Mode Conversion for Rossby Waves over Topography: Comments on "Localized Coupling between Surface- and Bottom-Intensified Flow over Topography"

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1. Introduction

Recently, Hallberg (1997) used multilayer quasigeostrophic models to demonstrate that topography induces a linear coupling between layers and, as a consequence, that the layer in which Rossby waves are intensified can change in the course of wave propagation. Thus, a bottom-intensified Rossby wave may become surface intensified (and vice versa) when propagating over variable topography.

The phenomenon is most easily understood in the two-layer case and using the WKB approximation, that is, assuming a separation between the scale of the waves and the scale of the topography. As described by Hallberg, the dispersion relation has two branches, neither of which is exclusively associated with surface- or bottom-intensified waves; rather, each branch may correspond to either type of wave, depending on the parameters (wavenumbers, topography slope, etc.). In other words, the physical nature of the waves changes along a branch. Provided that the WKB assumptions are satisfied, wave propagation takes place along a given branch. Thus, the physical nature of a wave can change when varying parameters are encountered during propagation; this is referred to as mode conversion. For Rossby waves over topography, it implies that the layer in which a wave is intensified may change during propagation.

However, it is also possible that the two branches of the dispersion relation become linearly coupled, leading to wave energy transmission between branches. This phenomenon, which occurs only when the WKB assumptions are violated because the two branches come asymptotically close together, has, in fact, been studied in detail in a variety of physical systems and is often also called, somewhat confusingly, mode conversion [see, e.g., Flynn and Littlejohn (1994) and references therein].[1] It is the purpose of this note to examine its relevance to Rossby-wave propagation over topography. Specifically, our aims are (i) to consider Rossby wave propagation in the light of the general theory of mode conversion and point out the pertinent literature; (ii) to apply this theory in the simple case of Rossby waves propagating over a one-dimensional (meridional) topography and, in particular, derive an expression for the amount of wave energy that is transmitted from one branch of the dispersion relation to the other; and (iii) to use the theory to clarify some of the Hallberg (1997) results. We emphasise that the coupling between the two branches of the dispersion relation should be regarded as an exceptional phenomenon, and we discuss Hallberg's (1997) mass-conservation argument, which suggests, on the contrary, that such a coupling is widespread.

2. Rossby waves over meridional topography

The linearized evolution equations for a two-layer quasigeostrophic flow over meridional topography are written

$$\partial_t[\nabla^2\psi_i + \lambda_i^{-2}(\psi_{3-i} - \psi_i)] + \beta_i\partial_x\psi_i = 0, \quad i = 1, 2, \qquad (2.1)$$

where ψ_i denotes the streamfunction in each layer, λ_i the internal radius of deformation based on the depth of each layer, and the index $i = 1, 2$ refers to the upper and lower layer respectively; $\beta_1 = \beta$ is the planetary vorticity gradient and $\beta_2 = \beta + f/h_2\partial_y\eta$ is the potential vorticity gradient in the lower layer. In the latter expression, f is the Coriolis parameter,

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[1] The theory of mode conversion has been recently applied to a different oceanographic problem, namely coupling between Kelvin and Yanai waves, by Kaufman et al. (1999); earlier work by Allen and Romea (1980) considered coupling between Kelvin and coastal-trapped waves but did not quantify the possible transmission between branches. [See also Grimshaw and Allen (1979, 1982) for a finite-dimensional analog.]

h_2 the lower-layer depth (taken as constant following the quasigeostrophic scaling), and η is the bottom topography height.

We assume that the lower-layer potential vorticity gradient varies slowly and write formally $\beta_2 = \beta_2(\epsilon y)$, where $\epsilon \ll 1$. This assumption, which is valid providing that the topography slope varies slowly, allows the WKB approximation to be employed. In this approximation, the streamfunction is written in the form

$$\psi_i(x, y, t) = \frac{\lambda_{3-i}}{\lambda_i} e^{i[kx+\theta(y)-\omega t]}\phi_i(\epsilon y), \qquad i = 1, 2,$$

where $\theta(y)$ is a (rapidly varying) phase. The zonal wavenumber k and the frequency ω have been taken constant because (2.1) is independent of x and t, and the factor λ_{3-i}/λ_i has been introduced for convenience. Substituting this expression into (2.1) leads to the eigenvalue problem

$$\mathbf{D\Phi} := \begin{pmatrix} \lambda_2\lambda_1^{-1}(k^2 + l^2 + \lambda_1^{-2} - s_1) & -(\lambda_1\lambda_2)^{-1} \\ -(\lambda_1\lambda_2)^{-1} & \lambda_1\lambda_2^{-1}(k^2 + l^2 + \lambda_2^{-2} - s_2) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0, \qquad (2.2)$$

where the meridional wavenumber $l = \partial_y\theta$ is the eigenvalue and $\mathbf{\Phi}$, which describes the vertical structure of the wave, is the eigenvector. In the above, we have defined $s_i = -k\beta_i/\omega$; we note that s_1 is a constant, whereas $s_2 = s_2(\epsilon y)$ depends slowly on the meridional coordinate.

The dispersion relation is derived from the solvability condition

$$\det\mathbf{D} = 0$$

and can be written

$$l^2 = \frac{1}{2}[l_1^2 + l_2^2 \pm \sqrt{(l_1^2 - l_2^2)^2 + 4(\lambda_1\lambda_2)^{-2}}],$$

where

$$l_i^2 = s_i - k^2 - \lambda_i^2, \qquad i = 1, 2,$$

are the squared meridional wavenumbers obtained for (uncoupled) one-layer flows whose characteristics are those of the upper and lower layer, respectively. In the above, we require that $l_i{}^2 > 0$, $i = 1, 2$, a sufficient condition for wave propagation.

Regarding k, ω, β, and λ_i as fixed parameters, the dispersion relation provides the relationship between the meridional wavenumber l and the topography slope—or equivalently $s_2(\epsilon y)$ or $l_2(\epsilon y)$—that should be satisfied along the propagation rays. Concentrating on the case $l > 0$ (the case $l < 0$ being similar), it is seen that the dispersion relation has two branches, separated by the two curves $l = l_1$ and $l = l_2(\epsilon y)$. This is illustrated in Fig. 1, which displays the wavenumber l as a function of l_2 used in place of the meridional coordinate y. Dimensionless variables are constructed by taking l_1 as inverse reference length; this leaves $\delta^2 = (\lambda_1\lambda_2 l_1^2)^{-2}$ as the only dimensionless parameter. This parameter controls the distance between the dispersion curves and the straight lines $l = l_1$ and $l = l_2$ (giving the dispersion relation in uncoupled layers) as well as the distance between both branches of the dispersion relation: for small δ, these distances are small—in fact, an estimate of the minimum distance between the two branches is found to be δ.

The vertical structure of the modes can be derived from the eigenvector $\mathbf{\Phi}$. It is easy to show that the ratio between the streamfunction in each layer satisfies

$$\left|\frac{\psi_1}{\psi_2}\right| = \frac{\lambda_2}{\lambda_1}\left|\frac{l^2 - l_2^2}{l^2 - l_1^2}\right|^{1/2}.$$

Thus, surface intensification corresponds to $l \approx l_1$, whereas bottom intensification corresponds to $l \approx l_2$. Clearly, then, each branch of the dispersion relation can be associated with either type of intensification. In other words, the physical nature of the modes changes as l_2 changes, that is, as the slope of the bottom topography changes. This is, in the simplified one-dimensional context, the physical phenomenon discussed by Hallberg (1997). Note that it is for small δ that the intensification of waves in a layer is the clearest since l becomes very close to either l_1 or l_2 when $l_1 \neq l_2$.

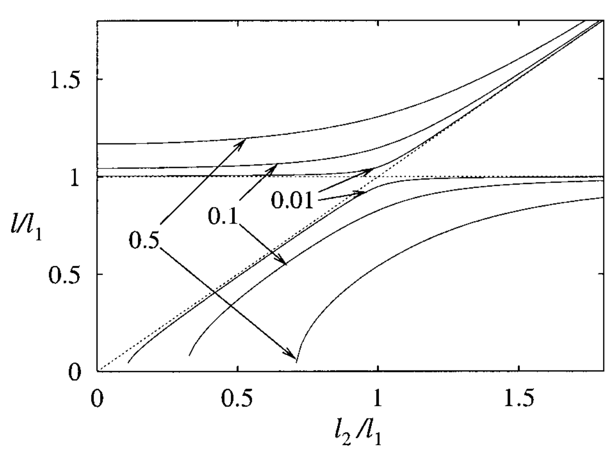

FIG. 1. Dispersion curves for $\delta^2 = (\lambda_1\lambda_2 l_1^2)^{-2} = 0.01$, 0.1, and 0.5. The dispersion curves $l = l_1$ and $l = l_2$ corresponding to uncoupled layers are also indicated (dashed lines).

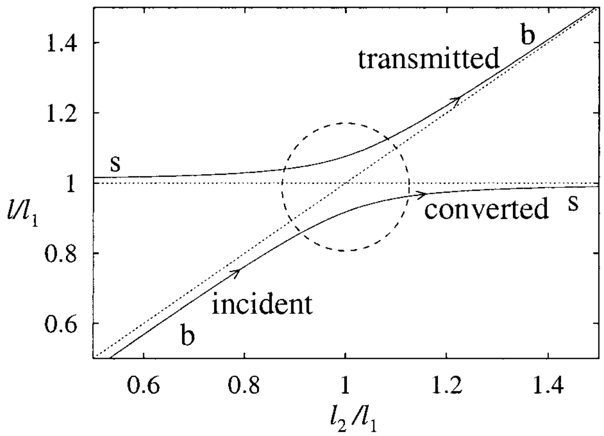

FIG. 2. Dispersion relation for $\delta^2 = 0.025$. Regions associated with surface- and bottom-intensified waves are identified by the letters s and b, respectively. The coupling region is located inside the dashed circle. The parts of the dispersion curves associated with incident, converted, and transmitted waves are indicated.

3. Mode conversion and transmission

We now use the theory of mode conversion, as developed, for instance, in Kaufman and Friedland (1987), Friedland et al. (1987), and Flynn and Littlejohn (1994), to discuss the possibility of energy transmission from one branch of the dispersion relation to the other. For definiteness, we consider the following situation illustrated in Fig. 2. A wave with $l_2/l_1 < 1$ initially concentrated in the bottom layer propagates meridionally over a topography in such a way that $l_2 = l_2(\epsilon y)$ increases. The point representing this *incident wave* on the dispersion curve starts in the left-hand corner of Fig. 2 and follows the lower branch toward the right. If the two branches are well separated, that is, if δ is $O(1)$ compared to the WKB parameter ϵ, then the WKB theory applies and the wave energy is confined to the lower branch. For $l_2 > l_1$, the wave is *converted*;[2] that is, its physical nature has changed and it is concentrated in the surface layer. In fact, the theory of mode conversion ensures that this scenario is valid, provided that $\delta \gg O(\epsilon^{1/2})$.

It is only when $\delta = O(\epsilon^{1/2})$ or smaller that the WKB approximation fails. It does so in the coupling region where the two branches of the dispersion relation come close together [dashed circle with radius $O(\delta)$ in Fig. 2]. This is because the vertical structure of the modes changes in this region on scales that are not long compared to the wavenumber l_2. As a result, there is nontrivial coupling between the two branches, with a transmission of energy between them. Because a proportion of the wave energy switches branch, a *transmitted* wave appears that is associated with the upper branch and is thus bottom intensified, and the conversion of the bottom-intensified incident wave into a surface-intensified wave is therefore not total.

The main achievement of the theory of mode conversion is the derivation of a scattering formula relating the amplitudes of the incident, converted, and transmitted waves [see, e.g., Kaufman and Friedland (1987); Friedland et al. (1987); Flynn and Littlejohn (1994)]. It is conveniently expressed in terms of wave energy (or action) meridional flux, that is, wave energy density times meridional group velocity. Let J_I, J_C, and J_T be the corresponding fluxes associated with the incident, converted, and transmitted waves. The scattering formula reads

$$|J_T| = T|J_I|, \qquad |J_C| = C|J_I|,$$

where the transmission coefficient T and conversion coefficient C satisfy $C = 1 - T$. The form of the matrix $\mathbf{D}$ in (2.2) is suitable for a direct application of the theory of Kaufman and Friedland (1987), Friedland et al. (1987), or Flynn and Littlejohn (1994).

First, we note that the center of the coupling region in the (y, l) plane is a saddle point of $\det\mathbf{D}$: its coordinates y_0 and l_0, say, solve the equations

$$\partial_y \det\mathbf{D} = \partial_l \det\mathbf{D} = 0,$$

which take the simpler form

$$l_0 = l_1 = l_2(\epsilon y_0),$$

or, more explicitly,

$$l_0 = \sqrt{s_1 - k^2 - \lambda_1^{-2}}, \qquad s_2(\epsilon y_0) = s_1 - \lambda_1^{-2} + \lambda_2^{-2}.$$

The coordinates (y_0, l_0) are useful since they can be employed to evaluate all the quantities involved in the scattering problem. [This is because the coupling region is very small, with radius $O(\epsilon^{1/2})$.] Applying formula (4.12) of Flynn and Littlejohn (1994) provides the transmission coefficient in the form

$$T = \exp(-2\pi\nu),$$

with

$$\nu = \frac{1}{2\epsilon(\lambda_1\lambda_2)^2 l_0 |s_2'(\epsilon y_0)|} = \frac{\delta^2 l_1^3}{2\epsilon |s_2'(\epsilon y_0)|},$$

where the prime denotes the derivative of $s_2(\epsilon y)$ with respect to its argument. The last expression clearly demonstrates that transmission is significant or, equivalently, that conversion is not total, only when $\delta = O(\epsilon^{1/2})$ or smaller, as announced. For $\delta = O(1)$, as will generically be the case, the transmission coefficient is exponentially small in ϵ and conversion is quasi total. In terms of the original variables, the dimensionless parameter ν controlling the strength of the transmission reads

$$\nu = \frac{\omega h_2}{2 f(\lambda_1\lambda_2)^2 k l_1 |\partial_{yy}^2 \eta|}, \qquad (3.1)$$

and is seen to be inversely proportional to the curvature of the topography.

[2] Hallberg (1997) speaks of a reflected wave; here we follow the standard terminology.

4. Discussion

The results derived above in the simple case of one-dimensional propagation illustrate the essence of the phenomenon of mode conversion for Rossby waves over topography. They could easily be extended to quantify the transmission between branches of the dispersion relation in the two-dimensional context studied by Hallberg (1997). The qualitative properties of mode conversion will, of course, be the same as those discussed here. An important property, emphasized by Flynn and Littlejohn (1994), is the nongeneric nature of transmission between branches; that is, that conversion should be regarded as the rule and transmission as an exception. In our physical system, this means that one should not expect wave energy to be confined in either the surface or bottom layer. The crucial point here is that conversion is total and transmission is negligible ($C = 1$ and $T = 0$ up to exponentially small corrections) unless $\delta = O(\epsilon^{1/2})$. Thus, if the separation between wave scale and topography scale is sufficient, that is, if ϵ is sufficiently small, transmission can be neglected and wave energy is confined to a single branch of the dispersion relation.

This conclusion is in contrast with Hallberg's (1997) claim that, in the Rossby wave problem, wave energy does not remain predominantly in a single mode. Hallberg argues that conservation of mass provides a strong constraint that limits the possible conversion between surface- and bottom-intensified waves and therefore leads to significant transmission. In fact, he derives an estimate [his Eq. (4.4)] for what is, up to a scaling factor, the square root of T and finds it to be very close to 1. That there is a problem with this argument should be clear, not only because it yields a large value for T, but also because this value does depend on the scale-separation parameter ϵ: for ϵ sufficiently small (smaller than δ^2), the WKB approximation is valid everywhere and T should be exponentially small, not close to 1. One can trace the failure of the mass conservation argument to the fact that for linear waves there is simply no net horizontal mass transport, regardless of the wave amplitude measured, for instance, by the streamfunction. This is because any sensible mass balance involves the computation of mass fluxes across surfaces that extend over many wavelengths [precisely a number of wavelengths of order $O(\delta\epsilon^{-1}) \gg 1$ since this is the spatial extent of the mode coupling region]; these fluxes are therefore exponentially small in ϵ and, in the WKB context, can be regarded as vanishing. It is only when the WKB approximation breaks down that the fluxes are nonzero, so mass conservation implies a transmission of wave energy between branches of the dispersion relation, consistent with the results of mode conversion theory. The quantification of the mass fluxes in the mode conversion region is delicate in this case because of the rapid change of the wave amplitude in each layer.

Of course, in realistic situations such as the ones studied numerically by Hallberg (1997), the scale-separation assumption (on which mode conversion theory as well as the WKB approximation rely) may not be strictly valid in most regions, and the conversion of energy between surface- and bottom-intensified waves may be limited by a number of non-WKB effects. These effects cannot be captured by mode conversion theory, which treats only the local failure of the WKB approximation that arises when two branches of the dispersion relation come asymptotically close together. Formula (3.1) and its possible extensions may nevertheless prove useful in estimating the strength of the coupling between modes that results from topographic effects.

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