Upwelling Layers in a Steady, Wind-Driven Stratified Ocean

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ABSTRACT

A theoretical study is made of a region of steady upwelling adjacent to the eastern boundary of a stratified ocean. A multi-layered structure is required to satisfy all the boundary conditions. The velocity and temperature fields are found by matching to Ekman and thermal layers of the interior region. In the principal upwelling layer, vertical advection of heat is balanced by horizontal diffusion of heat. Very close to the coast a narrow mixing region is required to complete the solution. It is shown that the onshore flow and the vertical velocity change sign with depth.

1. Introduction

A summary of earlier observations and theories of coastal upwelling and a discussion of the biological significance of the nutrients supplied by this upwelling is given by Smith (1968). More recent theoretical work has included a study of the transverse circulation near a coast by Blumsack (1972) based on an f -plane approximation and with all variables independent of the longshore coordinate. Hsueh and Kenney (1972) have considered coastal upwelling on an f plane with a linear vertical temperature gradient by the use of Fourier analysis. In this paper the solution is three-dimensional and based on a conventional β plane with a basic vertical temperature gradient that is exponential.

A considerable number of numerical models have been formulated to examine the upwelling region. For example, O'Brien and Hurlburt (1972) consider a two-layer model on an f plane. Further extensions of this work to include some longshore variations on a β plane and more layers are given in McNider and O'Brien.¹ Earlier numerical work by Bryan and Cox (1968) for a complete ocean basin has shown the significant upwelling that is present near the coast, although the grid size prevented any detailed analysis of the coastal layer. In particular, Bryan and Cox (Fig. 13) show that the vertical velocity changes sign with depth in the upwelling region, being upward near the surface and deep down but downward at intermediate depths.

In this paper an attempt is made to develop a three-dimensional model of an upwelling layer which may be matched onto an interior solution consisting of an Ekman layer, a thermal layer, and an abyssal layer. An alternative motive is to extend the model of Johnson (1971) to include the eastern boundary layer and there-

by determine the interior solution in a more rigorous manner. In order to satisfy all the boundary conditions the boundary layer is split into various parts. Near the surface it is found that a hydrostatic layer is required to satisfy the longshore velocity boundary condition and a buoyancy layer to carry the upwelling. The buoyancy layer is neither geostrophic nor hydrostatic and narrows as the surface is approached. Inshore of the buoyancy layer, a thin mixing region is required to complete the solution. There is no significant transport in this mixing layer and it is required merely to adjust the temperature field so that there is no heat flow into the coast.

An illustration of the solution for a particular wind stress is given in Section 7. It will be seen that the water that upwells to the surface comes only from the very top of the thermal layer. Three horizontal regions of fluid become apparent: a surface region in which upwelling from the thermal layer to the Ekman layer occurs; an intermediate zone in which downwelling occurs and the flow in the thermal layer changes from onshore to offshore; and finally, a deep region where water upwells from the abyssal layer to enter the lower part of the thermal layer where it flows offshore. This kind of behavior is shown numerically by Bryan and Cox (1968) and illustrated by Smith (1968).

2. Formulation

The β -plane approximation is used with the x axis eastward, y northward and z upward. The model ocean is confined between a horizontal surface $z=1$ and the bottom at $z=0$. The western and eastern boundaries are at $x=0$ and $x=1$, respectively. In this paper attention is confined to the region near the eastern boundary between the latitudes of zero wind-stress curl defining the subtropical gyre.

¹ Private communication.

The steady-state equations of motion are

$$\begin{aligned} \mathbf{u} \cdot \nabla \mathbf{u} - f\mathbf{v} &= -\frac{1}{\rho} \nabla p + \nu_H (\mathbf{u}_{xx} + \mathbf{u}_{yy}) + \nu_v \mathbf{u}_{zz}, \\ \mathbf{u} \cdot \nabla \mathbf{v} + f\mathbf{u} &= -\frac{1}{\rho} \nabla p + \nu_H (\mathbf{v}_{xx} + \mathbf{v}_{yy}) + \nu_v \mathbf{v}_{zz}, \\ 0 &= -\frac{1}{\rho} \nabla p - g, \\ u_x + v_y + w_z &= 0, \\ \mathbf{u} \cdot \nabla T &= K_H (T_{xx} + T_{yy}) + K_v T_{zz}, \\ \rho &= \rho_0 [1 - \alpha(T - T_0)], \end{aligned}$$

where p is the pressure, $f = f_0 + \beta y$ the Coriolis parameter, u, v, w the velocity components, ρ the density, T the temperature, ν_H, ν_v, K_H, K_v the eddy coefficients for horizontal and vertical diffusion of momentum and heat, respectively, g the apparent gravity, and α the coefficient of thermal expansion. In the linear equation of state, T_0, ρ_0 are the average bottom temperature and average bottom density, respectively; all such averages are taken over the region $0 \leq x \leq 1, 0 \leq y \leq 1$.

The dimensionless variables (indicated by primes) are as follows:

$$\left. \begin{aligned} T &= T_0 + (\Delta T) T' \\ p &= -\rho_0 g z + \rho_0 f_0 U L p' \\ \rho &= \rho_0 (1 + \rho'), \quad \rho' \ll 1 \\ u, v &= U(u', v'); \quad w = \frac{D}{L} U w' \\ x, y &= L(x', y'); \quad z = D z' \\ f &= f_0 f'; \quad \beta = f_0 \beta' / L \end{aligned} \right\}$$

Here ΔT is the difference between the average surface temperature and the average bottom temperature; L and D are the width and depth of the ocean; and U is a typical horizontal velocity scale. It is shown in Johnson (1971) that for a wind-driven ocean

$$U = \frac{\tau^*}{\rho_0 (f_0 \nu_v)^{1/2}},$$

where τ^* is a typical magnitude of the wind stress.

Substitution of the new variables into the above equations of motion, omitting primes, gives the dimensionless equations of motion:

$$\mathbf{R} \mathbf{u} \cdot \nabla \mathbf{u} - f\mathbf{v} = -\nabla p + E_H (\mathbf{u}_{xx} + \mathbf{u}_{yy}) + E \mathbf{u}_{zz}, \quad (1)$$

$$\mathbf{R} \mathbf{u} \cdot \nabla \mathbf{v} + f\mathbf{u} = -\nabla p + E_H (\mathbf{v}_{xx} + \mathbf{v}_{yy}) + E \mathbf{v}_{zz}, \quad (2)$$

$$0 = -\nabla p + S T, \quad (3)$$

$$u_x + v_y + w_z = 0, \quad (4)$$

$$\mathbf{R} \mathbf{u} \cdot \nabla T = -\frac{E_H}{\sigma_H} (T_{xx} + T_{yy}) + \frac{E}{\sigma} T_{zz}. \quad (5)$$

Here the Boussinesq approximation has been used, with ρ' neglected except in the buoyancy term; $E_H = \nu_H / (f_0 L^2)$ and $E = \nu_v / (f_0 D^2)$ are Ekman numbers; $R = U / (f_0 L)$ is a Rossby number and $S = g \alpha (\Delta T) D \rho_0 \nu_v^{1/2} / (f_0^{1/2} L \tau^*)$ is a stratification parameter; and $\sigma_H = \nu_H / K_H$ and $\sigma = \nu_v / K_v$ are Prandtl numbers.

The choice of dimensionless temperature requires that the average bottom temperature be zero and the average surface temperature unity. That is

$$\int_0^1 \int_0^1 T dx dy = 1; \quad z = 1. \quad (6a)$$

The other surface boundary conditions are

$$u_z = \frac{\tau^x}{E^1}, \quad v_z = \frac{\tau^y}{E^1}, \quad w = 0; \quad z = 1, \quad (6b)$$

where τ is the dimensionless applied wind stress. These boundary conditions are chosen since this paper is concerned with the wind-driven circulation. The use of a similarity solution in the thermal layer precludes the addition of an arbitrary choice of surface temperature distribution. In Johnson (1971) it is shown that conditions (6) can lead to a reasonable distribution of surface temperature.

At all rigid boundaries

$$u = v = w = 0, \quad \mathbf{n} \cdot \nabla T = 0, \quad (7)$$

where $\mathbf{n}$ is normal to the boundary. This condition implies no heat flux through the bottom or sides of the ocean.

To facilitate the solution of (1)–(7) the ocean is divided into three horizontal layers, a surface Ekman layer, a thermal layer, and an almost isothermal abyssal layer.

FIG. 1. Positions of the interior and upwelling layers.

At the eastern boundary an upwelling layer is added and divided up as in Fig. 1. In this paper most attention will be paid to the dynamics of the upwelling layer. In Section 3 a summary is given of the interior solution away from the eastern boundary.

3. The Ekman layer and thermal layer solution

The solution in the Ekman layer is well known. The horizontal velocities are $O(1)$ and the vertical velocity is $O(E^{1/2})$. The temperature is independent of depth in this layer. The results that are needed for matching with the upwelling layer are the Ekman suction condition for large ζ , i.e.,

$$w_E \approx E^{1/2} \mathbf{k} \cdot \text{curl} \boldsymbol{\tau} / f, \quad (8)$$

and the horizontal Ekman transport toward the eastern boundary which is

$$E^{1/2} \int_{\infty}^0 u_E d\zeta = E^{1/2} \tau^y / f, \quad (9)$$

where $\zeta = (1-z)/E^{1/2}$ is the Ekman layer vertical coordinate.

The thermal layer solution is the similarity solution with exponential decay given by many authors. In the notation of Johnson (1971) it may be written

$$T = T_T = T_0(x, y) \exp(-c\eta/f), \quad (10)$$

$$u = E^{1/2} S^{1/2} u_T$$

$$= -E^{1/2} S^{1/2} \left(\frac{T_{0y}}{c} + \frac{\beta T_0}{fc} + \frac{\beta}{f^2} \eta T_0 \right) \exp(-c\eta/f), \quad (11)$$

$$v = E^{1/2} S^{1/2} v_T = E^{1/2} S^{1/2} (T_{0x}/c) \exp(-c\eta/f), \quad (12)$$

$$w = E^{1/2} S^{1/2} w_T = E^{1/2} S^{1/2} \left[\frac{kc}{f} + \frac{\beta}{c^2} T_{0x} \exp(-c\eta/f) \right], \quad (13)$$

$$p = E^{1/2} S^{1/2} p_T = E^{1/2} S^{1/2} (fT_0/c) \exp(-c\eta/f), \quad (14)$$

where $\eta = (1-z)S^{1/2}/E^{1/2}$, $k = S^{1/2}E^{1/2}/\sigma R$, c is a constant, and

$$T_0(x, y) = \frac{c^2}{\beta} \int_1^x \mathbf{k} \cdot \text{curl} \boldsymbol{\tau} dx - \frac{kc^3}{\beta f} (x-1) + T^*(y). \quad (15)$$

The temperature and velocity fields decay exponentially in the thermal layer whose dimensionless thickness is $O(E^{1/2}S^{-1/2})$. The horizontal velocities are $O(E^{1/2}S^{1/2})$ and hence the horizontal transport is $O(E^{1/2})$, the same as the Ekman layer transport. The vertical velocity decays from its value just below the Ekman layer (given by the Ekman suction produced by the wind stress) to its asymptotic value which represents deep upwelling from the abyssal layer. The heat balance near the surface is principally advective and the surface temperature distribution is determined by advection with the circulation. Near the bottom of the thermal layer there

is a balance between downward diffusion of heat and upward advection of heat by the deep upwelling.

It may be seen that, in general, the solution (10)–(14) cannot satisfy the eastern boundary condition given by (7). In particular, no choice of the arbitrary function $T^*(y)$ can make u_T identically zero at $x=1$. Thus, upwelling layers are also required to redistribute the water within the thermal layer even if $\tau^y=0$ at $x=1$. The matching conditions to be applied at the seaward edge of the upwelling layer are

$$T_T(1, y, \eta) = T^*(y) \exp(-c\eta/f), \quad (16)$$

$$u_T(1, y, \eta) = - \left(\frac{T_y^*}{c} + \frac{\beta T^*}{fc} + \frac{\beta}{f^2} \eta T^* \right) \exp(-c\eta/f), \quad (17)$$

$$v_T(1, y, \eta) = \left(\frac{c}{\beta} \mathbf{k} \cdot \text{curl} \boldsymbol{\tau} - \frac{kc}{f} \right) \exp(-c\eta/f), \quad (18)$$

$$w_T(1, y, \eta) = \left(\mathbf{k} \cdot \text{curl} \boldsymbol{\tau} - \frac{kc}{f} \right) \exp(-c\eta/f) + \frac{kc}{f}. \quad (19)$$

The remainder of this paper is concerned with determining the structure and the solution of the upwelling layer which enables the boundary conditions (7) to be satisfied and which completes the circulation by providing the water for the offshore Ekman transport. In doing so the unknown function $T^*(y)$ is determined.

4. The general equation for the upwelling layer

To determine the structure of the upwelling layer at the eastern boundary of the thermal layer a single equation is derived for the upward velocity w . Certain assumptions will be made here which can be justified *a posteriori*. The boundary layer approximation requires that $\partial/\partial x \gg \partial/\partial y$, $\partial/\partial \eta$. This implies that the horizontal scale of the upwelling layer is smaller than the vertical scale of the thermal layer. It will also be assumed that the dimensionless upwelling velocity w is larger than the dimensionless longshore velocity v so that the continuity equation is

$$u_x = (S^{1/2}/E^{1/2}) w_{\eta}. \quad (20)$$

In other words, the incoming transport from the thermal layer is converted into upward motion. Another common boundary layer assumption is that the longshore variation of the pressure does not change across the boundary layer; that is, $p_{xy}=0$. This may be justified *a posteriori* and leads to the horizontal momentum equations

$$-fv = -p_x + E_H u_{xx}, \quad (21)$$

$$fu_x = E_H v_{xx}, \quad (22)$$

which implies that the onshore flow is not geostrophic. The longshore flow may be geostrophic in an outer layer in which the u term of (21) is small, but will not be

TABLE 1. Layers obtained by balancing pairs of terms in Eq. (27).

Name	Dimensionless width	Terms balanced in (27)	Restrictions
Hydrostatic	$(FT_T)^{\frac{1}{2}}$	(iii) & (iv)	$E_H^2 D^2 < (FT_T)^3$, $E_H < FT_T$
Stewartson	$(DE_H)^{\frac{1}{2}}$	(i) & (iv)	$E_H^2 D^2 > (FT_T)^3$, $E_H < D^2$
Upwelling	$E_H^{\frac{1}{2}}$	(ii) & (iv)	$E_H > FT_T$, $E_H > D^2$
Viscous hydrostatic	$E_H/(FT_T)^{\frac{1}{2}}$	(ii) & (iii)	$E_H < FT_T$, $E_H^2 > FT_T D^2$
Stokes	D	(i) & (ii)	$E_H > D^2$, $E_H^2 > FT_T D^2$
Buoyancy	$(DE_H)^{\frac{1}{2}}/(FT_T)^{\frac{1}{2}}$	(i) & (iii)	$E_H^2 D^2 < (FT_T)^3$, $E_H^2 < FT_T D^2$

geostrophic in any inner layer in which this term is important.

The third momentum equation must be modified from (4) if the boundary condition $w=0$ is to be satisfied. This involves including a lateral diffusion term so that the equation becomes

$$0 = (S^{\frac{1}{2}}/E^{\frac{1}{2}})p_{\eta} + ST + \delta^2 E_H w_{xx}, \quad (23)$$

where δ is the ratio of the vertical length scale to the horizontal length scale of the ocean. The component of Coriolis acceleration that is omitted from this equation always remains small compared with the last term under the conditions of interest here.

The heat equation is considerably simplified to

$$-R \frac{S^{\frac{1}{2}}}{E^{\frac{1}{2}}} w T_{T\eta} = \frac{E_H}{\sigma_H} T_{xx}, \quad (24)$$

a balance between vertical advection of heat and horizontal diffusion. The latter must be important to even out the temperature distribution so that the water just outside the upwelling layer is cooled by the rising cool water to prevent an unstable situation developing. The heat equation has become linear as a result of the combination of the large vertical velocity with the thermal layer $O(1)$ temperature gradient. Again it may be justified *a posteriori* that the neglected terms are indeed small.

Eqs. (20)–(24) form a closed set for the upwelling layer variables. If the variables are eliminated in favor of w the following equation is produced:

$$\frac{E^{\frac{1}{2}}}{S} \left[-\delta^2 E_H^2 w_{xxxxxx} + E_H^2 w_{xxxx\eta\eta} - S^{\frac{1}{2}} E^{\frac{1}{2}} \sigma_H R w_{xx} T_{T\eta} + f^2 w_{\eta\eta} \right] = 0. \quad (25)$$

To simplify the discussion of this equation two new parameters are introduced. Let

$$F = S^{\frac{1}{2}} E^{\frac{1}{2}} \sigma_H R, \quad D = E^{\frac{1}{2}} \delta / S^{\frac{1}{2}}, \quad (26)$$

Eq. (25) then becomes

$$D^2 E_H^2 w_{xxxxxx} + E_H^2 w_{xxxx\eta\eta} - F T_{T\eta} w_{xx} + f^2 w_{\eta\eta} = 0. \quad (27)$$

(i) (ii) (iii) (iv)

Note that D is a modified aspect ratio giving essentially the ratio of the thermal layer depth to the thermal layer width.

The various layers that may be obtained by balancing pairs of terms of (27) are summarized in Table 1, together with the restrictions on the various parameters involved. This table may be compared with that in Blumsack and Barcilon (1971) in which similar layers are found involving slightly different parameters for an axisymmetric linear vortex. A further discussion of similar layers is given in Blumsack (1972) for the transverse circulation near a coast when the solution is independent of the coordinate along the coast and the basic temperature gradient is linear. In this paper the flow is three-dimensional and the basic temperature field has exponential decay.

It is often assumed that for typical ocean values E_H and δ^2 are the same order. Thus, (26) implies that generally $E_H > D^2$. For this case the details given in Table 1 are summarized in Fig. 2. Notice how the layer structure conveniently merges onto the $E_H^{\frac{1}{2}}$ layer below the thermal layer. If the size of the parameters D , E_H and F are such that $E_H^2 > F D^2$, then the upper structure will be absent. However, the lowest layers will always appear at some depth since T_T will eventually become less than E_H/F . Detailed analysis shows that the layers associated with large vertical movements of water are the $E_H^{\frac{1}{2}}$, $E_H/(FT_T)^{\frac{1}{2}}$ and $(DE_H)^{\frac{1}{2}}/(FT_T)^{\frac{1}{2}}$ layers. The change in width of these layers with depth produces a crowding together of the streamlines as the surface is approached. Although the major upwelling that supplies the offshore Ekman transport comes from the very thin buoyancy layer, it is shown below that significant vertical velocities also occur in the thicker $(FT_T)^{\frac{1}{2}}$ layer.

If the horizontal eddy diffusivity $K_H = 10^3 \text{ m}^2 \text{ sec}^{-1}$, the horizontal eddy viscosity $\nu_H = 10^2 \text{ m}^2 \text{ sec}^{-1}$, and the other typical ocean values are those quoted in Johnson (1971), then $E_H^2/F D^2 \approx 6 \times 10^{-2}$ and $E_H/F \approx 2 \times 10^{-2}$. In the remainder of this paper only the upper layers will be examined in detail. The analysis given below shows that the actual upwelling into the surface Ekman layer comes only from the uppermost part of the thermal layer. The remaining parts of the upwelling region involve redistribution of water within the thermal layer and with deep upwelling from the abyssal layer.

5. The hydrostatic layer

To obtain the equations of motion for this layer and to justify some of the assumptions made in Section 4, a

FIG. 2. The layer structure of the upwelling region for $E_H > D^2$, and the associated temperature distribution T_T in the thermal layer near the eastern boundary.

formal expansion is used, in which:

$$\left. \begin{aligned} X &= (x-1)/F^{1/2}, \quad \eta = (1-z)S^{1/2}/E^{1/2} \\ u &= E^{1/2} S^{1/2} \left[u_T(x, y, \eta) + \frac{E_H}{F} \bar{u}(X, y, \eta) \right] \\ v &= E^{1/2} S^{1/2} (v_T + \bar{v}) \\ w &= E^{1/2} \left(w_T + \frac{E_H}{F^{3/2}} \bar{w} \right) \\ p &= E^{1/2} S^{1/2} (p_T + F^{1/2} \bar{p}) \\ T &= T_T + F^{1/2} \bar{T} \end{aligned} \right\}.$$

Substitution of these relations into (1), (2), (4), (5) and (23) leads to

$$f\bar{v} = \bar{p}_X, \quad (28)$$

$$f\bar{u}_X = \bar{v}_{XX}, \quad (29)$$

$$\bar{p}_\eta + \bar{T} = 0, \quad (30)$$

$$\bar{u}_X = \bar{w}_\eta, \quad (31)$$

$$\frac{c}{f} \bar{w} T_T = \bar{T}_{XX} \quad \text{as} \quad T_{T\eta} = -\frac{c}{f} T_T, \quad (32)$$

provided $E_H^{1/2} < (FT_T)^{1/2} < E_H^{1/2}$, $E_H^2 D^2 < (FT_T)^3$, $T_T \leq O(1)$. [This layer is replaced by an $E_H^{1/2}$ layer if $(FT_T)^{1/2} > E_H^{1/2}$.]

The longshore velocity correction $\bar{v}$ is of the same order as its thermal layer counterpart and (28) shows

that it is geostrophic. However, (29) shows that the small onshore correction is not geostrophic. The hydrostatic balance is maintained in (30) and the correction to the heat equation is a balance between vertical advection and lateral diffusion of heat. The vertical advection becomes important here because of the enhanced vertical velocity in the coastal layer.

The details of the solution for this outer hydrostatic layer are given in the Appendix, where it is shown that the main purpose of this layer is merely to bring the interior longshore velocity to zero at the coast. As the interior thermal layer and the hydrostatic layer both satisfy the thermal wind relation, the boundary condition

$$v_T + \bar{v} = 0 \quad \text{at} \quad x=1 \quad (X=0)$$

is equivalent to

$$T_{Tx} + \bar{T}_X = 0 \quad \text{at} \quad x=1 \quad (X=0).$$

Hence the interior temperature gradient is also brought to zero at the coast in this layer.

Although the vertical velocity in this layer is larger than the interior value, the vertical transport is only $O(E_H/F)$ and is therefore small. Hence the flow in this layer does not form part of the lowest order transverse circulation. It is shown in the Appendix that a small downward transport of water does occur as a secondary consequence of bringing the longshore flow to zero.

Thus, except for this secondary vertical transport, the incoming water from the thermal layer passes straight through the hydrostatic layer to enter the inner buoyancy layer.

6. The buoyancy layer

The detailed theory of the buoyancy layer, whose width is of order $(DE_H)^{1/2}/(FT_T)^{1/2}$, is given in this section. An illustration of the solution appears in Section 7.

a. Determination of upward velocity and total upwelling

In this layer further correction functions are introduced so that the boundary conditions on u and w may be satisfied and so that the upwelling into the Ekman layer may be provided. These requirements determine the magnitude of the corrections to the u and w variables and the rest then follows. The boundary condition on u requires that the correction has the same magnitude as u in the thermal layer. The requirement that the vertical transport be $O(E_H^{1/2})$ to supply the upwelling to the Ekman layer determines the size of the vertical velocity in this layer.

Let

$$\left. \begin{aligned} \xi &= (x-1)(FT_T)^{1/2}/(DE_H)^{1/2} \\ u &= E_H^{1/2} S^{1/2} \left[u_T(x, y, \eta) + \frac{E_H}{F} \bar{u}(X, y, \eta) + \hat{u}(\xi, y, \eta) \right] \\ v &= E_H^{1/2} S^{1/2} \left[v_T + \bar{v} + \frac{D}{(FT_T)^{1/2}} \hat{v} \right] \\ w &= E_H^{1/2} \left[w_T + \frac{E_H}{F^{1/2}} \bar{w} + \frac{(FT_T)^{1/2}}{(DE_H)^{1/2}} \hat{w} \right] \\ p &= E_H^{1/2} S^{1/2} \left[p_T + F^{1/2} \bar{p} + \frac{(FT_T)^{1/2} E_H^{1/2}}{D^{1/2}} \hat{p} \right] \\ T &= T_T + F^{1/2} \bar{T} + \frac{(FT_T)^{1/2} D^{1/2}}{E_H^{1/2}} \hat{T} \end{aligned} \right\}, \quad (33)$$

and substitute in Eqs. (1), (2), (4), (5) and (23). The equations for the correction functions in this layer are found to be

$$0 = -\hat{p}_\xi + \hat{u}_{\xi\xi}, \quad (34)$$

$$f\hat{u} = \hat{v}_{\xi\xi}, \quad (35)$$

$$\hat{T} + \hat{w}_{\xi\xi} = 0, \quad (36)$$

$$\hat{u}_\xi - \hat{w}_\eta + \frac{c}{4f} \xi \hat{w}_\xi = 0, \quad (37)$$

$$\frac{c}{f} \hat{w} = \hat{T}_{\xi\xi}, \quad (38)$$

provided $T_T > E_H/F$, E_H^2/FD^2 and $FE_H^2 < D^2 < E_H$. The extra term in (37) arises because ξ depends on T_T and hence on η . Eqs. (34) to (36) show that the flow in this layer is both non-geostrophic and non-hydrostatic.

Eqs. (36) and (38) lead to the usual buoyancy layer relation

$$\hat{w}_{\xi\xi\xi\xi} + \frac{c}{f} \hat{w} = 0, \quad (39)$$

which has the following bounded solution as $\xi \rightarrow -\infty$:

$$\begin{aligned} \hat{w} &= a(y, \eta) \exp \left[(1+i) \left(\frac{c}{4f} \right)^{1/2} \xi \right] \\ &+ b(y, \eta) \exp \left[(1-i) \left(\frac{c}{4f} \right)^{1/2} \xi \right]. \end{aligned} \quad (40)$$

In this layer, as w has its largest value, the boundary condition at $\xi=0$ is $\hat{w}=0$; hence $b=-a$.

To determine the function $a(y, \eta)$ it is necessary to satisfy the boundary condition on u . From (37) and (40)

$$\begin{aligned} \hat{u} &= \frac{\partial}{\partial \eta} \left[\left(\frac{f}{4c} \right)^{1/2} a \{ (1-i) \exp[(1+i)(c/4f)^{1/2} \xi] \right. \\ &\quad \left. - (1+i) \exp[(1-i)(c/4f)^{1/2} \xi] \} \right] \\ &- \frac{c}{4f} \xi a \{ \exp[(1+i)(c/4f)^{1/2} \xi] \\ &\quad - \exp[(1-i)(c/4f)^{1/2} \xi] \}. \end{aligned} \quad (41)$$

The boundary condition on u is, to lowest order,

$$u_T + \hat{u} = 0 \quad \text{at } x=1, \xi=0,$$

and therefore, from (41) and (17),

$$-i \left(\frac{4f}{c} \right)^{1/2} \frac{\partial a}{\partial \eta} = \left(\frac{T_y^*}{c} + \frac{\beta T^*}{fc} + \frac{\beta}{f^2} \eta T^* \right) e^{-c\eta/f}.$$

Thus,

$$\begin{aligned} a(y, \eta) &= \hat{a}(y) - i \left(\frac{c}{4f} \right)^{1/2} \\ &\times \left(\frac{f}{c^2} T_y^* + \frac{2\beta T^*}{c^2} + \frac{\beta}{fc} \eta T^* \right) e^{-c\eta/f}. \end{aligned} \quad (42)$$

As the buoyancy layer is thinner than $E_H^{1/2}$ the Ekman suction condition does not apply at $\eta=0$. However, there is a continuity of volume flux condition, or in other words an upwelling condition, which requires from (9) that

$$\begin{aligned} \int_{-\infty}^0 \hat{w}|_{\eta=0} d\xi &= -\tau^y/f|_{x=1} \\ &= -(\text{Ekman layer transport at } x=1) \\ &= -ia(y, 0) \left(\frac{4f}{c} \right)^{1/2}. \end{aligned}$$

Hence, using (42),

$$\hat{a} = i \left(\frac{c}{4f} \right)^{\frac{1}{2}} \left(-\frac{f}{c^2} T_y^* + \frac{2\beta T^*}{c^2} - \frac{\tau_E^y}{f} \right), \quad (43)$$

where $\tau_E^y = \tau^y(1, y)$ is the northward component of wind stress at the eastern boundary.

Substituting (43) and (42) into (40) gives

$$\begin{aligned} \hat{w} = & \left[-\left(\frac{f}{c^2} T_y^* + \frac{2\beta T^*}{c^2} - \frac{\tau_E^y}{f} \right) \right. \\ & \left. + \left(\frac{f}{c^2} T_y^* + \frac{2\beta T^*}{c^2} + \frac{\beta}{f^2} \eta T^* \right) e^{-c\eta/f} \right] \\ & \times \left(\frac{4c}{f} \right)^{\frac{1}{2}} \exp \left(\frac{c}{4f} \right)^{\frac{1}{2}} \xi \sin \left(\frac{c}{4f} \right)^{\frac{1}{2}} \xi, \quad (44) \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^0 \hat{w} d\xi = & \left(\frac{f}{c^2} T_y^* + \frac{2\beta T^*}{c^2} - \frac{\tau_E^y}{f} \right) \\ & - \left(\frac{f}{c^2} T_y^* + \frac{2\beta T^*}{c^2} + \frac{\beta}{f^2} \eta T^* \right) e^{-c\eta/f}. \quad (45) \end{aligned}$$

These two equations give the upward velocity and the total upwelling at each depth in terms of the surface temperature T^* and the wind stress at the coast τ_E^y . In general, T^* and τ_E^y will not be independent as the surface temperature is determined largely by the amount of upwelling which, in turn, is determined by the magnitude of τ_E^y . Notice in (45) as $\eta \rightarrow \infty$ that there will be deep upwelling from lower layers into the region under consideration. The presence of the sine term in (44) means that there will be a small amount of downward flow offshore of the main upflow.

b. Determination of temperature distribution

The temperature distribution in the buoyancy layer may be found from (36) as

$$\begin{aligned} \hat{T} = & \left[\left(\frac{f}{c^2} T_y^* + \frac{2\beta T^*}{c^2} - \frac{\tau_E^y}{f} \right) \right. \\ & \left. - \left(\frac{f}{c^2} T_y^* + \frac{2\beta T^*}{c^2} + \frac{\beta}{f^2} \eta T^* \right) e^{-c\eta/f} \right] \\ & \times \frac{1}{2} \left(\frac{4c}{f} \right)^{\frac{1}{2}} \exp \left(\frac{c}{4f} \right)^{\frac{1}{2}} \xi \cos \left(\frac{c}{4f} \right)^{\frac{1}{2}} \xi. \quad (46) \end{aligned}$$

From (33),

$$T_x = T_{Tx} + \bar{T}_x + \frac{FT_T}{E_H} \hat{T}_x,$$

and as $T_T > E_H/F$ in this layer, the no-heat-flux boundary condition requires that $\hat{T}_x = 0$ at the coast.

This is clearly not possible from (46) and therefore an inner region in which diffusion is dominant is required to satisfy this boundary condition. This region will be discussed further in Section 9. For the moment it is sufficient to ensure that there is *no net flux of heat* into this inner layer. This requires that

$$\int_0^\infty \hat{T}_x|_{\xi=0} d\eta = 0, \quad (47)$$

which is satisfied if

$$fT_y^* + \frac{3}{2}\beta T^* = 2c^2 \frac{\tau_E^y}{f}. \quad (48)$$

This equation determines $T^*(y)$ in terms of τ_E^y and may be integrated to give

$$T^* = \frac{1}{f^{\frac{1}{2}}} \left[A + \int_0^y 2c^2 \frac{\tau_E^y}{f^{\frac{1}{2}}} dy' \right]. \quad (49)$$

The constant A may be found from the integral condition (6) that

$$\int_0^1 \int_0^1 T_0 dx dy = 1.$$

Using (15) and (49) this integral becomes (after integration by parts)

$$\begin{aligned} \frac{2}{\beta} \left[1 - \frac{1}{(1+\beta)^{\frac{1}{2}}} \right] A = & 1 + \frac{c^2}{\beta} \int_0^1 \int_0^1 x \mathbf{k} \cdot \text{curl} \frac{\boldsymbol{\tau}}{f} dx dy \\ & - \frac{kc^3}{2\beta^2} \log(1+\beta) - \frac{4c^2}{\beta} \int_0^1 \frac{\tau_E^y}{f} \left[1 - \frac{f^{\frac{1}{2}}}{(1+\beta y)^{\frac{1}{2}}} \right] dy. \quad (50) \end{aligned}$$

Hence $\hat{w}$ and $\hat{T}$ may be determined in terms of τ_E^y . Some examples of the solution are discussed in Section 7 for a particular wind stress. The details of the solution for $\hat{u}$ and $\hat{v}$ are given in Section 8.

7. Illustration of buoyancy layer solution

To illustrate the solution obtained above, set $c=0.6$. This is the value found in Johnson (1971) to give the most satisfactory distribution of surface temperature over the ocean gyre. Let the surface wind stress be $\tau^x = \tau^y = -\cos \pi y$ to represent the westerlies and trade winds. With these values, (45) with (49) and (50) gives the upward transport at each depth, from which the vertical velocity distribution may be deduced using (44). The other variable of interest is u_T at $x=1$, the onshore velocity component just outside the upwelling layer. If (48) is substituted in (17), then

$$u_T(1, y, \eta) = \left(-\frac{2c}{f^2} \tau_E^y + \frac{\beta T^*}{2fc} - \frac{\beta}{f^2} \eta T^* \right) \exp(-c\eta/f).$$

Use of (49) and (50) then gives the required velocity.

FIG. 3. Velocity profiles for a case of surface upwelling at $y=0$. The left-hand side shows the onshore velocity u_T in the thermal layer at the edge of the upwelling region. The right-hand side shows the upward transport as a function of the thermal layer coordinate η .

The result of calculating

$$u_T(1, y, \eta) \quad \text{and} \quad \int_{-\infty}^0 \hat{w} d\xi$$

at two different latitudes is shown in Figs. 3 and 4.

Fig. 3 shows a region of surface upwelling where the wind stress at the coast has a southward component. Of particular interest is the fact that the upwelling to the surface only comes from the upper part of the thermal layer. The downwelling below gives the isotherms the observed downward bowing. The profile for vertical velocity near the coast, which is the same as the profile for vertical transport in Fig. 3, compares favorably with the corresponding profile in Fig. 13 of Bryan and Cox (1968) for a numerical solution using the full nonlinear equations.

FIG. 4. Velocity profiles for a case of surface downwelling at $y=1$. The notation is the same as in Fig. 3.

Fig. 4 shows velocity profiles in a region where there is no surface upwelling as the wind stress has a northward component near the coast. There is downwelling from the Ekman layer but again considerable deep upwelling from the abyssal layer. The thermal layer flow is entirely away from the coast. Comparison with the appropriate figure in Bryan and Cox (1968) is satisfactory.

The deep upwelling indicated in Fig. 3 requires that the flow in the abyssal layer be toward the coast. This is illustrated diagrammatically in Fig. 5. Notice that the flow normal to the coast is offshore in the surface Ekman layer and then reverses twice in the thermal layer and is ultimately onshore in the abyssal layer. This kind of reversal of the onshore velocity is also apparent in Fig. 3 of Smith (1968) which presents observations made off Oregon in an early stage of upwelling.

8. Horizontal velocities within the upwelling layers

The horizontal velocities within the buoyancy layer may be calculated from (41) and (35). They are

$$\begin{aligned} \hat{u} = & \left(\frac{2c}{f^2} \tau_E^y + \frac{\beta T^*}{2fc} + \frac{\beta}{f^2} \eta T^* \right) e^{-c\eta/f} e^{\hat{\xi}} (\cos \hat{\xi} - \sin \hat{\xi}) \\ & + \frac{1}{2} \hat{\xi} \left[\left(\frac{\beta T^*}{2fc} + \frac{c\tau_E^y}{f^2} \right) \right. \\ & \left. - \left(\frac{2c^2}{f^3} \tau_E^y - \frac{\beta T^*}{2f^2} + \frac{c\beta}{f^3} \eta T^* \right) e^{-c\eta/f} \right] e^{\hat{\xi}} \sin \hat{\xi}, \\ \hat{v} = & \left(\frac{f}{c} \right)^{\frac{1}{2}} \left(\frac{2c}{f} \tau_E^y + \frac{\beta T^*}{2c} + \frac{\beta}{f} \eta T^* \right) e^{-c\eta/f} e^{\hat{\xi}} \cos \hat{\xi} \\ & + \left[\left(\frac{\beta T^*}{2c} + \frac{c\tau_E^y}{f} \right) - \left(\frac{2c^2}{f^2} \tau_E^y - \frac{\beta T^*}{2f} + \frac{c\beta}{f^2} \eta T^* \right) e^{-c\eta/f} \right] \\ & \times \left(\frac{f}{c} \right)^{\frac{1}{2}} e^{\hat{\xi}} [(1 - \hat{\xi}) \cos \hat{\xi} + \sin \hat{\xi}], \quad (51) \end{aligned}$$

where

$$\hat{\xi} = (c/4f)^{\frac{1}{2}} \xi = (FT^*c/4fD^2E_H^2)^{\frac{1}{2}} \exp(-c\eta/4f).$$

From the expression for $\hat{v}$ it may be shown that

$$\int_{-\infty}^0 \hat{v} d\xi = 0;$$

thus, there is no net northward flow in this layer.

From the form of the boundary layer coordinate given in (51) it is clear that the thickness of this boundary layer grows with depth and that $\hat{u}$ tends to zero as $\eta \rightarrow \infty$. Moreover, as the correction to v provided

by this layer is

$$v_c = E_H^{\frac{1}{2}} S^{\frac{1}{2}} \frac{D}{(FT^*)^{\frac{1}{2}}} e^{c\eta/(2f)} \hat{\psi}, \quad (52)$$

it may be seen that v_c increases with depth. However, in the $E_H/(FT_T)^{\frac{1}{2}}$ layer below the buoyancy layer, the correction to v is of order

$$E_H^{\frac{1}{2}} S^{\frac{1}{2}} E_H/(FT_T), \quad (53)$$

and (52) and (53) have the same magnitude when $T_T = O(E_H^2/FD^2)$. Thus, the solution in the buoyancy layer merges into its counterpart in the viscous hydrostatic layer at the appropriate depth.

In a similar way in the $E_H^{\frac{1}{2}}$ upwelling layer the correction to v is of order $E_H^{\frac{1}{2}} S^{\frac{1}{2}}$ which is the value attained by (53) when $T_T = O(E_H/F)$. Thus, the $E_H/(FT_T)^{\frac{1}{2}}$ layer acts as a transition region between the near-surface upwelling layer and the deep, almost isothermal, $E_H^{\frac{1}{2}}$ upwelling layer.

9. Inner diffusion layer

An inner layer is required to satisfy the no-heat-flux through the coast boundary condition which could not be satisfied in the buoyancy layer. The reason that a diffusion layer is needed is that w tends to zero as the coast is approached and therefore near the coast the

horizontal diffusion cannot be balanced by the vertical advection but must be balanced by vertical diffusion instead. The simplest way to try to model this is by choosing eddy diffusivities that are different in this region than further away from the coast.

From (45) and (46)

$$\hat{T}_\xi|_{\xi=0} = -\frac{c}{f} \int_{-\infty}^0 \hat{w} d\xi,$$

which may be estimated from the graphs in Figs. 3 and 5. At $y=0$, for example, there is a flux of heat into the diffusion layer near the surface and deep down, with a compensatory outward flux at mid-depths. The transfer of this heat vertically must be carried out by turbulent mixing in this innermost layer. If the ratio of vertical to horizontal eddy diffusivity is large enough then the heat equation reduces to

$$\tilde{T}_{\tilde{x}\tilde{x}} + \tilde{T}_{\eta\eta} = 0,$$

where $\tilde{x}$ is the approximate stretched coordinate. This may be attained if K_H decreases near the coast. Thus, a solution is required of Laplace's equation that is bounded for large $\tilde{x}$ and η and which satisfies

$$\tilde{T}_{\tilde{x}} + \hat{T}_\xi = 0 \quad \text{at} \quad \xi = \tilde{x} = 0.$$

Solutions to this quarter-plane problem are obtainable either by Fourier sine transforms or by mapping on to the half-plane. As the solution does not affect the determination of the solution elsewhere no details will be given.

APPENDIX

The Solution for the Hydrostatic Layer

Elimination of the variables in (28)–(32) in favor of $\bar{w}$ produces

$$\frac{c}{f} T_T \bar{w}_{XX} + f^2 \bar{w}_{\eta\eta} = 0. \quad (A1)$$

The introduction of a new independent variable

$$s = e^{-c\eta/(2f)} \quad \text{so that} \quad T_T(1, y, \eta) = T^* s^2$$

changes (A1) into

$$\frac{4T^*}{fc} \bar{w}_{XX} + \bar{w}_{ss} + \frac{1}{s} \bar{w}_s = 0. \quad (A2)$$

Looking for a solution of (A2) of the form

$$\bar{w} = \sum_{n=1}^{\infty} W_n(s, y) \exp[-\lambda_n(y)X]$$

leads to

$$W_n'' + \frac{1}{s} W_n' + \frac{4T^* \lambda_n^2}{fc} W_n = 0,$$

FIG. 5. Schematic representation of the transverse circulation at $y=0$.

which has the solutions

$$W_n = A_n J_0 \left[\left(\frac{4T^*}{fc} \right)^{\frac{1}{2}} \lambda_n s \right] + B_n Y_0 \left[\left(\frac{4T^*}{fc} \right)^{\frac{1}{2}} \lambda_n s \right].$$

However, since $s \rightarrow 0$ as $\eta \rightarrow \infty$, Y_0 is unbounded for large η . This requires $B_n = 0$ and the general solution for $\bar{w}$ is

$$\bar{w} = \sum_{n=1}^{\infty} A_n e^{-\lambda_n X} J_0 \left[\left(\frac{4T^*}{fc} \right)^{\frac{1}{2}} \lambda_n s \right]. \quad (A3)$$

The Ekman suction condition at $\eta = 0$ (or $s = 1$) is $\bar{w} = 0$ as $\bar{w} > O(E^{\frac{1}{2}})$ in this layer. This gives, from (A3),

$$\left(\frac{4T^*}{fc} \right)^{\frac{1}{2}} \lambda_n = j_n, \quad (A4)$$

the positive zeros of J_0 . Hence

$$\bar{w} = \sum_{n=1}^{\infty} A_n \exp \left[- \left(\frac{fc}{4T^*} \right)^{\frac{1}{2}} j_n X \right] J_0 [j_n e^{-c\eta/(2f)}]. \quad (A5)$$

Notice that as $J_0(0) = 1$, $\bar{w}$ does not approach zero as $\eta \rightarrow \infty$ and there is deep upwelling, although the actual upward transport is small compared with the interior.

The unknown function A_n may be found by satisfying the boundary condition $V_T + \bar{v} = 0$ at $x = 1$ (or $X = 0$). However, as this layer and the interior both satisfy the thermal wind relation

$$f v_{T\eta} = -T_{Tx}, \quad f \bar{v}_{\eta} = -\bar{T}_X,$$

it is simpler here to satisfy the condition

$$T_{Tx} + \bar{T}_X = 0 \quad \text{at } x = 1, X = 0. \quad (A6)$$

Use of (32) and (A5) gives

$$\bar{T} = 4 \left(\frac{T^*}{f} \right)^2 e^{-c\eta/f} \sum_{n=1}^{\infty} \frac{A_n}{j_n^2} e^{-\lambda_n X} J_0 [j_n e^{-c\eta/(2f)}] \quad (A7)$$

which does approach zero as $\eta \rightarrow \infty$. Application of the boundary condition (A6) to (A7) and (10) gives

$$\frac{2c^{\frac{1}{2}}}{f^{\frac{3}{2}}} T^{*\frac{3}{2}} \sum_{n=1}^{\infty} \frac{A_n}{j_n} J_0(j_n s) = T_{0x}|_{x=1}.$$

Multiplying both sides by $s J_0(j_m s)$, integrating with respect to s over $(0, 1)$, and using the orthogonal property

$$\int_0^1 s J_0(j_n s) J_0(j_m s) ds = \frac{1}{2} [J_1(j_m)]^2 \delta_{mn},$$

we have

$$A_n = \frac{f^{\frac{3}{2}} T_{0x}|_{x=1}}{c^{\frac{1}{2}} T^{*\frac{3}{2}} J_1(j_n)},$$

where, from (15),

$$T_{0x}|_{x=1} = -\frac{c^2}{\beta} \mathbf{k} \cdot \text{curl} \left[\frac{\tau}{f} \right]_{x=1} - \frac{kc^3}{\beta f}.$$

The upward transport in this layer is

$$\begin{aligned} \int_{-\infty}^0 \bar{w} dX &= -\frac{f}{cT^*} T_{0x}|_{x=1} \sum_{n=1}^{\infty} \frac{2J_0(j_n s)}{j_n J_1(j_n)} \\ &= -\frac{f}{cT^*} T_{0x}|_{x=1} \quad \text{for } \eta > 0, \end{aligned} \quad (A8)$$

as

$$\sum_{n=1}^{\infty} \frac{2J_0(j_n s)}{j_n J_1(j_n)} = 1 \quad \text{for } 0 \leq s < 1.$$

In the subtropical gyre where the surface temperature increases eastward over the interior of the ocean, T_{0x} is negative and (A8) shows that there is a small downward transport in the $F^{\frac{1}{2}}$ layer that is independent of depth.

The horizontal velocity components may be determined from (31) and (29) to be

$$\begin{aligned} \bar{u} &= \frac{T_{0x}|_{x=1}}{T^*} e^{-c\eta/(2f)} \sum_{n=1}^{\infty} e^{-\lambda_n X} \frac{J_0'[j_n e^{-c\eta/(2f)}]}{J_1(j_n)}, \\ \bar{v} &= -\frac{4}{c} T_{0x}|_{x=1} e^{-c\eta/(2f)} \sum_{n=1}^{\infty} e^{-\lambda_n X} \frac{J_0'[j_n e^{-c\eta/(2f)}]}{j_n^2 J_1(j_n)}, \end{aligned}$$

with λ_n given by (A4). The Fourier-Bessel series required to check that $\bar{v} + v_T = 0$ at $X = 0$ is

$$\sum_{n=1}^{\infty} \frac{J_0(j_n s)}{j_n^3 J_1(j_n)} = \frac{1-s^2}{8}.$$

Thus, this outer hydrostatic layer merely brings the interior longshore velocity to zero at the coast and in so doing smooths out the interior temperature gradient T_{Tx} to zero at the coast. A secondary effect is the production of a small downward transport of water associated with vertical velocities that are significant compared with the seaward interior flow, but not large enough to enter the lowest order transverse circulation.

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