On the Use of the DuFort-Frankel Finite-Difference Approximation for Simulation of Diffusion in Geophysical Fluids[1]

JAMES E. OVERLAND

Dept. of Meteorology and Oceanography, New York University, Bronx, N. Y. 10453

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The DuFort-Frankel scheme for numerical simulation of diffusion does not necessarily conserve scalar properties. Consider the temperature equation commonly used in the Bénard cell problem:

$$\frac{T_{ij}^{N+1}-T_{ij}^{N-1}}{2\Delta t}=-J^*(\psi^N,T^N)+\frac{K}{(\Delta S)^2}\delta^N, \qquad (1)$$

where J^* is a conservative Jacobian operator, ΔS the

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space discretization, Δt the time step, K the thermal diffusivity, and δ a diffusion operator. Two numerically stable diffusion operators in common use are forward differencing with a one time step lag, i.e.,

$$\overset{N}{\delta}_F = \overset{N-1}{T}_{i+1,j} + \overset{N-1}{T}_{i-1,j} + \overset{N-1}{T}_{i,j+1} + \overset{N-1}{T}_{i,j-1} - 4\overset{N-1}{T}_{ij}, \quad (2)$$

and the DuFort-Frankel operator

$$\overset{N}{\delta}_{DF} = \overset{N}{T}_{i+1,j} + \overset{N}{T}_{i-1,j} + \overset{N}{T}_{i,j+1} + \overset{N}{T}_{i,j-1} - 2\overset{N+1}{T}_{ij} - 2\overset{N-1}{T}_{ij}. \quad (3)$$

Subscripts i and j refer to space points, while the superscript refers to time level.

Further, consider adjacent boxes A and B, with all sides insulated except for the interface. For the forward-difference scheme, the diffusive flux from A to B per time step is given by

$$\gamma[\overset{N-1}{T}_A - \overset{N-1}{T}_B], \quad (4)$$

with

$$\gamma \equiv \frac{2K\Delta t}{(\Delta S)^2}. \quad (5)$$

By symmetry the flux from B to A per time step is equal and opposite, which conserves the scalar T. For the DuFort-Frankel scheme, the flux from A to B is given by

$$\gamma\left[\overset{N}{T}_A - \left(\frac{\overset{N+1}{T}_B + \overset{N-1}{T}_B}{2}\right)\right]; \quad (6)$$

this is not necessarily the same magnitude as the flux from B to A, which is given by

$$\gamma\left[\overset{N}{T}_B - \left(\frac{\overset{N+1}{T}_A + \overset{N-1}{T}_A}{2}\right)\right]. \quad (7)$$

In certain geophysical problems the non-conservative nature of the DuFort-Frankel scheme may be restrictive. Salinity intrusion in an estuary has several features in common with the Bénard problem, with the major differences being that a density source is maintained at a side boundary and that the height-to-length ratio is very small. Numerical experiments using the DuFort-Frankel scheme on the estuary problem consistently lost salt. The loss of salt was estimated to be 0.60% of the salt in the system per time iteration. It is felt that the DuFort-Frankel scheme did not perform well for this problem because the interior was far removed from the source-sink boundaries.

The use of the forward-difference scheme appears to be necessary when strict conservation of a scalar is required, although the required small time step may be prohibitive. The other recourse is the use of implicit methods, which can be handled easily in one dimension (Richtmyer and Morton, p. 199), but are much more cumbersome in large multi-dimensional systems. It is apparent that the inclusion of diffusion in numerical simulations is a non-trivial exercise.

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REFERENCE

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