitable for the far boundary and do not affect the solution in the upwelling zone.

While no wind stress and no N-S pressure gradient at the far boundary are not the most characteristic choices we could have made for the real ocean, this is unimportant. What is important is that there exists a Sverdrup interior in the sense that (20) holds. The role of the condition on v_{xx} at the far boundary is considered in more detail in Section 5c.

Previous investigators have obtained results similar to (28). As discussed in the introduction, Yoshida (1967) examines a three-dimensional, linear, steady-state model of coastal upwelling on a β -plane which neglects lateral and bottom friction. By adding the vorticity equations of his two-layer model and using mass continuity, he finds that only the Sverdrup balance remains:

$$v_1 H_1 + v_2 H_2 - \frac{1}{\rho \beta} \text{curl}_z \tau = 0. \quad (36)$$

Garvine (1971) considers a linear, homogeneous, steady-state model on an x, z plane with an imposed N-S sea surface slope, h_y . Assuming $\text{curl}_z \tau = 0$, $h_{xy} = 0$, flat topography ($H = \text{constant}$), and neglecting bottom friction, he finds that

$$\int_{-H}^0 v dz = 0. \quad (37)$$

We find that in the absence of wind stress curl near the coast and for flat bottom topography, a similar constraint on the longshore flow holds even for the more general time-dependent and nonlinear problem in which $h_{xy} = \beta v$.

Eq. (36) can be derived from our model vorticity equations in the downwelling zone, but not in the upwelling zone. In both (7) and (10) only the interfacial stress terms are smaller than βv in the upwelling zone. Furthermore, we know the Sverdrup balance

FIG. 9. Effects of positive wind stress curl near the coast at day 15. Distribution of τ_{sy} (left), pycnocline and free surface height anomalies (center), and longshore velocity (right) versus distance off shore for curl (solid) and no curl (dashed) near the coast. In both cases an equatorward surface jet and a poleward undercurrent are observed.

must break down in the viscous boundary layer, as confirmed in Fig. 9 which shows the effects of positive wind stress curl near the coast. The curl in Fig. 9 has a maximum of $12 \times 10^{-8} \text{ dyn cm}^{-3}$ at the coast and an e -folding width of 25 km. Thus, 25 km off shore the barotropic component of a Sverdrup balance would be 11 cm sec^{-1} . Clearly, a Sverdrup balance does not hold in this case, although the upwelling undercurrent is enhanced by the positive wind stress curl and the surface current is diminished by it. At 100 km from the coast the offshore transport is the same for both wind stress profiles, indicating that shoreward of this point the total vertical transport is the same for the two cases.

Fig. 9 dramatically demonstrates the importance of wind stress curl in the upwelling zone. Unfortunately, present observations are insufficient for determination of the most realistic wind stress profile in most coastal regions. Wind observations off the coast of Peru do suggest a large positive curl within 100 km of the coast in some regions (Compania Administradora del Guano Departamento de Oceanografía e Ictología, 1960).² For either stress profile considered in Fig. 9, the longshore flow is in qualitative agreement with observations shown by Mooers *et al.* (1972) in terms of vertical structure, current speed, width scale, and position.

4. Sharp shelf case

The second case to be considered in detail is similar to the flat bottom case, except that the bottom topography varies off shore. The topography is shown in Fig. 10, and is characterized by a very shallow (64 m deep) flat shelf region and a sharp incline 100 km off shore with maximum slope 10^{-2} . The associated distribution of upwelling is shown every 5 days up to day 25 in Fig. 11. The interface surfaces in 28 days under the wind stress forcing of Fig. 2. A new feature observed is the secondary upwelling zone over the shelf slope.

² Monthly means based on data from 1939–53 on a $\frac{1}{2}^\circ$ mesh over a strip about 150 km wide.

Secondary upwelling has been observed off the western coast of Africa in the southern Benguela Current region by Bang (1971) over topography similar to our sharp shelf case. A snapshot of the velocity profiles at day 25 is shown in Fig. 12. Note that the longshore flow

FIG. 10. Bottom topography for the sharp shelf case. Initially the upper layer is 50 m thick.

FIG. 11. Pycnocline height anomaly for the sharp shelf case as a function of distance off shore at 5-day intervals up to day 25.

FIG. 12. Velocity component profiles as a function of distance off shore after 25 days. The profiles for the lower layer are dashed. The longshore flow in the upwelling zone exhibits an equatorward jet in the upper layer: $u_1 < 0$, $u_2 > 0$.

is dramatically different from the flat bottom case. Outside the shallow shelf the longshore flow is still weak. Over the shelf but outside the primary upwelling zone there is a nearly barotropic equatorward flow of about -15 cm sec^{-1} . A strong equatorward jet in the upper layer and a very weak poleward upwelling undercurrent in the lower layer remain in the primary upwelling zone. The maximum equatorward flow occurs 9 km off shore, but the maximum poleward flow is only 3 km off shore. Additional interesting characteristics of the longshore flow are revealed by a composite of the upper and lower layer longshore velocity profiles every 5 days up to 25 days as shown in Fig. 13. Most notable is the quasi-steady state in the longshore flow on the shallow shelf outside the upwelling zone after approximately 5 days of model integration.

These new features can be understood by considering mass continuity and by re-examining the term balances in the primitive equations. As before, the longshore flow is essentially geostrophic, and the offshore flow in the

FIG. 13. Longshore velocity profiles for the sharp shelf case as a function of distance off shore at 5-day intervals up to day 25.

upper layer is slightly weaker than Ekman drift. The offshore velocity is nearly the same outside the shallow shelf as over it. Hence, mass continuity requires that the onshore flow in the lower layer be greater (by a factor h_{2F}/h_{2s}) over the shallow shelf than outside it. The symbols h_{2s} and h_{2F} refer to the thickness of the lower layer over the shallow shelf and outside of it, respectively. From (20),

$$g \frac{\partial}{\partial y} (h_1 + h_2 + D) = \int_{-L}^{-L_u} \beta v_1 dx = \frac{\tau_{sy}}{\rho(h_1 + h_{2F})}; \quad (38)$$

hence, the sea surface slope no longer balances the onshore flow over the shallow shelf.

Initially an inertial balance exists over the shallow shelf, i.e.,

$$\frac{\partial v_2}{\partial t} \approx -f u_2. \quad (39)$$

After 5 days the longshore flow has developed sufficiently so that the onshore flow is approximately balanced by the bottom stress; *viz.*

$$f u_2 \approx -\frac{\tau_{By}}{\rho h_2} = -\frac{C_B v_2 |v_2|}{h_2}. \quad (40)$$

From (33) we expect the flow to be essentially barotropic outside the primary upwelling zone. In the nearly barotropic region a scaling relation can be obtained from (23), omitting advection, the viscous terms, and assuming barotropy. Where $\text{curl}_z \mathbf{v} = 0$ this relation describes the development of the longshore flow to within $\pm 10\%$ for $\tau_{sy} < 0$:

$$v_1 \approx v_2 \approx \frac{h_1 + h_2}{2C_B t} - \left[\left(\frac{h_1 + h_2}{2C_B t} \right)^2 - \left(\frac{h_{2F} - h_2}{h_1 + h_{2F}} \right) \frac{\tau_{sy}}{\rho C_B} + \frac{h_1 + h_2}{C_B} \epsilon \right]^{1/2}, \quad (41)$$

where

$$\epsilon(x) = \int_{-L_u}^x \beta v_1 dx,$$

and represents the contribution to the N-S pressure gradient shoreward of L_u . For $\epsilon = 0$, (41) predicts $v_2 = -12.5 \text{ cm sec}^{-1}$ at day 5 and $v_2 = -17 \text{ cm sec}^{-1}$ at day 25 over the shelf, in good agreement with the model solution.

Recalling (18) we conclude that the N-S sea surface slope over the shallow shelf must continually increase in magnitude toward the coast. Since

$$\int_{-L}^x \beta (v_1 - v_2) dx = g' \frac{\partial h_1}{\partial y}, \quad (42)$$

the baroclinic N-S pressure gradient is very small out-

side the upwelling zone. Inside the zone the slope of the pycnocline is opposite that of the sea surface [$(h_1)_y < 0$]. From (41) we find $v_2 < 0$ in the nearly barotropic region over the shelf. Thus,

$$\int_{-L}^x \beta v_2 dx = g \frac{\partial}{\partial y} (h_1 + h_2 + D) - g' \frac{\partial h_1}{\partial y} \quad (43)$$

shows that the net N-S pressure gradient over the shelf must increase in magnitude toward the coast at least to the upwelling zone. This is manifested by the slight tilt in the v_2 profile in that region (see Figs. 11 and 13) also in agreement with (41) and the lower layer y -momentum equation. The moderate increase in the magnitude of the N-S pressure gradient in the lower layer, coupled with the marked decrease of most other terms in the viscous boundary layer of the y -momentum equation, allows the N-S pressure gradient force to be dominant very near the coast. The necessity of the N-S pressure gradient in the formation of the upwelling undercurrent is discussed in Section 5b.

From (33) we would not anticipate the strong equatorward surface jet in the primary upwelling zone shown in Figs. 12 and 13, but only that the flow should be baroclinic there. In fact, no strong currents are associated with the other baroclinic regions, such as the secondary upwelling zone or the edges of the downwelling zone. A dynamical consistency relation which clearly demands a jet structure is conservation of potential vorticity:

$$\frac{d}{dt} \left[\frac{\frac{\partial v_1}{\partial x} + f}{h_1} \right] = 0, \quad (44)$$

$$\frac{d}{dt} \left[\frac{\frac{\partial v_2}{\partial x} + f}{h_2} \right] = 0. \quad (45)$$

Outside the viscous boundary layer and where $\text{curl}_z \tau = 0$, conservation of potential vorticity in our model is applicable in qualitative dynamical arguments. The first integral of (44) and (45) shows the necessity of a jet structure in the primary upwelling zone:

$$\frac{\partial v_1}{\partial x} = f \left[\frac{h_1 - H_1}{H_1} \right], \quad (46)$$

$$\frac{\partial v_2}{\partial x} = f \left[\frac{h_2 - H_2}{H_2} \right], \quad (47)$$

where $H_2 = H_2(x)$. See discussions by Charney (1955), Csanady (1968, 1972), and O'Brien and Hurlburt (1972) for further details on applications of this concept to wind-driven coastal jets. Note that conserva-

tion of potential vorticity does not reveal the vertical structure of the jet, but only shows that one must exist in at least one layer.

Some mechanistic understanding of the jet can be gained by considering the upper layer y -momentum equation. Outside the upwelling zone there is a near balance given by (19). As we approach the coast $u_1 \rightarrow 0$ to satisfy the kinematic boundary condition, but the sum of the geopotential gradient and the wind stress term changes very little. Hence, at least initially $(v_1)_t < 0$ in the upwelling zone, and a jet develops outside the viscous boundary layer.

During the first five days it is also true that $(v_1)_t < 0$ outside the upwelling zone over the shallow shelf. This is explained on the basis of the lower layer dynamics and the geostrophic constraint (33). In the y -momentum equation $f u_1$ has the same sign as $(v_1)_t$. Hence, during the first five days the offshore transport over the shallow shelf is slightly weaker than outside it. As a result a secondary upwelling zone develops over the shelf slope. Note in Figs. 11 and 13 that the growth of the secondary upwelling zone ceases as $(v_1)_t \rightarrow 0$. Given the longshore flow, the secondary upwelling is also demanded for dynamical consistency by conservation of potential vorticity (O'Brien and Hurlburt, 1972) as given by (46), but not (47). Recall that (46) and (47) were derived by integrating a material derivative, and thus are valid in the Eulerian sense only when advection is unimportant. Over the shelf slope the advective and divergence terms in the lower layer continuity equation are nearly the same size. Eqs. (46) and (41) show that the maximum in the secondary upwelling should be shifted shoreward of the maximum shelf slope in agreement with Figs. 10 and 11.

From Figs. 12 and 13 the diagnostic relations (34) and (35) do not hold shoreward of the foot of the shelf. Recall that their derivation required the sum of all other terms in (23) to be $\ll h_1(v_{1t} + A v_{1xx})$ and $h_2(v_{2t} + A v_{2xx})$. Over the sharp shelf $g(h_1 + h_2)(h_1 + h_2 + D)_y - (\tau_{sy}/\rho)$ does not vanish as required in the derivation.

5. Additional remarks

a. Model dynamics and parameter space

Thus far we have discussed our model via two specific cases. We have chosen to concentrate on cases which are most applicable to upwelling in the real ocean, but which are as simple as possible. These have revealed most of the important aspects of the model dynamics, but discussion of the model parameter space has been limited. At this point we will consider the roles of certain physical processes in the model, examining regions of the parameter space of particular interest.

The x -momentum equations are dominated by geostrophy even when $f = 10^{-5} \text{ sec}^{-1}$, except in a thin viscous boundary layer, $O(300 \text{ m})$, with negligible transport. The tendency terms are important in the

geostrophic adjustment process. No other terms in these equations play an important role for realistic values of the specified parameters.

The fundamental dynamics of the y -momentum equations have been discussed in detail in Sections 3 and 4. Some additional aspects of the dynamics deserve further consideration. In these equations only the advective terms play no important qualitative role. For realistic values of C_I , the interfacial stress terms are unimportant except in time integrations much longer than those considered here. The diffusive terms are important only in the viscous boundary layer, which for realistic stratification and values of A , is much narrower than the upwelling zone. The viscous width scale in the upper layer can be estimated from $[Av_1\rho H_1(H_1+H_2)/H_2\tau_{sy}]^{1/2}$. Some characteristics of the flat bottom case for two different values of A are compared in Table 3.

Over the 19-day period considered, the bottom stress plays no significant role in the flat bottom case, but is very important in the sharp shelf case. Its role was considered in detail in Section 4. As C_B is decreased, the longshore flow over the outer portion of the shallow shelf becomes stronger, as shown by (41). The poleward undercurrent also becomes much stronger, but the coastal surface jet becomes weaker. The role of the remaining terms has been discussed in previous sections.

Comparisons between the nonlinear model and a linearized version reveal no major qualitative differences, except for the effects of topography. Some characteristics of linear and nonlinear solutions for the flat bottom case are compared in Table 3.

b. β -plane vs f -plane solution

We saw in Fig. 6 and from (18) that β exerts a major influence on the longshore flow by inducing a N-S sea surface slope. Since βv is a small term in the vorticity equations, we can conclude from (20) that varying β should have little effect on the solution as long as the major contribution to the integral occurs in the downwelling zone. For $\beta = 2 \times 10^{-13} \text{ cm}^{-1} \text{ sec}^{-1}$ that contribution is about 80% of the total in the flat bottom case. The β -plane solution rather than the f -plane solution will hold for any value of β such that the Sverdrup balance is valid in the downwelling zone.

Using the lower layer y -momentum equation, we can demonstrate that without the N-S pressure gradient no poleward undercurrent can develop in our model. Initially, there is an inertial balance. Thus, onshore flow implies an equatorward acceleration. After several days there occurs a balance between the Coriolis force and bottom friction, which also requires equatorward flow. In the steady state $u \rightarrow 0$, and there is a balance between bottom friction and the interfacial stress outside the viscous boundary layer, again implying equatorward flow.

The N-S pressure gradient can be included in the

f -plane model of O'Brien and Hurlburt (1972) as a specified parameter. If we specify the N-S sea surface slope to be $\tau_{sy}/\rho(h_1+h_2)$ as given by the β effect, then their model yields nearly the same solution as that obtained from the β -plane model. If we set the N-S sea surface slope to $\alpha\tau_{sy}/\rho g(h_1+h_2)$, where α is an arbitrary constant, and specify the pycnocline slope such that

$$g' \frac{\partial h_1}{\partial y} = \frac{g(h_1+h_2)}{h_2} \frac{\partial(h_1+h_2+D)}{\partial y} - \frac{\tau_{sy}}{\rho h_2}, \quad (48)$$

then the longshore flow still vanishes outside the upwelling zone. For $\alpha=2$ the equatorward surface jet and the poleward undercurrent are little changed for the same height anomaly at the coast as in the previous case, and the vertical velocity near the coast is reduced by one-third.

Fig. 14 compares f -plane and β -plane solutions for the sharp shelf case. While the secondary upwelling zone at day 15 has nearly disappeared in the f -plane solution, it has slightly increased in the β -plane solution. Note that for the f -plane solution the disappearance of the secondary upwelling zone is accompanied by a flattening of the longshore velocity profiles over the shelf slope. These differences can be explained by comparing (41) to a similar relation for the f -plane solution. For barotropic flow on an f -plane with $\tau_{sy} < 0$,

$$v_1 \approx v_2 \approx \frac{h_1+h_2}{2C_{Bt}} - \left| \frac{(h_1+h_2)^2}{2C_{Bt}} - \frac{\tau_{sy}}{\rho C_B} \right|^{1/2}. \quad (49)$$

During the early spin-up the longshore flow is predicted by

$$v_1 \approx v_2 \approx \frac{\tau_{sy} l}{\rho(h_1+h_2)}. \quad (50)$$

Although the spin-up is faster in the shallower water, the quasi-steady state is independent of h_2 as shown by (49). The remaining dynamics have been discussed in detail in Section 4. For further discussion of the f -plane solution for the sharp shelf case see O'Brien and Hurlburt (1972). Secondary upwelling over a shelf break is also discussed by Yoshida (1967). Alternatively, Hsueh and O'Brien (1971) propose a mechanism for secondary upwelling which involves the interaction of bottom friction and offshore currents.

c. Westward intensification

For the model to exhibit westward intensification we must specify a non-zero value of $v_{xx}(-L)$. Consistent with this specification, we can formulate a quasi-balanced initialization from the vorticity equations, noting that $u \rightarrow 0$ in the steady state. Applying the appropriate boundary conditions, $v(0) = v(-L) = 0$ and $Av_{xx}(-L) = g(h_1+h_2+D)_y|_{-L} - \tau_{sy}(-L)/\rho H_1$, and ignoring curl in the stress terms, we obtain a barotropic

TABLE 3. Flat bottom case at day 15.

	A	v_1 jet Velocity (cm sec ⁻¹)	Position (km)	v_2 jet Velocity (cm sec ⁻¹)	Position (km)	e -folding width of upwelling zone (km)	Interface height anomaly at coast (km)
Nonlinear	10^7	-10.4	18	1.9	18	35.1	11.8
Nonlinear	10^6	-33.6	8	4.5	9	14.9	38.2
Linear	10^6	-20.0	10	5.4	11	17.8	30.8

longshore velocity for the initial state similar to the Munk (1950) solution:

$$v = -\frac{2K}{\sqrt{3}\beta} v_{xx}(-L) \times \exp\left[-\frac{K}{2}(x+L)\right] \sin\left[\frac{\sqrt{3}}{2}K(x+L)\right], \quad (51)$$

where $K = (\beta/A)^{1/2}$.

For the flat bottom case and the wind stress of Fig. 2 the longshore mass transport due to westward intensification exactly balances the Sverdrup transport

when we set

$$Av_{xx}(-L) = g \frac{\partial}{\partial y} (h_1 + h_2 + D) \Big|_{-L} = \frac{\tau_{sy}}{\rho(h_1 + h_2)} = -2.5 \times 10^{-5}.$$

In the western region a maximum v of 40 cm sec⁻¹ occurs 22 km from the boundary and a minimum of -6.5 cm sec⁻¹ occurs at 83 km. Using this quasi-balanced initialization with an imposed N-S pressure gradient at the far boundary, we obtain virtually the

FIG. 14. The f -plane versus the β -plane solution for the sharp-shelf case. The top row shows pycnocline and free surface height anomalies and the bottom row shows the longshore velocities.

same solution in the upwelling zone as obtained previously by setting $g(h_1+h_2+D)_y|_{-L}=0$.

Additional insight into the role of the condition on v_{xx} at the western boundary, the dynamics of the western boundary region, and their possible effects on the eastern boundary circulation can be gained, if the y -momentum equations for the model are rewritten using (18):

$$\frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_1}{\partial x} + f u_1 = -g \frac{\partial}{\partial y} (h_1 + h_2 + D) \Big|_{-L} - \int_{-L}^x \beta v_1 dx + \frac{\tau_{sy} - \tau_{ly}}{\rho h_1} + A \frac{\partial^2 v_1}{\partial x^2}, \quad (52)$$

$$\frac{\partial v_2}{\partial t} + u_2 \frac{\partial v_2}{\partial x} + f u_2 = -g \frac{\partial}{\partial y} (h_1 + h_2 + D) \Big|_{-L} + g' \frac{\partial h_1}{\partial y} \Big|_{-L} - \int_{-L}^x \beta v_2 dx + \frac{\tau_{ly} - \tau_{by}}{\rho h_2} + A \frac{\partial^2 v_2}{\partial x^2}. \quad (53)$$

Here we will only note that regardless of the wind stress and N-S pressure gradient imposed at the western boundary, any initial imbalance will result in the development of western boundary currents and Rossby waves in the model such that (20) will hold in the eastern region, provided the basin is sufficiently wide. This can be seen by noting as in Section 5b that any zonal motion in the lower layer not geostrophically balanced will result in the development of longshore flow. From (18) the longshore flow will induce a N-S pressure gradient tending to oppose the development of longshore flow of the same sign east of that point.

d. Steady state

The model steady state is not physically realistic, but objections to the steady state are not objections to any of the spin-up shown here. In the steady state the vertically integrated flow in each layer normal to the coast (i.e., u_1 and u_2) is zero. In order to represent a more realistic steady state it would be necessary to include some parameterization of thermohaline mixing (Yoshida, 1967), and to continue integration after the interface between the layers had surfaced. This sort of work is better left to multi-layer models.

In reality, upwelling is not a steady-state problem even though some aspects of it may appear quasi-steady. For example, the upwelling undercurrent has been found to vary little with time in this model and conservation of potential vorticity suggests that the width of the upwelling zone should be almost independent of time. This work, that of O'Brien and Hurlburt (1972), and even more dramatically that of Thompson and O'Brien (1973) show that upwelling responds greatly to variations in the wind on a time scale of a few days. Observations such as current

records and spectra (e.g., Mooers *et al.*, 1972) also display this high degree of time variability in the upwelling circulation.

6. Summary and conclusions

We have included the β effect in a two-layer x, z plane wind-driven model of an eastern ocean circulation. The model is both nonlinear and time-dependent. Since the longshore flow is known to be nearly geostrophic in the real ocean, it is possible to neglect longshore derivatives of the velocity field in the model equations for the case of wind stress independent of latitude, while retaining both the barotropic and baroclinic N-S pressure gradients. For several configurations of bottom topography and wind stress the governing equations are solved using a semi-implicit numerical scheme.

The β effect exerts a dominant influence on the longshore flow by inducing a N-S sea surface slope. For a flat bottom ocean driven by the equatorward wind stress shown in Fig. 2, the model predicts upwelling adjacent to the coast with a mean vertical velocity of 10^{-3} to 10^{-2} cm sec $^{-1}$ and an e -folding width of 15 km, comparable to that given by the baroclinic Rossby radius of deformation, but modified by frictional effects. Due to the N-S sea surface slope the offshore transport outside the upwelling zone in the upper layer is slightly weaker than Ekman drift. Since a Sverdrup balance exists outside the upwelling zone, the subsurface onshore flow there is balanced by the N-S sea surface slope. Inside the upwelling zone this combination of geostrophy and Ekman drift breaks down, allowing an equatorward acceleration in the upper layer and a poleward acceleration in the lower layer. For the special case of flat topography the net longshore mass transport vanishes locally where $\text{curl}_z \tau = 0$, even in the upwelling zone where the Sverdrup balance is invalid. In this case diagnostic relations can be obtained which allow direct calculation of the longshore velocities from the baroclinic E-W pressure gradients.

Strong positive wind stress curl in the coastal upwelling zone sharply attenuates the equatorward surface jet and increases the upwelling undercurrent. The effects of bottom topography depend on the effective depth of the Sverdrup balance. As the ocean depth becomes shallower than this depth, the onshore flow becomes supergeostrophic. This results in a temporary equatorward acceleration, eventual frictional balance, and the formation of a secondary upwelling zone over points of large bottom slope, such as just off the edge of a continental shelf. Very near shore the flow again becomes subgeostrophic and a poleward acceleration occurs.

The model we have described is quite crude. It considers no thermodynamic effects, such as thermohaline mixing between layers, and does not adequately model pycnocline surfacing. We are restricted to con-

sidering the onset of upwelling in an idealized ocean. Despite these limitations, our results agree with recent observations (Mooers *et al.*, 1972). From our results these conclusions can be drawn:

First, the spatial and temporal distribution of wind stress over the upwelling region strongly affects the upwelling and the longshore velocity. Thus, field experiments such as the NSF-IDOE³ Coastal Upwelling Experiment (CUE) must adequately resolve the wind distribution in space and time.

Second, we have found that our model can produce a realistic subsurface poleward flow within the upwelling zone. However, we find no evidence for a similar undercurrent outside this zone. We suggest that poleward flowing undercurrents observed in the real ocean outside the upwelling zone are produced by mechanisms other than those included in our model, such as thermodynamic effects and the large-scale ocean circulation.

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APPENDIX

List of Symbols

A	horizontal eddy viscosity
C_I, C_B	drag coefficients for interfacial and bottom stress
D	height of the bottom topography above a reference level
E	horizontal Ekman number
f	Coriolis parameter
g	acceleration due to gravity
g'	reduced gravity, $g(\rho_2 - \rho_1)/\rho_2$
h_1, h_2	instantaneous local thickness of the layers
h_{2S}, h_{2F}	thickness of the lower layer over the sharp shelf and at the foot of the shelf
H_1, H_2	initial thickness of the layers
L	total width of the model region

L_u	width of the region of positive height anomaly
t	time
u_1, u_2	x -directed components of current velocity
U_1	characteristic speed normal to the coast
v_1, v_2	y -directed components of current velocity
w_I	vertical velocity at the pycnocline
$\bar{w}_I$	time mean vertical velocity of the pycnocline
x, y, z	tangent plane Cartesian coordinates: x positive eastward, y positive northward, z positive upward
β	df/dy
δ	aspect ratio, ocean depth to width
ΔH	pycnocline height anomaly at the coast
Δt	time increment in the numerical integration
Δx	horizontal grid increment
ϵ	contribution to the N-S pressure gradient over the shelf
λ	baroclinic Rossby radius of deformation
ρ, ρ_1, ρ_2	densities of sea water
$\tau_{sx}, \tau_{Ix}, \tau_{Bx}$	x -directed tangential stresses at the surface, interface, and bottom
$\tau_{sy}, \tau_{Iy}, \tau_{By}$	y -directed tangential stresses at the surface, interface, and bottom

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³ International Decade of Ocean Exploration.

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