Isopycnal Averaging at Constant Height. Part II: Relating to the Residual Streamfunction in Eulerian Space

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ABSTRACT

In Part I, the ‘‘vertical’’ transport streamfunction was defined as resulting from isopycnic averaging at constant height in the same way that the meridional streamfunction results from averaging at constant latitude. Part II here discusses the relationship between these two isopycnic streamfunctions and the Eulerian residual streamfunction that arises from the transformed Eulerian mean (TEM). It is known that the meridional isopycnic streamfunction can be approximated by a Taylor expansion to give an Eulerian residual streamfunction involving the horizontal eddy flux. This Taylor expansion approximation works well in the interior, removing the spurious mixing associated with the simple Eulerian-averaged streamfunction. However, it fails near the surface where isopycnals outcrop to the surface. It can be shown in a similar way that the *vertical* isopycnic streamfunction can formally be approximated by a residual streamfunction involving the *vertical* eddy flux. However, if horizontal isopycnal displacements are large, this approximation fails even in the ocean interior. Inspired by the two different residual streamfunctions, a more general form of TEM formulation is explored. It is shown that the different TEM residual streamfunctions arise from decomposing the eddy flux into a component along isopycnals, which leads to advective flow, and a remaining diffusive component, which is oriented either vertically or horizontally. In theory the diffusive flux can be oriented in any direction, although in practice the orientation should be such that neither the advective flow nor the diffusive flux cross any boundary (surface, sidewalls, and bottom). However, it is not clear how to merge the continuously changing orientation in a physically meaningful way. A variety of approaches are discussed.

1. Introduction

In Nurser and Lee (2004, henceforth Part I) we showed how properties may be zonally averaged along isopycnals on the ‘‘horizontal’’ plane instead of the ‘‘vertical’’ plane. For simplicity, both in Part I and here, we assume that potential density is equivalent to potential temperature so that isopycnals and isotherms are coincident. Then this new horizontal isopycnal averaging defines a mean temperature field $\theta^*(y, z)$ chosen so as to retain the area colder than any θ on any horizontal plane and hence conserve water masses on horizontal planes. In contrast, conventional vertical isopycnal averaging (de Szoeke and Bennett 1993; McDougall and McIntosh 2001, hereinafter MM01) defines the mean temperature field $\tilde{\theta}$ so as to conserve the area colder than θ (and so water masses) on vertical planes.

Associated with this new averaging, an isopycnal *vertical* transport streamfunction $\psi^*(\theta, z)$ may be defined as the total upward flow across the horizontal plane at constant z of all fluid colder than the given temperature θ. This may be contrasted with the conventional isopycnal meridional transport streamfunction $\tilde{\psi}(y, \theta)$, defined as the total northward flow across the vertical plane at constant y of fluid colder than θ.

These isopycnic streamfunctions can be remapped into Eulerian space. The vertical transport streamfunction $\psi^*(\theta, z)$ is remapped to $\Psi^*(y, z)$ by identifying temperature at a given depth with the meridional position of the horizontally averaged isotherm θ^*. This is analogous to remapping the conventional meridional isopycnal transport streamfunction $\tilde{\psi}(y, \theta)$ to $\tilde{\Psi}(y, z)$ by identifying temperature at a given latitude with the vertical position of the vertically averaged isotherm $\tilde{\theta}$. Both of these remapped streamfunctions have the useful property that their spatial gradients give mean horizontal and vertical velocities; namely, they are ‘‘true’’ streamfunctions. Moreover, the isopycnally averaged quantities $\tilde{\theta}$ and θ^* are advected in y, z space by the remapped streamfunctions $\tilde{\Psi}$ and Ψ^* in exactly the same way as the unaveraged θ is advected by the unaveraged flow.

Although this is an exact and elegant approach, it is often more convenient to analyze models in terms of simple Eulerian averages. The remapped isopycnal quantities $\tilde{\Psi}$, $\tilde{\theta}$, and so on may be approximated in terms

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of such simple Eulerian averages. McIntosh and McDougall (1996) and MM01 derived Taylor series approximations for the vertically isopycnally averaged fields $\tilde{\Psi}$ and $\tilde{\theta}$. In the zonal mean they give

$$\tilde{\theta} \sim \theta^{\#} = \overline{\theta} - \frac{\partial}{\partial z}\left(\frac{\phi}{\overline{\theta}_z}\right) \quad \text{and} \tag{1.1a}$$

$$\tilde{\Psi} \sim \Psi^{\#} = \overline{\Psi} - \frac{\overline{\upsilon'\theta'}}{\overline{\theta}_z}L(y, z), \tag{1.1b}$$

where θ' denotes a deviation from the mean $\overline{\theta}$, $\phi = \frac{1}{2}\overline{\theta'^2}$ is one-half of the temperature variance, $\overline{\Psi}$ is the Eulerian zonally integrated streamfunction, $\overline{\theta}_z$ is the mean stratification, and $L(y, z)$ is the zonal length of the channel. An approximation to the evolution equation for $\tilde{\theta}$ can then be constructed, involving the advection of $\theta^{\#}$ by $\Psi^{\#}$. This approximation removes most of the spurious mixing involved in the advection of $\overline{\theta}$ by $\overline{\Psi}$. However, the approximations for both $\tilde{\theta}$ and $\tilde{\Psi}$ (and so for the advection equation) break down near the surface.

The alternative approach is to deal directly with the Eulerian mean temperature and so accept the loss of watermass conservation properties. In the transformed Eulerian mean (TEM) formalism, the eddy flux divergence is mathematically manipulated into the form of an eddy-induced advection of mean temperature, plus the divergence of a diffusive flux. Conventionally (Andrews and McIntyre 1976; Andrews et al. 1987), the zonal-mean eddy-induced flow is given by the meridional streamfunction, $-\overline{\upsilon'\theta'}/\overline{\theta}_z$. This expression also appears in (1.1b) for the Taylor expansion of $\Psi^{\#}$. This conventional TEM approach treats the horizontal direction of eddy flux as special, although there is no obvious reason for this. Unfortunately, this streamfunction is not zero at the sea surface, and so it gives rise to an eddy ''flow'' across the sea surface. Although not derived by a Taylor expansion it still has problems near the surface.

Andrews and McIntyre (1978) show that there is more than one way of decomposing the zonal-mean eddy flux into advective and diffusive components. They argue that the most natural choice involves an eddy streamfunction using the component of the eddy flux along the (sloping) zonal-mean isotherms. Unfortunately, like $-\overline{\upsilon'\theta'}/\overline{\theta}_z$ this is not zero at the sea surface. Instead, Held and Schneider (1999, hereinafter HS99) used the different eddy streamfunction $\overline{w'\theta'}/\overline{\theta}_y$. This is automatically zero at the sea surface, and so works well where the conventional meridional eddy streamfunction $-\overline{\upsilon'\theta'}/\overline{\theta}_z$ fails.

Here we show that the remapped *vertical* streamfunction Ψ^* can be approximated by a Taylor series in the horizontal direction. In the zonal-mean case this gives

$$\Psi^* \sim \Psi^{\dagger} = \overline{\Psi} + \frac{\overline{w'\theta'}}{\overline{\theta}_y}L(y, z). \tag{1.2}$$

The zonal-mean streamfunction $\overline{w'\theta'}/\overline{\theta}_y$ used by HS99 contributes to $\Psi^{\dagger}$ in the same way that $-\overline{\upsilon'\theta'}/\overline{\theta}_z$ from the conventional TEM contributes to the Taylor series approximation $\Psi^{\#}$. The horizontally averaged θ^* can be approximated similarly. These approximations do not fail near the surface like those for $\Psi^{\#}$ and $\theta^{\#}$, but do not hold well where the eddies are of large amplitude.

In this study we formulate the TEM decomposition as generally as possible. The eddy flux is decomposed into an advective component along the isopycnals and the remaining diffusive flux across the isopycnals. The direction of the eddy diffusive flux is arbitrary: differently oriented diffusive flux components give different residual streamfunctions. Upward or horizontally oriented diffusive components (thus the streamfunctions $-\overline{\upsilon'\theta'}/\overline{\theta}_z$ and $\overline{w'\theta'}/\overline{\theta}_y$) are just two special cases.

We discuss this decomposition in the context of the three-dimensional temporal average as well as the zonal mean and link it to the temporal residual mean (TRM) of MM01–Taylor series approximations for the vertically isopycnally *time*-averaged streamfunctions and for $\tilde{\theta}$. However, we illustrate our ideas using the temporal–zonal averages from the zonal-channel model used in Part I.

2. Approximation of isopycnally averaged quantities

In this section, we discuss how the residual streamfunction approximates the isopycnically averaged streamfunctions described in Part I. The most physically transparent approach is that of McDougall and McIntosh (McIntosh and McDougall 1996; McDougall 1998; MM01). The variables $\tilde{\theta}$ and $\tilde{\Psi}$ are expressed directly in (x, y, z) and then expanded as Taylor series.

In the appendix we rederive the Taylor series expansions, using a more formal scaling analysis than that employed by McDougall and McIntosh, and extend the analysis [using an approach similar to that of Kushner and Held (1999)] to approximate the horizontally averaged fields Ψ^* and θ^*. In this section we simply state the key results.

a. *The approximation of* $\tilde{\Psi}$ *and* Ψ^*

The zonally integrated vertically averaged isopycnal streamfunction $\tilde{\Psi}$ is related *exactly* to the Eulerian-mean streamfunction $\overline{\Psi}$ by

$$\tilde{\Psi}(y, z_a) = \overline{\Psi}(y, z_a) + \overline{\int\int_{z_a}^{z_a+z_a'} \upsilon\, dz\, dx}. \tag{2.1}$$

The integral is approximated as a Taylor series expansion, and isotherm perturbation height z_a' is linked to temperature perturbation at constant height θ' and mean stratification by

$$z_a' \sim -\frac{\theta'}{\overline{\theta}_z}. \tag{2.2}$$

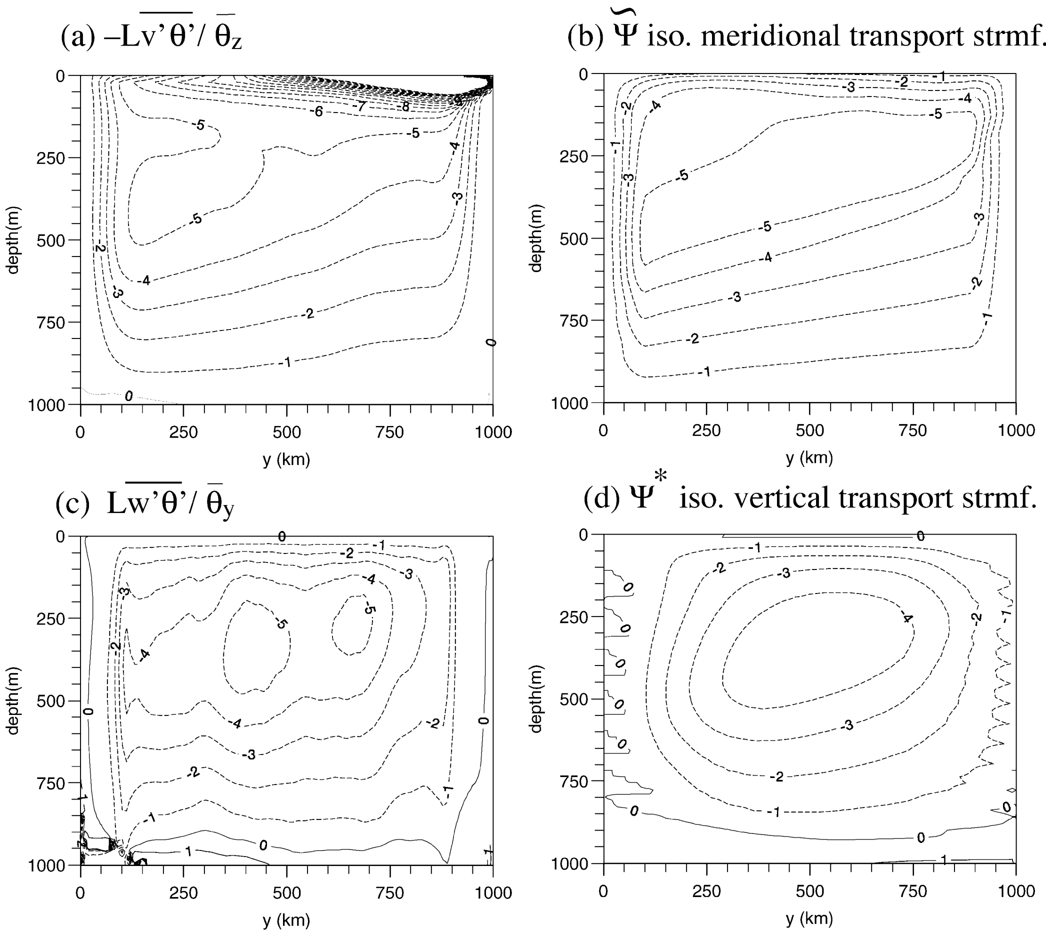

FIG. 1. Comparison between the isopycnal streamfunctions (Sv ≡ 10^6 m^3 s^{-1}) and their Taylor expansion approximations. Compare (a) the approximation $-L\overline{v'\theta'}/\overline{\theta}_z$ (near the surface, only plotting values greater than −20 Sv) with (b) the isopycnically averaged meridional transport streamfunction $\tilde{\Psi}$. Compare (c) the approximation $L\overline{w'\theta'}/\overline{\theta}_y$ with (d) the isopycnically averaged vertical transport streamfunction Ψ^*. The plots in (b) and (d) are reproduced from Fig. 9 in Part I.

This gives [McIntosh and McDougall 1996; see also (A.15) in the appendix] the approximation (1.1b), which is accurate to second order,

$$\tilde{\Psi}(y, z) \sim \Psi^{\#} = \overline{\Psi} - \frac{\overline{v'\theta'}}{\overline{\theta}_z}L(y, z).$$

Similar approximations relate $\tilde{\theta}$ to $\overline{\theta}$ and lead to (1.1a), derived as (A.6):

$$\tilde{\theta} \sim \theta^{\#} = \overline{\theta} - \frac{\partial}{\partial z}\left(\frac{\phi}{\overline{\theta}_z}\right).$$

Figure 1a shows the approximation (1.1a) diagnosed from the zonal channel model (the same model as in Part I). In our channel model without wind forcing, the zonal-mean meridional velocity disappears almost completely, and so the Eulerian mean streamfunction $\overline{\Psi} \approx 0$. There is good agreement between the approximation $-L\overline{v'\theta'}/\overline{\theta}_z$ (Fig. 1a) and the exact meridional transport streamfunction $\tilde{\Psi}$ (Fig. 1b), except near the surface. The problem near the surface is that the approximations (1.1a,b) require (2.2) to link isotherm perturbation height z'_a to temperature perturbation at constant height θ'. If Z' is the typical vertical displacement of isotherms by eddies, this link holds only at depths below Z', where isotherms do not outcrop. In our model runs, Z' is typically 100–200 m (see Fig. 4 in Part I).[1] At the surface we should have $\tilde{\Psi} = \overline{\Psi}$ (equal to the net barotropic flow). So the nonzero (and relatively large) surface values of $-\overline{v'\theta'}/\overline{\theta}_z$ are a serious problem.

The zonally integrated isopycnically averaged verti-

[1] This is larger than would be implied by an application of McDougall’s estimate of Z' as $L_\rho S_y$, where L_ρ is the Rossby radius and S_y the large-scale isopycnal slope. For our model runs, writing $L_\rho = NH/f \sim 50$ km, where the buoyancy frequency $N \sim 5 \times 10^{-3}$ s^{-1}, the depth $H = 1000$ m, the Coriolis parameter $f = 10^{-4}$ s^{-1}, and the slope $S_y \sim 500$ m/10^6 m $\sim 5 \times 10^{-4}$, we have $L_\rho S_y \sim 25$ m. This underestimate is also seen in the real ocean, where Killworth (2001) estimates $L_\rho S_y \sim 20$ m, but MM01 quote observations of typical isopycnal heaving of ∼200 m in the Southern Ocean.

cal transport streamfunction Ψ^* can be approximated by an analogous Taylor expansion but in the horizontal rather than the vertical. We show in the appendix (A.15) that

$$\Psi^* \sim \Psi^\dagger = \overline{\Psi} + \frac{\overline{w'\theta'}}{\overline{\theta}_y} L(y, z).$$

This expansion does not fail catastrophically near the sea surface and, indeed, correctly gives $\Psi = \overline{\Psi}$ at the sea surface. The reversed sign relative to the expression (1.1b) for $\Psi^\#$ results from assuming that the cold water lies to the north in evaluating Ψ^* so that the isotherm marks the lower limit in y of the integral, in contrast to (1.1b), where the isotherm is the upper limit in z of the integral making up the transport.

To obtain (1.2) we need to link *horizontal* perturbations in the isotherm position, y'_a, to temperature perturbations θ' by

$$y'_a \sim -\frac{\theta'}{\overline{\theta}_y}. \qquad (2.3)$$

However, (2.3) does not hold for realistic density fields, with large meanders and closed eddies. In this case $\partial\theta'/\partial y \sim \partial\overline{\theta}/\partial y$, so the Taylor expansion used to derive (2.3) fails. Consequently (2.3) cannot hold in any realistically eddying field. Indeed, in our vigorously eddying model runs (again $\overline{w}$ is vanishingly small for the zonal channel), we find that the approximation $\Psi^\dagger$ (Fig. 1c), though similar, differs almost everywhere from the vertical transport streamfunction Ψ^* (Fig. 1d).

So, although the Taylor expansion of Ψ^* is more robust than that of $\tilde{\Psi}$ at the ocean surface, it does not work as well as that of $\tilde{\Psi}$ in the interior because of the finite amplitude of the eddies in the horizontal.

b. The approximated temperature advection equation

Using these approximations to $\tilde{\theta}$ and $\tilde{\Psi}$ we follow MM01 and approximate the evolution equation for $\tilde{\theta}$ in Part I [(6.2)] by

$$\left(\frac{\partial}{\partial t} + \upsilon^\# \frac{\partial}{\partial y} + w^\# \frac{\partial}{\partial z}\right)\theta^\# = Q^\# + \text{third-order terms}, \qquad (2.4)$$

where

$$(\upsilon^\#, w^\#) = \frac{1}{L}\left(\frac{\partial \Psi^\#}{\partial z}, -\frac{\partial \Psi^\#}{\partial y}\right) \qquad (2.5)$$

and the isopycnally averaged heating $\tilde{Q}$ is approximated (A.19) by another Taylor series

$$Q^\# = \overline{Q} - \frac{\partial}{\partial z}\left(\frac{\overline{Q'\theta'}}{\overline{\theta}_z}\right) + \frac{\partial}{\partial z}\left[\frac{\phi \overline{Q}_z}{(\overline{\theta}_z)^2}\right]. \qquad (2.6)$$

As discussed above, the approximations required for (2.4) hold away from the surface. However, they, and hence (2.4) itself, fail in the upper ~100 m, a vital area in the ocean.

Similarly, we can approximate the advection equation for the horizontally isopycnally averaged temperature, θ^*, by

$$\left(\frac{\partial}{\partial t} + \upsilon^\dagger \frac{\partial}{\partial y} + w^\dagger \frac{\partial}{\partial z}\right)\theta^\dagger = Q^\dagger + \text{third-order terms}, \qquad (2.7)$$

where further Taylor expansions in the horizontal give (A.8)

$$\theta^\dagger = \overline{\theta} - \frac{\partial}{\partial y}\left(\frac{\phi}{\overline{\theta}_y}\right) \quad \text{and} \qquad (2.8)$$

$$(\upsilon^\dagger, w^\dagger) = \frac{1}{L}\left(\frac{\partial \Psi^\dagger}{\partial z}, -\frac{\partial \Psi^\dagger}{\partial y}\right), \qquad (2.9)$$

and (A.20) an approximation to the horizontally isopycnally averaged heating Q^*:

$$Q^\dagger = \overline{Q} - \frac{\partial}{\partial y}\left(\frac{\overline{Q'\theta'}}{\overline{\theta}_y}\right) + \frac{\partial}{\partial y}\left[\frac{\phi \overline{Q}_y}{(\overline{\theta}_y)^2}\right]. \qquad (2.10)$$

The problem here is that the approximations $\Psi^\dagger$ and $\theta^\dagger$ and, hence, the equation itself are only valid for a field with weak eddies. Although these approximations are never very accurate because of the finite amplitude of the eddies, at the surface they are far better than the approximations $\Psi^\#$ and $\theta^\#$ for $\tilde{\Psi}$ and $\tilde{\theta}$.

So using the Taylor expansion approach, the different streamfunctions $\Psi^\#$ and $\Psi^\dagger$ advect around different temperature fields $\theta^\#$ and $\theta^\dagger$ with different forcings $Q^\#$ and $Q^\dagger$. The problem however is that the $\Psi^\#$, $\theta^\#$ pairing fails at the surface, while the $\Psi^\dagger$, $\theta^\dagger$ pairing fails everywhere for finite-amplitude eddies, and always at the meridional boundaries. Moreover, it is unclear what is happening where they fail. In the next section we consider the transformed Eulerian mean approach, which is less intuitive, but perhaps more robust.

c. Temporal means of 3D fields: The temporal residual mean

To compare with the three-dimensional treatment of the TEM in the next section, note that the above approximations may be generalized (MM01) to the eddy effect on a three-dimensional temporal-mean field. In this case $\theta^\#$ is forced by $Q^\#$ and advected by the 3D temporal residual mean velocity field:

$$(u^\#, \upsilon^\#) = \frac{\partial}{\partial z}(\Psi^\#_{[x]}, \Psi^\#_{[y]}), \qquad w^\# = -\frac{\partial \Psi^\#_{[x]}}{\partial x} - \frac{\partial \Psi^\#_{[x]}}{\partial y}. \qquad (2.11)$$

Here the time-mean meridional overturning streamfunction (A.16)

$$\Psi^\#_{[y]} = \overline{\Psi} - \frac{\overline{\upsilon'\theta'}}{\overline{\theta}_z} + \frac{\phi \overline{\upsilon}_z}{(\overline{\theta}_z)^2} \qquad (2.12)$$

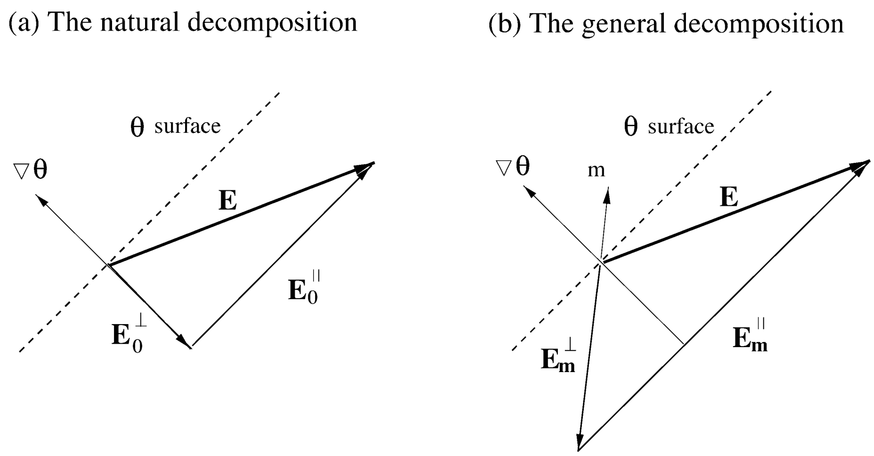

FIG. 2. Schematic illustrating (a) the natural decomposition of eddy flux **E** into a component along the isotherm, $\mathbf{E}_0^{\parallel}$, and a component perpendicular to the isotherm, $\mathbf{E}_0^{\perp}$, and (b) the more general decomposition into an arbitrarily oriented $\mathbf{E}_{\mathbf{m}}^{\perp}$ and a component along the isotherm, $\mathbf{E}_{\mathbf{m}}^{\parallel}$.

is similar to the zonally integrated meridional streamfunction (1.1a) but with overbars now denoting time averages. Since the time-mean velocity is in general much larger relative to the transient velocities than was the zonal-mean velocity relative to zonal variations in velocity, $\Psi^{\#}_{[y]}$ includes an extra term resulting from the vertical variation of time-mean velocity (see MM01 or the appendix). The zonal overturning streamfunction $\Psi^{\#}_{[x]}$ is the same except that u replaces v. In terms of a vector streamfunction $\mathbf{A}^{\#}$:

$$\mathbf{u}^{\#} = (u^{\#}, v^{\#}, w^{\#}) = \nabla \times \mathbf{A}^{\#}, \tag{2.13}$$

where (with **k** the unit upward vector)

$$\mathbf{A}^{\#} = (\Psi^{\#}_{[y]}, -\Psi^{\#}_{[x]}, 0) = -\mathbf{k} \times (\Psi^{\#}_{[x]}, \Psi^{\#}_{[y]}, 0). \tag{2.14}$$

The horizontally isopycnally averaged temperature $\theta^{\dagger}$ is forced by $Q^{\dagger}$ and advected by

$$(u^{\dagger}, w^{\dagger}) = -\frac{\partial}{\partial y}(\Psi^{\dagger}_{[x]}, \Psi^{\dagger}_{[z]}), \qquad v^{\dagger} = \frac{\partial \Psi^{\dagger}_{[x]}}{\partial x} + \frac{\partial \Psi^{\dagger}_{[z]}}{\partial z}. \tag{2.15}$$

In this case, the time-mean vertical meridional overturning streamfunction (A.18),

$$\Psi^{\dagger}_{[z]} = \overline{\Psi} + \frac{\overline{w'\theta'}}{\overline{\theta}_y} - \frac{\overline{w}_y \phi}{(\overline{\theta}_y)^2}, \tag{2.16}$$

is similar to the approximated (1.2) zonally integrated vertical streamfunction $\Psi^{\dagger}$. Again, $\Psi^{\dagger}_{[z]}$ includes an extra term, this time resulting from the meridional variation of time-mean vertical velocity. The zonal/meridional streamfunction $\Psi^{\dagger}_{[x]}$ is the same except that u replaces w. In terms of a vector streamfunction $\mathbf{A}^{\dagger}$

$$\mathbf{u}^{\dagger} = (u^{\dagger}, v^{\dagger}, w^{\dagger}) = \nabla \times \mathbf{A}^{\dagger}, \tag{2.17}$$

where (with **j** the unit northward vector)

$$\mathbf{A}^{\dagger} = (\Psi^{\dagger}_{[z]}, 0, -\Psi^{\dagger}_{[x]}) = \mathbf{j} \times (\Psi^{\dagger}_{[x]}, 0, \Psi^{\dagger}_{[z]}). \tag{2.18}$$

3. The transformed Eulerian mean

a. *The time mean*

We consider the time-mean of a three dimensional field, and later specialize to the simpler zonal mean case. The Eulerian-mean temperature $\overline{\theta}$ is advected by the mean velocity $\overline{\mathbf{u}}$ and forced by the mean forcing $\overline{Q}$ and the convergence of the (three-dimensional) eddy flux $\mathbf{E} \equiv \overline{\mathbf{u}'\theta'}$:

$$\frac{\partial \overline{\theta}}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{\theta} = -\nabla \cdot \mathbf{E} + \overline{Q}. \tag{3.1}$$

The crucial idea here is that, if the eddy heat flux lies along a (mean) isothermal surface, it cannot change the volume of water of given mean temperature, or the variance, and so its effect on the mean field may be regarded as simply due to an eddy-driven advection. So we decompose the eddy flux into such an along-isothermal component (which drives an advection) and the remainder (which we hope is relatively small) whose effect *cannot* in general be treated as advective.

1) THE NATURAL DECOMPOSITION

As in Andrews and McIntyre (1978), we break up the three-dimensional eddy flux $\mathbf{E} \equiv \overline{\mathbf{u}'\theta'}$ into a component $\mathbf{E}_0^{\parallel}$ lying along the $\overline{\theta}$ surface (i.e., with $\mathbf{E}_0^{\parallel} \cdot \nabla \overline{\theta} = 0$), and a component $\mathbf{E}_0^{\perp}$ perpendicular to the $\overline{\theta}$ surface (Fig. 2a):

$$\mathbf{E} = \mathbf{E}_0^{\perp} + \mathbf{E}_0^{\parallel}, \tag{3.2}$$

where

$$\mathbf{E}_0^{\perp} = \frac{\mathbf{E} \cdot \nabla \overline{\theta}}{(\nabla \overline{\theta})^2} \nabla \overline{\theta} \quad \text{and} \quad \mathbf{E}_0^{\parallel} = \frac{\nabla \overline{\theta} \times \mathbf{E}}{(\nabla \overline{\theta})^2} \times \nabla \overline{\theta}.$$

This uses the vector product identity $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ with $\mathbf{a} = \mathbf{c} = \nabla \overline{\theta}$ and $\mathbf{b} = \mathbf{E}$.

If we define a vector streamfunction

$$\mathbf{A}_0 = \frac{\nabla\overline{\theta} \times \mathbf{E}}{(\nabla\overline{\theta})^2}, \tag{3.3}$$

we can write

$$\mathbf{E}_0^{\parallel} = \mathbf{A}_0 \times \nabla\overline{\theta} = \overline{\theta}\nabla \times \mathbf{A}_0 - \nabla \times (\overline{\theta}\mathbf{A}_0).$$

Define an eddy-induced velocity, $\hat{\mathbf{u}}_0$ as the curl of the vector streamfunction

$$\hat{\mathbf{u}}_0 = \nabla \times \mathbf{A}_0. \tag{3.4}$$

The parallel component can then be written as the sum of an advective flux $\overline{\theta}\hat{\mathbf{u}}_0$ and a rotational component $\nabla \times (\overline{\theta}\mathbf{A}_0)$:

$$\mathbf{E}_0^{\parallel} = \overline{\theta}\hat{\mathbf{u}}_0 - \nabla \times (\overline{\theta}\mathbf{A}_0). \tag{3.5}$$

The eddy flux divergence is then

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{E}_0^{\perp} + \hat{\mathbf{u}}_0 \cdot \nabla\overline{\theta},$$

and the Eulerian mean equation becomes

$$\frac{\partial\overline{\theta}}{\partial t} + (\overline{\mathbf{u}} + \hat{\mathbf{u}}_0) \cdot \nabla\overline{\theta} = -\nabla \cdot \mathbf{E}_0^{\perp} + \overline{Q}. \tag{3.6}$$

The mean isotherm is advected by the Eulerian-mean velocity and eddy-induced velocity $\hat{\mathbf{u}}_0$. Although $\mathbf{E}_0^{\parallel}$ is uniquely set, $\mathbf{A}_0$ is not: a component $\lambda\nabla\overline{\theta}$ may be added to $\mathbf{A}_0$ without changing $\mathbf{E}_0^{\parallel}$. This nonuniqueness of $\mathbf{A}_0$ does not affect the $\overline{\theta}$ equation since the additional induced velocity $\nabla \times (\lambda\nabla\overline{\theta}) = \nabla\overline{\theta} \times \nabla\lambda$ lies on the $\overline{\theta}$ surface and so cannot advect $\overline{\theta}$. Nor does the nondivergent component, $-\nabla \times (\overline{\theta}\mathbf{A}_0)$, influence the evolution of $\overline{\theta}$, although it helps to satisfy the boundary conditions.

The mean temperature in (3.6) is still forced by the convergence of a diffusive flux, $-\nabla \cdot \mathbf{E}_0^{\perp}$. This means that the residual velocity $\overline{\mathbf{u}} + \hat{\mathbf{u}}_0$ may cross the mean isotherms even if there is no diabatic forcing. This is in contrast to the evolution equations for $\tilde{\theta}$ and θ^*, (6.2) and (6.7) in Part I, in which the eddy effect is purely advective, so the total (mean + eddy) flow always follows the mean $\tilde{\theta}$ and θ^* isotherms unless there is diabatic forcing.

The processes driving eddy diffusion $\mathbf{E}_0^{\perp}$ are evident from the eddy temperature variance equation (HS99; MM01):

$$\frac{\partial\phi}{\partial t} + \overline{\mathbf{u}} \cdot \nabla\phi + \nabla \cdot \overline{(\mathbf{u}'\phi)} = -\mathbf{E} \cdot \nabla\overline{\theta} + \overline{Q'\theta'}. \tag{3.7}$$

Downgradient eddy flux $|\mathbf{E}_0^{\perp}| = |\mathbf{E} \cdot \nabla\overline{\theta}|/|\nabla\overline{\theta}|$ is balanced by accumulation and advection of eddy variance, eddy flux of eddy variance, and eddy-forcing correlation. For the zonal mean, these processes are generally weak in the ocean interior, and so $\mathbf{E}_0^{\perp}$ is generally small although $\mathbf{E}_0^{\parallel}$ need not be small.

The decomposition, $\mathbf{E} = \mathbf{E}_0^{\perp} + \mathbf{E}_0^{\parallel}$, is unique and mathematically elegant. It breaks up the eddy flux $\mathbf{E}$ into the generally large $\mathbf{E}_0^{\parallel}$, which can be treated as an advection, and the smallest possible $\mathbf{E}_0^{\perp}$ that must be treated as a diffusion. However it is not necessarily the

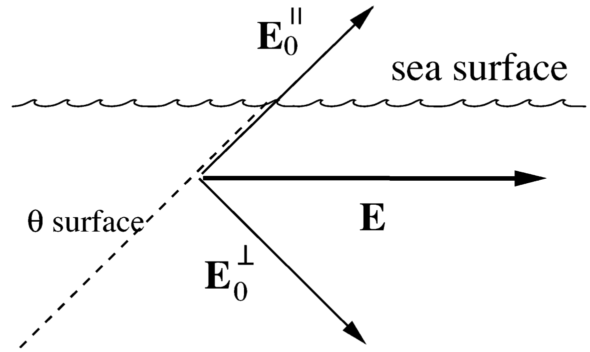

FIG. 3. Schematic illustrating that, although near the sea surface the total eddy flux $\mathbf{E}$ is along the surface, both $\mathbf{E}_0^{\perp}$ and $\mathbf{E}_0^{\parallel}$ cross the surface.

most useful decomposition. Suppose that the isotherms strike the sea surface as in the schematic Fig. 3, but that the eddy flux along the sea surface does not disappear. Unless the isotherm is vertical where it strikes the sea surface, $\mathbf{E}_0^{\perp}$ must then have a component across the sea surface, which cancels the across sea surface component of $\mathbf{E}_0^{\parallel}$. This leads to a nonzero, varying, streamfunction at the surface advecting heat through the surface, balanced by a spurious diffusive flux across the surface. Both these consequences seem unnatural and unsatisfactory. Additionally, if we want to represent as much as possible of the effect of the eddies as an advection, it is $\nabla \cdot \mathbf{E}^{\perp}$ rather than $\mathbf{E}^{\perp}$ itself that we wish to minimize.

2) THE GENERAL DECOMPOSITION

We can give ourselves all the flexibility we might want (and more!) if we drop the requirement that $\mathbf{E}^{\perp}$ be perpendicular to the $\overline{\theta}$ surface, and merely require that $\mathbf{E}^{\perp} \cdot \nabla\overline{\theta} = \mathbf{E} \cdot \nabla\overline{\theta}$. We may define $\mathbf{E}^{\perp}$ in terms of its (arbitrary) direction $\mathbf{m}$. Another way of viewing this decomposition is that $\mathbf{E}$ is broken up into an $\mathbf{E}_{\mathbf{m}}^{\parallel}$, which is arbitrary (so long as it lies on the isotherm), plus a remainder $\mathbf{E}_{\mathbf{m}}^{\perp}$ whose orientation defines $\mathbf{m}$ (Fig. 2b). We thus write

$$\mathbf{E} = \mathbf{E}_{\mathbf{m}}^{\perp} + \mathbf{E}_{\mathbf{m}}^{\parallel}, \tag{3.8}$$

where

$$\mathbf{E}_{\mathbf{m}}^{\perp} = \frac{\mathbf{E} \cdot \nabla\overline{\theta}}{\mathbf{m} \cdot \nabla\overline{\theta}}\mathbf{m} \quad \text{and} \quad \mathbf{E}_{\mathbf{m}}^{\parallel} = \mathbf{A}_{\mathbf{m}} \times \nabla\overline{\theta}, \tag{3.9}$$

and the vector streamfunction

$$\mathbf{A}_{\mathbf{m}} = \frac{\mathbf{m} \times \mathbf{E}}{\mathbf{m} \cdot \nabla\overline{\theta}}. \tag{3.10}$$

This uses again the vector product identity $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ but now with $\mathbf{a} = \mathbf{m}$, $\mathbf{b} = \mathbf{E}$ and $\mathbf{c} = \nabla\overline{\theta}$.

Note that we are free to choose $\mathbf{m}$ (i.e., the orientation of $\mathbf{E}_{\mathbf{m}}^{\perp}$) exactly as we wish. This $\mathbf{m}$ is a vector field, which may vary in space, though only its direction matters.

We of course recover the natural split (3.2) in which $\mathbf{E}_\mathbf{m}^\perp$ really is perpendicular to the isotherm, by choosing $\mathbf{m} = \boldsymbol{\nabla}\overline{\theta}$. Although $\mathbf{A_m}$ as defined above must satisfy $\mathbf{A_m} \cdot \mathbf{E} = 0$, we can make it completely arbitrary by adding a nonadvecting component, $\lambda\boldsymbol{\nabla}\overline{\theta}$ (which does not affect $\mathbf{E}_\mathbf{m}^\parallel$).

The parallel part $\mathbf{E}_\mathbf{m}^\parallel$ is again broken up into nondivergent and advective components:

$$\mathbf{E}_\mathbf{m}^\parallel = \overline{\theta}\hat{\mathbf{u}}_\mathbf{m} - \boldsymbol{\nabla} \times (\overline{\theta}\mathbf{A_m}), \tag{3.11}$$

where the eddy-induced velocity

$$\hat{\mathbf{u}}_\mathbf{m} = \boldsymbol{\nabla} \times \mathbf{A_m}. \tag{3.12}$$

The Eulerian mean equation now becomes, similarly to (3.6),

$$\frac{\partial\overline{\theta}}{\partial t} + (\overline{\mathbf{u}} + \hat{\mathbf{u}}_\mathbf{m}) \cdot \boldsymbol{\nabla}\overline{\theta} = -\boldsymbol{\nabla} \cdot \mathbf{E}_\mathbf{m}^\perp + \overline{Q}. \tag{3.13}$$

The mean isotherm is now advected by the mean and eddy-induced velocity and forced by the eddy diffusive divergence $-\boldsymbol{\nabla} \cdot \mathbf{E}_\mathbf{m}^\perp$.

Where the eddies only advect and the eddy flux lies on isopycnals, $\mathbf{E}^\perp = 0$, $\mathbf{E}^\parallel = \mathbf{E}$. In this case, for any choice of $\mathbf{m}$, $\mathbf{E}_\mathbf{m}^\parallel = \mathbf{E}$, implying $\mathbf{A_m}$ is unique up to $\lambda\boldsymbol{\nabla}\overline{\theta}$. As before, although $\hat{\mathbf{u}}_\mathbf{m}$ is not unique, the heat advection is unique since $\boldsymbol{\nabla} \times (\lambda\boldsymbol{\nabla}\overline{\theta})$ gives a velocity that lies on the isotherm and so cannot advect $\overline{\theta}$. Thus, in the ocean interior where $\mathbf{E}_0^\perp$ is small, the heat advection is independent of $\mathbf{m}$.

In the ocean, isothermal slopes are generally very small, so in practice there is little difference between choosing $\mathbf{m} = \boldsymbol{\nabla}\overline{\theta}$ and $\mathbf{m} = \mathbf{k}$ (the unit vector in the z direction). The standard choice (Andrews and McIntyre 1976; Andrews et al. 1987) $\mathbf{m} = \mathbf{k}$ has the advantage of giving simple forms for $\mathbf{E}_\mathbf{m}^\perp$ and $\mathbf{A_m}$, and is probably the most natural and convenient choice. However, it suffers from the same problem near the sea surface as does the natural decomposition, $\mathbf{m} = \boldsymbol{\nabla}\overline{\theta}$ (Fig. 3).

3) Ocean boundaries

On the ocean boundaries—the sea surface, walls, and ocean floor—we wish to decompose $\mathbf{E}$ into $\mathbf{E}^\perp$ and $\mathbf{E}^\parallel$ in such a manner that both $\mathbf{E}^\perp$ and $\mathbf{E}^\parallel$ lie along the boundary, so there is no spurious heat advection or diffusion through the boundary. This is assured simply by taking $\mathbf{m}$ (and hence $\mathbf{E}_\mathbf{m}^\perp$) to lie along the boundary.

For if $\mathbf{m}$ lies along the boundary, then by (3.10) the vector streamfunction $\mathbf{A_m}$ (and so $\mathbf{A_m}\overline{\theta}$) must be aligned perpendicular to the boundary. By Stokes' theorem neither the advective heat flux $(\boldsymbol{\nabla} \times \mathbf{A_m})\theta = \hat{\mathbf{u}}_\mathbf{m}\theta$ nor the nondivergent heat flux $\nabla \times (\mathbf{A_m}\theta)$ cross the boundary.

So good behavior near the boundary (no normal flow, no normal diffusive flux, and no nondivergent flux) is assured by requiring $\mathbf{m}$ to lie along the bounding surface. On the upper-ocean surface this implies that $\mathbf{m}$ must be horizontal; this is quite a different direction from the natural choice $\mathbf{m} = \boldsymbol{\nabla}\overline{\theta} \approx \mathbf{k}$ in the interior.

There is still freedom in specifying the direction of $\mathbf{m}$ and $\mathbf{E}_\mathbf{m}^\perp$ *along* the boundary. For example, setting $\mathbf{m} \parallel \mathbf{E}$ and so $\mathbf{E}^\perp = \mathbf{E}$ on the boundary ensures that $\mathbf{A_m} = 0$, and the streamfunction disappears. Alternatively we might seek to be consistent with the earlier interior split with $\mathbf{m} = \boldsymbol{\nabla}\overline{\theta}$ and minimize the magnitude of $\mathbf{E}^\perp$ by choosing it to lie along the projection of $\boldsymbol{\nabla}\overline{\theta}$ onto the bounding surface and set $\mathbf{m}$ parallel to $\boldsymbol{\nabla}\overline{\theta} - (\mathbf{n} \cdot \boldsymbol{\nabla}\overline{\theta})\mathbf{n}$, where $\mathbf{n}$ is the unit vector normal to the boundary.

4) Comparison with the TRM

We compare the TEM advection equation for $\overline{\theta}$, (3.13), with vertically oriented diffusive flux, $\mathbf{m} = \mathbf{k}$, to the TRM advection equation for $\theta^\#$ [(2.4) extended to 3D], the approximate form of the equation [(6.2) of Part I] for "vertically isopycnally averaged" temperature, $\tilde{\theta}$. Comparison of (3.10) with $\mathbf{m} = \mathbf{k}$ with (2.14) immediately shows that the vector streamfunction $\mathbf{A_k}$ gives the $\overline{u'\theta'}$ and $\overline{v'\theta'}$ terms in the TRM streamfunction $\mathbf{A}^\#$. By inspection, or (MM01) by approximations involving the eddy variance equation (3.7), the $\boldsymbol{\nabla} \cdot \mathbf{E}_\mathbf{k}^\perp$ term in the TEM advection equation in (3.13) reappears in the TRM as terms involving the advection of $\overline{\theta}$ by the $\overline{u}_z$ and $\overline{v}_z$ terms in $\mathbf{A}^\#$, $D/Dt(\theta^\# - \overline{\theta})$ (associated with advection of eddy variance) and contributions to the approximated vertically isopycnally averaged heating, $Q^\#$ (2.6). Near the surface, of course, the Taylor approximations behind this TRM break down and this reexpression of $\boldsymbol{\nabla} \cdot \mathbf{E}_\mathbf{k}^\perp$ fails: the problems associated with the TEM $\mathbf{m} = \mathbf{k}$ at the sea surface emerge in a catastrophic form in the TRM interpretation.

Similarly, the TEM equation in (3.13) with *horizontally* oriented diffusive flux, $\mathbf{m} = \mathbf{j}$, should be compared with the TRM advection equation for $\theta^\dagger$ [(2.7) in 3D], the approximate form of the equation [(6.7) of Part I] for "horizontally isopycnally averaged" temperature, θ^*. The vector streamfunction $\mathbf{A_j}$ now gives the $\overline{w'\theta'}$ and $\overline{u'\theta'}$ terms in the alternative TRM streamfunction $\mathbf{A}^\dagger$ (2.18). The $\boldsymbol{\nabla} \cdot \mathbf{E}_\mathbf{j}^\perp$ term in the TEM advection equation in (3.13) reappears in the TRM as various similar terms, including contributions to the approximated *horizontally* isopycnally averaged heating, $Q^\dagger$ (2.10).

The horizontal Taylor series approximations behind the TRM advection equation for $\theta^\dagger$ remain valid near the surface, corresponding to the nice properties of the TEM when we make the natural choice $\mathbf{m} = \mathbf{j}$ of a diffusive flux along the surface.

5) Merging the field

So there seems one natural decomposition of $\mathbf{E}$ in the interior and another on the boundaries. The question is how best to move from one to the other so as to minimize in some sense the diffusive eddy flux divergence $\boldsymbol{\nabla} \cdot \mathbf{E}^\perp$.

By our choice of $\mathbf{m}$ we can produce an arbitrary $\mathbf{E}^\parallel$ so long as it is aligned along the isotherm. In theory we can choose $\mathbf{E}^{\parallel}$ so that locally $\nabla \cdot \mathbf{E}^{\parallel} = \nabla \cdot \mathbf{E}$—implying that $\nabla \cdot \mathbf{E}^{\perp} = 0$. If we could do this everywhere, then the effect of the eddies could be rewritten solely as an extra eddy-driven flow, even though there might be nonzero downgradient eddy fluxes, $\mathbf{E}^{\perp} \neq 0$.

However, in practice we cannot do this everywhere, at least for the physically sensible boundary condition that $\mathbf{E}^{\perp}$ should lie along the boundaries, $\mathbf{E}^{\perp} \cdot \mathbf{n} = 0$. Integrating $\nabla \cdot \mathbf{E}^{\perp}$ throughout the volume underneath a given isotherm gives

$$\int \nabla \cdot \mathbf{E}^{\perp}\, dV = \int_{\overline{\theta}} \mathbf{E}^{\perp} \cdot \mathbf{n}_{\theta}\, dA = \int_{\overline{\theta}} \mathbf{E} \cdot \mathbf{n}_{\theta}\, dA, \qquad (3.14)$$

where $\mathbf{n}_{\theta}$ is the vector normal to the $\overline{\theta}$ surface. In general this is not equal to zero. Physically this is because eddies can drive mixing. Since the above holds for any isotherm, it also holds between any two isotherms; viz $\nabla \cdot \mathbf{E}^{\perp}$ integrated within any isopycnal layer is in general nonzero. Only by allowing $\mathbf{E}^{\perp}$ (and hence $\mathbf{E}^{\parallel}$) to cross the boundaries could we remove the diffusive forcing $\nabla \cdot \mathbf{E}^{\perp}$.

Although we cannot completely remove the effect of the downgradient eddy flux, we can redistribute the forcing term $\nabla \cdot \mathbf{E}^{\perp}$ over an isothermal layer. For the zonal-mean case, Gille and Davis (1999) minimized the diffusive eddy forcing in a mean square sense by smearing out $\nabla \cdot \mathbf{E}^{\perp}$ to its thickness-weighted mean on each layer. It is not clear whether this is desirable, though. The downgradient fluxes are largest in certain regions, for example, near the surface. It seems unphysical to spread their effect throughout the isothermal layer into regions where they are, in fact, small. Also if there really are processes (e.g., surface forcing) driving upgradient fluxes in one region and other processes (e.g., molecular diffusion) driving downgradient fluxes in another, it seems incorrect to attempt to cancel them out. Instead it would seem best to retain the natural choice $\mathbf{m} = \nabla\overline{\theta} \approx \mathbf{k}$ in the interior where downgradient fluxes are small, but to somehow move smoothly from this choice to the surface choice over some boundary layer where the downgradient fluxes are substantial.

We now see how these ideas work out in the simpler zonal averaging case.

b. The zonal mean

The above discussion is fully three-dimensional and is valid for time averaging of the three-dimensional field. We now apply it to the special case of zonal averaging. In this case, the vector streamfunction $\mathbf{A}_{\mathbf{m}}(y, z)$ is a function of (y, z) only. We are only interested in the $\mathbf{i}$ component ($\mathbf{i}$ is the unit vector in the x direction) of $\mathbf{A}_{\mathbf{m}}$ because the $\mathbf{j}$ (the unit vector in the y direction) and $\mathbf{k}$ components give rise to zonal flow which does not advect $\overline{\theta}(y, z)$. This is consistent with the natural choice that $\mathbf{m}$ (the direction of $\mathbf{E}^{\perp}_{\mathbf{m}}$) should lie on the $\mathbf{j}$–$\mathbf{k}$ plane. The two simplest choices of $\mathbf{m}$ are either $\mathbf{m} = \mathbf{k}$ (Andrews and McIntyre 1976; Andrews et al. 1987) or $\mathbf{m} = \mathbf{j}$ (e.g., HS99).

When $\mathbf{m} = \mathbf{k}$, the parallel and the perpendicular fluxes are (Fig. 4a)

$$\mathbf{E}^{\parallel}_{\mathbf{k}} = \overline{v'\theta'}(\mathbf{j} + s\mathbf{k}) \quad \text{and}$$

$$\mathbf{E}^{\perp}_{\mathbf{k}} = \frac{\mathbf{E} \cdot \nabla\overline{\theta}}{\overline{\theta}_z}\mathbf{k} = (\overline{w'\theta'} - s\overline{v'\theta'})\mathbf{k}, \qquad (3.15)$$

where $s = -(\partial\overline{\theta}/\partial y)/(\partial\overline{\theta}/\partial z)$ is the slope of mean isotherms. The zonal-mean vector streamfunction is

$$\mathbf{A}_{\mathbf{k}} = \psi_{\mathbf{k}}\mathbf{i} = -\frac{\overline{v'\theta'}}{\overline{\theta}_z}\mathbf{i}, \qquad (3.16)$$

and the eddy-induced velocity is

$$(\hat{v}_{\mathbf{k}}, \hat{w}_{\mathbf{k}}) = \left(\frac{\partial\psi_{\mathbf{k}}}{\partial z}, -\frac{\partial\psi_{\mathbf{k}}}{\partial y}\right). \qquad (3.17)$$

When $\mathbf{m} = \mathbf{j}$, the parallel and the perpendicular fluxes are (Fig. 4b)

$$\mathbf{E}^{\parallel}_{\mathbf{j}} = \overline{w'\theta'}(s^{-1}\mathbf{j} + \mathbf{k}) \quad \text{and}$$

$$\mathbf{E}^{\perp}_{\mathbf{j}} = \frac{\mathbf{E} \cdot \nabla\overline{\theta}}{\overline{\theta}_y}\mathbf{j} = (\overline{v'\theta'} - s^{-1}\overline{w'\theta'})\mathbf{j}. \qquad (3.18)$$

The zonal-mean vector streamfunction is

$$\mathbf{A}_{\mathbf{j}} = \psi_{\mathbf{j}}\mathbf{i} = \frac{\overline{w'\theta'}}{\overline{\theta}_y}\mathbf{i}, \qquad (3.19)$$

and the eddy-induced velocity is

$$(\hat{v}_{\mathbf{j}}, \hat{w}_{\mathbf{j}}) = \left(\frac{\partial\psi_{\mathbf{j}}}{\partial z}, -\frac{\partial\psi_{\mathbf{j}}}{\partial y}\right). \qquad (3.20)$$

Where the eddy flux is along isopycnals, $\mathbf{E} \cdot \nabla\overline{\theta} = 0$, and so

$$-\frac{\overline{v'\theta'}}{\overline{\theta}_z} = \frac{\overline{w'\theta'}}{\overline{\theta}_y}; \qquad (3.21)$$

the two streamfunctions are identical, $\psi_{\mathbf{k}} = \psi_{\mathbf{j}}$. This is not surprising as we have argued in the time-averaging case that if $\mathbf{E} \cdot \nabla\overline{\theta} = 0$, the vector streamfunction is unique up to $\lambda\nabla\overline{\theta}$. Since $\nabla\overline{\theta}(y, z)$ has no $\mathbf{i}$ component, all vector streamfunctions must have the same $\mathbf{i}$ component. For example, another streamfunction that is also identical to $\psi_{\mathbf{k}}$, if $\mathbf{E} \cdot \nabla\overline{\theta} = 0$, is $\psi_0 = (\overline{w'\theta'}\,\overline{\theta}_y - \overline{v'\theta'}\,\overline{\theta}_z)/|\nabla\overline{\theta}|^2$, the $\mathbf{i}$ component of the natural vector streamfunction $\mathbf{A}_0$ in (3.3). This ψ_0 was first used in Andrews and McIntyre (1978).

We have already plotted $\psi_{\mathbf{k}}$ and $\psi_{\mathbf{j}}$ for our model run in Figs. 1a and 1c. The Eulerian mean $\overline{\Psi}$ is insignificant, so we are in effect looking at the total residual streamfunction. The $\psi_{\mathbf{k}}$ and $\psi_{\mathbf{j}}$ are similar in the interior where the eddy flux is largely along isopycnals, as seen by plotting $\mathbf{E} \cdot \nabla\overline{\theta}$ (Fig. 5a). However, they differ near the ocean surface (where $\psi_{\mathbf{k}}$ becomes large and noisy) and the northern and southern relaxation regions (where $\psi_{\mathbf{j}}$

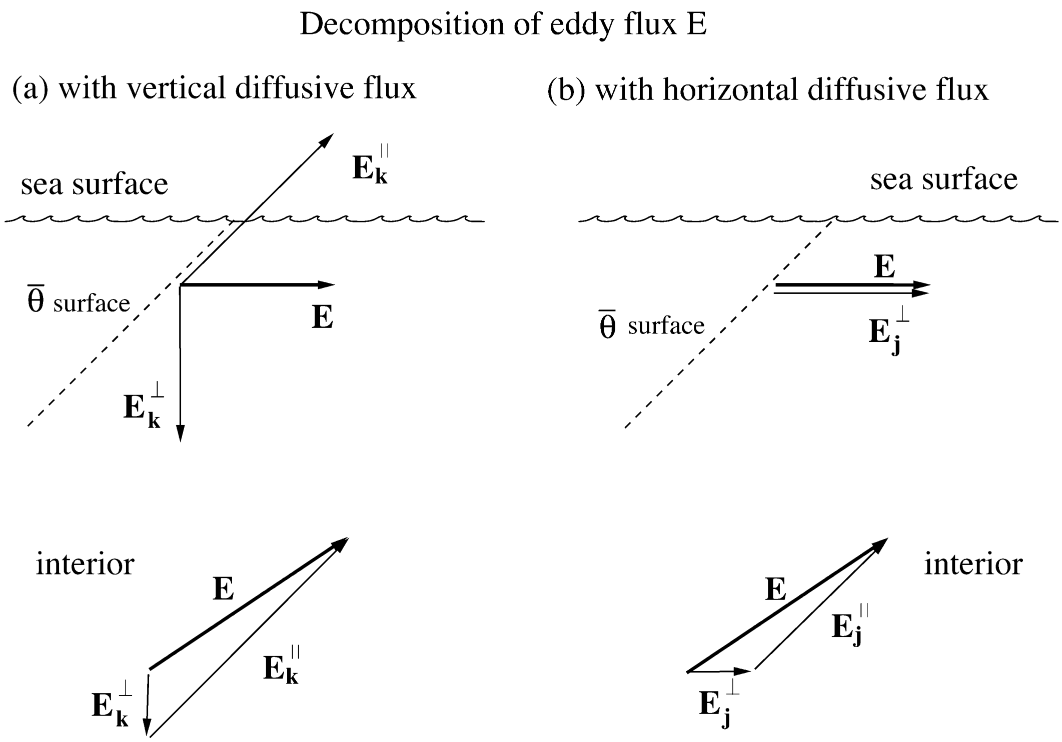

FIG. 4. (a) The decomposition using $\mathbf{m} = \mathbf{k}$. (b) The decomposition using $\mathbf{m} = \mathbf{j}$.

is large) and walls (where $\psi_{\mathbf{j}}$ is noisy). The differences at the boundaries can be understood as follows: clearly, there can be no normal flow at the boundary, and so $\overline{w'\theta'} = 0$ at the upper surface (so $\psi_{\mathbf{j}} = 0$ there) and $\overline{v'\theta'} = 0$ at the sidewalls (so $\psi_{\mathbf{k}} = 0$ there). However, there is no obvious reason why $\overline{v'\theta'}$ should be zero at the upper boundary or $\overline{w'\theta'}$ should be zero at sidewalls.

These eddy fluxes along the boundary cross isotherms intersecting the boundary, so there is a nonzero downgradient eddy flux. The eddy effect on the flow here cannot be thought of as solely advective. The processes driving nonzero downgradient eddy flux $\mathbf{E} \cdot \boldsymbol{\nabla}\overline{\theta}$ (plotted for our model in Fig. 5a) are evident from the eddy variance equation in (3.7). In our model run, the mean velocity is almost zero, and the model is in statistical steady state, so variance advection and changes are small. The eddy flux term is third order. In the interior, the heating/temperature correlation $\overline{Q'\theta'}$ is small, so $\mathbf{E} \cdot \boldsymbol{\nabla}\overline{\theta}$ is small—the eddy flux is along isopycnals, and the two streamfunctions are very similar. However $\overline{Q'\theta'}$ is important in the south, north, and surface relaxation regions, permitting downgradient eddy fluxes and differing streamfunctions.

So this zonal-mean case has shown quite clearly that, with a fixed $\mathbf{m}$, there are likely to be problems at some boundaries. Having said this, different streamfunctions give different flow across boundaries, and different diffusive forcing, $-\boldsymbol{\nabla} \cdot \mathbf{E}^{\perp}$. The smaller these are, the better. Near the sea surface, $\psi_{\mathbf{k}}$ does not work well. Since $\overline{v'\theta'} > 0$ (is northward) at the sea surface and the isotherms slope upward to the north, we have an along-isotherm heat flux $\mathbf{E}_\mathbf{k}^{\parallel}$ with a component up through the surface, balanced by a consistently downward heat flux $\mathbf{E}_\mathbf{k}^{\perp}$ [Fig. 4; (3.15)]. This downward heat flux $c_p\rho\mathbf{E} \cdot \boldsymbol{\nabla}\overline{\theta}/\overline{\theta}_z$ is typically about $c_p\rho(6/0.02)$°C m s^{-1} ~ 800 W m^{-2}; this is large when compared with the ~300 W m^{-2} of mean surface cooling. The nonzero $\mathbf{E}_\mathbf{k}^{\parallel}$ implies that $\psi_{\mathbf{k}}$ is nonzero on the surface. In fact, $\mathbf{E}_\mathbf{k}^{\parallel}$ and $\psi_{\mathbf{k}}$ vary along the sea surface, so $\hat{w}_{\mathbf{k}} \neq 0$ and heat is advected across the surface: warm water is advected up through the surface in the southern part of the channel and returns as cold water in the northern corner. Moreover, the nondivergent component, $\nabla \times (\mathbf{A}_\mathbf{k}\overline{\theta})$, gives a vertical heat flux across the surface, $\partial/\partial y(\psi_{\mathbf{k}}\overline{\theta})$, and a meridional heat flux, $-\partial/\partial z(\psi_{\mathbf{k}}\overline{\theta})$. Although the vertical flux makes no net contribution to the heat budget integrated over the upper surface, the meridional nondivergent component of the heat flux has a net depth integral:

$$-\int \frac{\partial(\psi_{\mathbf{k}}\overline{\theta})}{\partial z} = -[\psi_{\mathbf{k}}\overline{\theta}]_{z=-H}^{z=0} = \left[\frac{\overline{v'\theta'}\,\overline{\theta}}{\overline{\theta}_z}\right]_{z=-H}^{z=0} \neq 0. \qquad (3.22)$$

This must be added to the depth-integrated advective heat flux $\int \hat{v}_{\mathbf{k}}\overline{\theta}\, dz$ to give the total correct depth-integrated heat flux $\int\overline{v'\theta'}\, dz$.

The weakening stratification toward the northern end of the channel gives very large values of $\psi_{\mathbf{k}}$ there. Such large near-surface values of $\psi_{\mathbf{k}}$ would occur everywhere where there is a surface mixed layer.

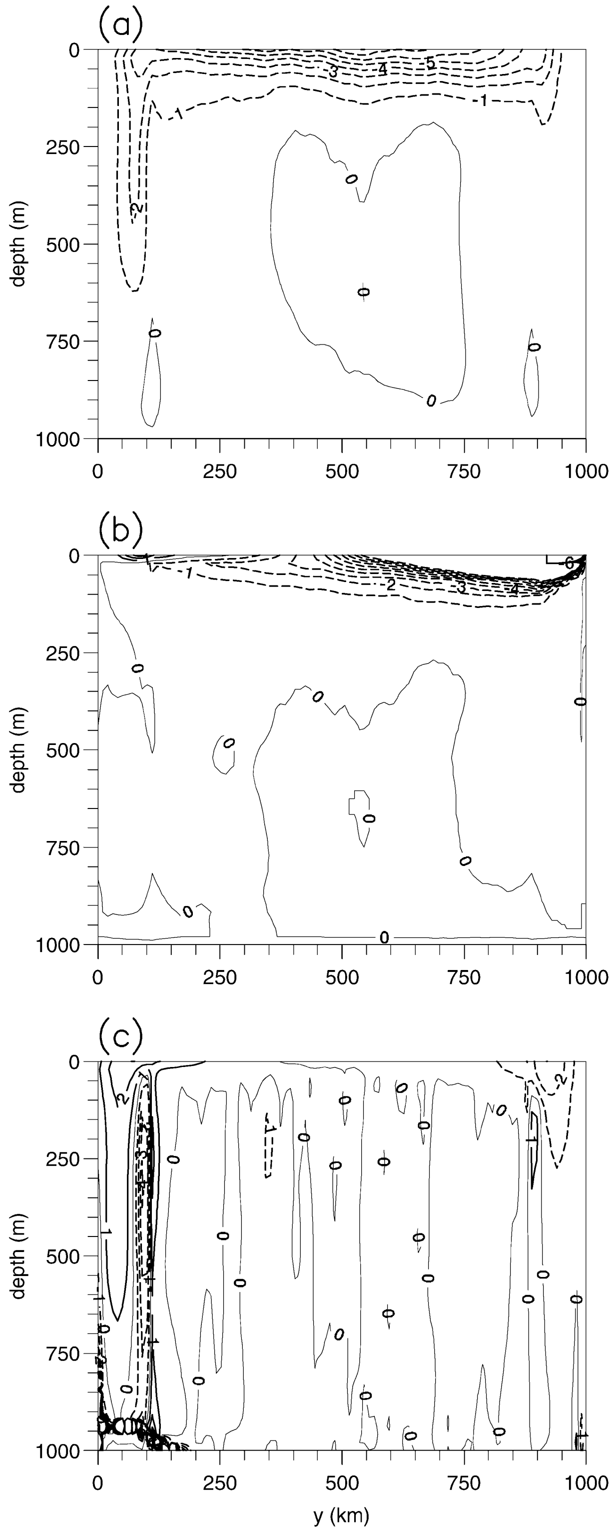

FIG. 5. (a) $\mathbf{E} \cdot \boldsymbol{\nabla}\overline{\theta}$ (10^{-6}°C^2 s^{-1}), (b) $\boldsymbol{\nabla} \cdot \mathbf{E}_{\mathbf{k}}^{\perp}$ (10^{-6}°C s^{-1}), and (c) $\boldsymbol{\nabla} \cdot \mathbf{E}_{\mathbf{j}}^{\perp}$ (10^{-6}°C s^{-1}).

Using $\psi_{\mathbf{j}}$ instead gives much better results near the sea surface (Fig. 1c). For in 2D, choosing $\mathbf{m}$ to lie along the sea surface implies that $\mathbf{E} = \mathbf{E}^{\perp}$ there, so $\psi_{\mathbf{j}} = 0$ on the sea surface, and there are no advective (or even nondivergent) heat fluxes through the surface. As a corollary, the depth-integrated nondivergent heat flux vanishes. Also $\mathbf{E}_{\mathbf{j}}^{\perp}$ is parallel to the sea surface, and so does not act as a net heat source. Additionally, weak surface stratification causes no problems.

Moreover, $\boldsymbol{\nabla} \cdot \mathbf{E}_{\mathbf{j}}^{\perp}$ is much smaller than $\boldsymbol{\nabla} \cdot \mathbf{E}_{\mathbf{k}}^{\perp}$ close to the surface (Figs. 5b,c). For (HS99), the $\mathbf{k}$-divergence term gives

$$-\boldsymbol{\nabla} \cdot \mathbf{E}_{\mathbf{k}}^{\perp} = -\frac{\partial}{\partial z}\left(\frac{\mathbf{E} \cdot \boldsymbol{\nabla}\overline{\theta}}{\overline{\theta}_z}\right) \sim \frac{|\mathbf{E}|D}{\delta L}, \qquad (3.23)$$

where we assume (i) that $\mathbf{E} \cdot \boldsymbol{\nabla}\overline{\theta} \sim |E \cdot \partial\overline{\theta}/\partial y|$ near the surface, where $\overline{w'\theta'}$ may be neglected; (ii) that the slope of the isotherms near the surface $-(\partial\overline{\theta}/\partial y)/(\partial\overline{\theta}/\partial z) \sim D/L$ where D and L are the width and the depth of the channel; and (iii) that δ is the depth scale of the region close to the surface over which $\mathbf{E} \cdot \boldsymbol{\nabla}\overline{\theta}/\overline{\theta}_z$ is substantial. So in this near-surface region $\boldsymbol{\nabla} \cdot \mathbf{E}_{\mathbf{k}}^{\perp}$ is of the same order as the advection $\hat{\upsilon}_{\mathbf{k}}\overline{\theta}_y$ (and tends to cancel it).

This is much larger than the $\mathbf{j}$-divergence term,

$$-\boldsymbol{\nabla} \cdot \mathbf{E}_{\mathbf{j}}^{\perp} = -\frac{\partial}{\partial y}\left(\frac{\mathbf{E} \cdot \boldsymbol{\nabla}\overline{\theta}}{\overline{\theta}_y}\right) \sim \frac{|\mathbf{E}|}{L}. \qquad (3.24)$$

The ratio of the $\mathbf{j}$ divergence to the $\mathbf{k}$ divergence (HS99) is δ/D, which is $\ll 1$, so long as the surface layer δ is much shallower than the large-scale depth D (as it is in our model and probably also in reality). Hence the $\mathbf{j}$ divergence is much smaller than advection near the sea surface here.

On the other hand, $\psi_{\mathbf{k}}$ works much better on the side boundaries where $\mathbf{k}$ lies along the boundary, so $\psi_{\mathbf{k}} = 0$, unlike $\psi_{\mathbf{j}}$ here. By similar arguments to the above, $\boldsymbol{\nabla} \cdot \mathbf{E}_{\mathbf{k}}^{\perp}$ should be smaller than $\boldsymbol{\nabla} \cdot \mathbf{E}_{\mathbf{j}}^{\perp}$ here; indeed this is the case in our run (see again Figs. 5b and 5c).

In the interior it should not matter which streamfunction we choose, as $\psi_{\mathbf{k}}$ and $\psi_{\mathbf{j}}$ should be the same. However, in practice $\psi_{\mathbf{k}}$ is the less noisy streamfunction in the interior because $\partial\overline{\theta}/\partial z$ and $\overline{\upsilon'\theta'}$ are larger, well-defined quantities, whereas $\partial\overline{\theta}/\partial y$ and $\overline{w'\theta'}$ are much smaller. Indeed $\partial\overline{\theta}/\partial y$ may well be zero or change sign. This is consistent with the idea that, away from boundaries, it is preferable to choose $\mathbf{m}$ so as to minimize $\mathbf{E}^{\perp}$, that is, choose $\mathbf{m} = \nabla\overline{\theta}$. Given the weak isothermal slopes in the ocean ($\sim 10^{-3}$) this is well approximated by $\mathbf{m} = \mathbf{k}$.

So the different streamfunctions with constant $\mathbf{m}$ are useful in different regions. As discussed in the preceding section, the natural choice of $\mathbf{m}$ would seem to be $\mathbf{m} = \mathbf{k}$ in the interior, with $\mathbf{m} = \mathbf{j}$ on the ocean surface and $\mathbf{m}$ lying along the ocean walls and floor—here with vertical walls and flat bottom, simply $\mathbf{m} = \mathbf{k}$ on the sidewalls. The difficult task is how best to merge the interior and boundary values.

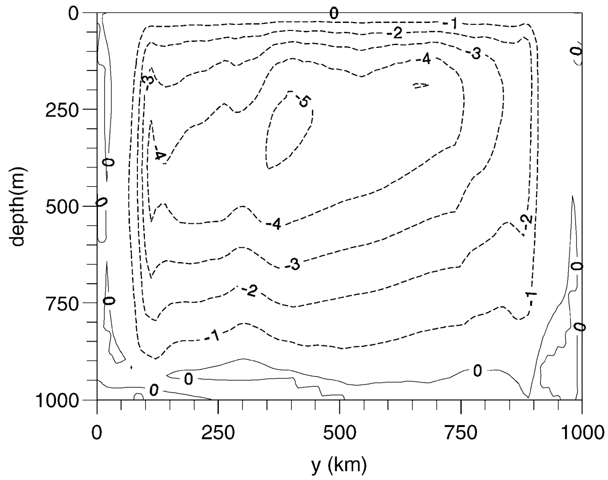

FIG. 6. The field of the minimum of the two streamfunctions $\psi_{\mathbf{k}}$ and $\psi_{\mathbf{j}}$.

Since **m** need not be continuous, we could simply choose $\mathbf{m} = \mathbf{j}$ on the upper and lower boundaries only. This ensures that $\psi_{\mathbf{m}} = 0$ everywhere on the boundaries and also that $\mathbf{E}_{\mathbf{m}}^{\perp} \cdot \mathbf{n} = 0$ on all the boundaries, and so there is no pseudodiffusive heat influx. It also ensures that the depth-integrated nonadvective heat flux disappears. However, $\psi_{\mathbf{m}}$ is still nonzero just below the surface, and this finite transport carried within an infinitesimally thin sheet [Killworth's (2001) δ function, which assures the correct heat transport] implies infinite velocity at the boundary. Also, $\mathbf{E}_{\mathbf{m}}^{\perp} = \mathbf{E}_{\mathbf{k}}^{\perp}$ immediately below this sheet implies infinitely strong δ-function diffusive forcing within this surface sheet, which balances this advection. So in the continuous case, such a reinterpretation makes little difference, except to the depth-integrated advective flux.

For a model with finite vertical resolution this approach leads to partial cancellation in the top grid box. However, it still gives a messy field with near-surface diffusive cooling opposing advection (HS99; Gille and Davis 1999). In the particular case where the model is so coarse that $\Delta z > \delta$, the grid point at $z = -\Delta z$ lies in the interior, and so has no upward diffusive flux associated with it. In this case the messy cancellation is avoided. However there must still be a horizontal flux associated with the choice $\mathbf{m} = \mathbf{j}$ at the surface.

The above argument suggests that choosing $\mathbf{m} = \mathbf{j}$ only on the upper boundaries gives discontinuous ψ and so leads to problems except in a very coarse resolution model.

The next simplest approach is to ensure that ψ is continuous. HS99 suggest simply choosing whichever of $\mathbf{m} = \mathbf{k}$ or $\mathbf{m} = \mathbf{j}$ gives the smallest magnitude of ψ. This seems to work well in practice (we have done this with the ψ from our model runs in Fig. 6). Obviously it ensures that $\psi = 0$ on the boundaries, and, in our model runs, it ensures that $\psi_{\mathbf{j}}$ is used throughout the surface layer and $\psi_{\mathbf{k}}$ within the north and south layers. The resulting combined streamfunction is quite similar to the meridional streamfunction $\tilde{\Psi}$ of Fig. 1b. However the physical basis behind the criterion that the magnitude of the streamfunction should be smallest is unclear. Indeed the combined streamfunction has taken up the noisy behavior of $\psi_{\mathbf{j}}$ over the southern relaxation zone.

A better idea might be to use $\mathbf{m} = \mathbf{k}$ at sidewalls, floor, and in the interior where $\mathbf{E}^{\perp}$ was "small," and to choose $\mathbf{m} = \mathbf{j}$ on the sea surface. Within the surface boundary layer **m** would be chosen so as to minimize the diffusive eddy forcing in a mean square sense, that is, minimize $\int (\boldsymbol{\nabla} \cdot \mathbf{E}^{\perp})^2\, dV$ in the surface boundary layer. This differs from the approach of Gille and Davis (1999), who minimized $\int (\boldsymbol{\nabla} \cdot \mathbf{E}^{\perp})^2\, dV$ over the whole channel. As discussed at the end of the last section, Gille and Davis's (1999) approach seems unnatural since it gives a ψ in the interior that may differ substantially from the natural ψ_0 or $\psi_{\mathbf{k}}$.

4. Discussion and conclusions

Most of the effect of geostrophic eddies on the density field (or neglecting salinity, the temperature field) can be described as advection. As discussed in Part I, zonal averages of temperatures can be defined so as to retain volumetric properties in the vertical ($\tilde{\theta}$) and in the horizontal (θ^*). The temperature field $\tilde{\theta}$ is advected by the "vertically isopycnally averaged" velocity fields ($\tilde{v}$, $\tilde{w}$), while θ^* is advected by the "horizontally isopycnally averaged" velocity fields (v^*, w^*). By definition these velocity fields include the "eddy advection."

If we know only Eulerian zonal (or time) mean quantities, we can represent the eddy advection in two ways. The approach of McIntosh and McDougall (1996) and McDougall and McIntosh (2001) is to take Taylor expansions of the isopycnally averaged flow and temperature fields to give approximate forms ($v^{\#}$, $w^{\#}$, $\theta^{\#}$, $Q^{\#}$). The resulting advection equation is similar to that of ($\tilde{v}$, $\tilde{w}$, $\tilde{\theta}$, $\tilde{Q}$) [cf. (2.4) with (6.2) in Part I]. This approach is very illuminating and is physically intuitive. It works very well in the interior, but fails near the sea surface, within the region where eddies may cause isopycnals to outcrop. This is anywhere within the mixed layer and perhaps 100–200 m below it—an important part of the ocean. Unfortunately, the approximated set ($v^{\dagger}$, $w^{\dagger}$, $\theta^{\dagger}$, $Q^{\dagger}$) for the (v^*, w^*, θ^*, Q^*) fails in the ocean interior wherever eddies are closed.

The alternative approach is to use the transformed Eulerian mean formalism. In this case, we work with the Eulerian mean temperature $\overline{\theta}$. Since $\overline{\theta}$ does not retain the volumetric properties of the unaveraged temperature field θ, the effect of eddies on $\overline{\theta}$ is inevitably both advective and diffusive. In the TEM approach the eddy heat flux is first broken up into a component along the isotherms—the advective part—and the remainder—the diffusive part. Second, the along-isotherm advective component is broken up into a component that advects $\overline{\theta}$ directly plus a nondivergent component that does not affect $\overline{\theta}$.

The orientation of the diffusive part is in principle arbitrary, so there is an infinite number of ways of breaking up the eddy heat flux into advective and diffusive parts. The most natural separation is to minimise the diffusive component by taking it to be perpendicular to mean isotherms (Andrews and McIntyre 1978). Given the small slopes in the ocean this is equivalent to taking the diffusive component to be vertical. Such a vertically aligned diffusive flux gives rise to an advective eddy flux with the conventional meridional zonal-mean streamfunction $\psi_{\mathbf{k}} = -\overline{v'\theta'}/\overline{\theta}_z$. This streamfunction is the leading-order component of the approximation to the meridional eddy-driven transport contribution to $\tilde{\Psi}$ in McDougall and McIntosh's interpretation.

However, this separation is not useful near the sea surface. In general the eddy flux here is largely horizontal and, if the isotherms slope, it follows that both diffusive and advective fluxes have components through the sea surface. This is physically unappealing since there cannot be any flow through the sea surface. Even if the isotherms are horizontal at the sea surface, the streamfunction $\psi_{\mathbf{k}}$ can still be nonzero at the surface because the nondivergent component can have a component through the surface. Additionally, the divergence of the diffusive flux is a strong source term in the evolution equation for $\overline{\theta}$, of the same order as the advection by $\psi_{\mathbf{k}}$. All of these problems, described by Held and Schneider (1999), are very evident in our zonally averaged channel runs.

Near the sea surface, it is most natural to take the diffusive component to lie horizontally along the sea surface. For zonal averaging, this means that the advective component vanishes at the surface; for temporal averaging, an advective component parallel to the outcropping isotherms, along the sea surface, is allowed. In neither case, however, does either component of the advective flux (the directly advective or the nondivergent) cross the sea surface. In the zonal average, mean temperature $\overline{\theta}$ is advected by the flow field associated with the streamfunction $\psi_{\mathbf{j}} = \overline{w'\theta'}/\overline{\theta}_y$. This streamfunction is the eddy contribution to $\Psi^{\dagger}$, the approximation produced by a Taylor expansion in the horizontal for the vertical streamfunction Ψ^*. It vanishes at the surface and gives an intuitively plausible flow field, similar to the remapped $\tilde{\Psi}$ and Ψ^* fields. Of course, it cannot be directly compared with $\tilde{\Psi}$ near the surface because $\tilde{\Psi}$ advects $\tilde{\theta}$ instead of $\overline{\theta}$ and $\tilde{\theta}$ may be several degrees warmer. This extra heat flux is carried by the diffusive component, which is now horizontal. However, the divergence of this horizontal heat flux is smaller than the divergence of the vertical heat flux near the surface in the conventional decomposition.

Unfortunately, the $\psi_{\mathbf{j}}$ decomposition also has its own problems. Formally it has the same problems near the northern and southern boundaries—implied flow through the boundary, and so on—as does the $\psi_{\mathbf{k}}$ decomposition at the sea surface. In our model runs the eddy flux actually disappeared at these boundaries, and so $\psi_{\mathbf{j}}$ was in fact well behaved. Problems instead arose over the relaxation zones, close to the boundaries, where the cross-isotherm eddy flux was large. More serious perhaps is the division by $\overline{\theta}_y$ in the expression for $\psi_{\mathbf{j}}$, which makes the field ill-defined (or at best noisy) wherever $\overline{\theta}_y$ is weak. This latter problem is associated with the interpretation of $\psi_{\mathbf{j}}$ as the leading-order component of the approximation to the vertical transport streamfunction Ψ^*—an approximation that fails whenever the temperature field has closed eddies or large meanders on horizontal surfaces.

The remedy is to blend the best of the two streamfunctions by using $\psi_{\mathbf{j}}$ near the sea surface and $\psi_{\mathbf{k}}$ in the interior. We can do this since the orientation of the diffusive component can be a function of space (or even of time). Away from strong forcing in the interior, the two streamfunctions $\psi_{\mathbf{k}}$ and $\psi_{\mathbf{j}}$ should in principle differ little since the cross-isotherm eddy flux is weak. Where they differ is over a near-surface boundary layer within which the cross-isotherm eddy flux is significant as a result of strong mixing or surface restoring flux conditions.

There are many possible ways of matching $\psi_{\mathbf{k}}$ to $\psi_{\mathbf{j}}$ through this near-surface layer. The crudest is to only use $\psi_{\mathbf{j}}$ at the surface—equivalent to forcing a boundary condition $\psi_{\mathbf{k}} = 0$ at the surface. This only works for a low-resolution model that does not resolve the rest of the boundary layer. It should be noted that this implies that a horizontal diffusive flux is required through the surface gridbox. An alternative is to use the minimum value of $\psi_{\mathbf{j}}$ and $\psi_{\mathbf{k}}$—this is equivalent to using $\psi_{\mathbf{j}}$ throughout the surface layer and works reasonably well both for Held and Schneider (1999) and in our model runs. The actual criterion, however, makes little sense physically and can lead to problems in other parts of the domain. We instead suggest using $\psi_{\mathbf{k}}$ in the interior and blending toward $\psi_{\mathbf{j}}$ at the surface by minimizing the mean squared divergence of the diffusive flux through this surface layer. This should give smooth fields and is a physically logical criterion. It differs from the approach of Gille and Davis (1999), who minimize the same quantity throughout the domain, which has the undesirable consequence of spreading the diffusive flux throughout the domain.

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APPENDIX

Approximation of Zonally and Temporally Isopycnically Averaged Quantities

In this appendix we discuss how the residual streamfunction approximates the zonally isopycnically averaged streamfunction shown in Part I. We also discuss approximation of the temporally averaged streamfunctions. The validity of the approximation is usually discussed in terms of the magnitude of the perturbation relative to the mean. Traditionally, there is no distinction between perturbation magnitudes for velocity or density. They are all expressed as, say, α. Here, we derive the approximation again but distinguish separately each perturbation magnitude. This distinction is important, as we will see, for understanding when the approximation should work.

a. Approximation of $\tilde{\theta}$

Assume that the θ_a isotherm lies at a height $z = z_a + z'_a$, where z_a is the zonal-mean height of θ_a and z'_a is the deviation from the mean. By definition (in Part I) we have $\tilde{\theta}(z_a) = \theta_a$ and $\overline{z'_a} = 0$.

Taylor's theorem gives

$$\begin{aligned}\tilde{\theta}(z_a) &= \theta(x, z_a + z'_a)\\ &= \theta(x, z_a) + \frac{\partial\theta}{\partial z}(x, z_a)z'_a + \frac{1}{2}\frac{\partial^2\theta}{\partial z^2}(x, z_a)z'^2_a\\ &\quad + \frac{1}{6}\frac{\partial^3\theta}{\partial z^3}(x, z_r)z'^3_r,\end{aligned} \tag{A.1}$$

where $|z'_r| < |z'_a|$.

The approximation requires that the higher-order terms in z'_a are in some sense "small." To estimate the size of these terms, we introduce the following scaling:

- Z' — a scaling for the isothermal height perturbation z'_a,
- Θ — the "significant" change in $\overline{\theta}$,
- $\alpha\Theta$ — a scaling for θ',
- D — the depth scale over which $\overline{\theta}$ varies significantly, and
- D' — the depth scale over which θ' varies.

We break up θ and its derivatives into mean and perturbation parts and scale all terms in (A.1) by Θ to give

$$\begin{aligned}\tilde{\theta}(z_a) - \overline{\theta} = &\underbrace{\theta'}_{O(\alpha)} + \underbrace{\frac{\partial\overline{\theta}}{\partial z}z'_a}_{O(Z'/D)} + \underbrace{\frac{\partial\theta'}{\partial z}z'_a}_{O(\alpha\cdot Z'/D')}\\ &+ \underbrace{\frac{1}{2}\frac{\partial^2\overline{\theta}}{\partial z^2}(x, z_a)z'^2_a}_{O(Z'^2/D^2)} + O\left(\alpha\frac{Z'^2}{D'^2}\right) + O\left(\frac{Z'^3}{D^3}\right).\end{aligned} \tag{A.2}$$

For this expansion to be useful, we require that

$$Z'/D \ll 1 \quad \text{and} \quad Z'/D' \ll 1:$$

the vertical scales over which the mean and perturbation temperatures vary must be larger than the vertical displacements of the isotherms.

Zonal averaging of (A.2) gives

$$\begin{aligned}\tilde{\theta}(z_a) - \overline{\theta} = &\underbrace{\overline{\frac{\partial\theta'}{\partial z}z'_a}}_{O(\alpha\cdot Z'/D')} + \underbrace{\frac{1}{2}\frac{\partial^2\overline{\theta}}{\partial z^2}(x, z_a)\overline{z'^2_a}}_{O(Z'^2/D^2)}\\ &+ O\left(\alpha\frac{Z'^2}{D'^2}\right) + O\left(\frac{Z'^3}{D^3}\right).\end{aligned} \tag{A.3}$$

This implies the balance for the first-order terms in (A.2) must be

$$\underbrace{\theta'}_{O(\alpha)} \sim \underbrace{-\frac{\partial\overline{\theta}}{\partial z}z'_a}_{O(Z'/D)}. \tag{A.4}$$

So, we have

$$\alpha \sim Z'/D. \tag{A.5}$$

Note that this automatically implies that

$$\frac{\partial\theta'}{\partial z}z'_a \ll \frac{\partial\overline{\theta}}{\partial z}z'_a;$$

that is, the mean stratification is greater than the perturbation to the stratification, which is a key assumption of quasigeostrophy.

We then use (A.4) to rewrite (A.3) to give

$$\tilde{\theta} = \theta^{\#} + O\left(\alpha\frac{Z'^2}{D'^2}\right) + O(\alpha^3),$$

where

$$\theta^{\#} = \overline{\theta} - \frac{\partial}{\partial z}\left(\frac{\phi}{\overline{\theta}_z}\right) \quad \text{and} \quad \phi = \frac{1}{2}\overline{\theta'^2}. \tag{A.6}$$

If we assume that $D \sim D'$ so that the depth scale of the eddies is the same as that of the mean flow, then $Z'/D' \sim Z'/D \sim \alpha$. Hence, $\tilde{\theta} = \overline{\theta} + O(\alpha^2)$ and $\tilde{\theta} = \theta^{\#} + O(\alpha^3)$. This gives the result of MM01. However, this assumption is stronger than necessary. From (A.3), we can see that $\theta^{\#}$ is still the leading-order approximation to $\tilde{\theta}$, so long as

$$\alpha\frac{Z'^2}{D'^2} \leq \frac{Z'^2}{D^2};$$

that is, $D'/D \gtrsim \alpha^{1/2}$.

The approximation (A.4) for θ' fails wherever $z_a > -Z'$ and z'_a has been "chopped off" by striking the sea surface. The closer to the surface, the worse the failure is here. Thus, (A.6) is only useful at depths sufficient that the $\tilde{\theta}$ isopycnal does not strike the surface; that is, $z < -Z'$. At the surface $\tilde{\theta} = \overline{\theta} + O(\theta') = \overline{\theta} + O(\alpha)$ (MM01; Killworth 2001). So, within a boundary layer of thickness $O(Z' \sim \alpha D)$, $\tilde{\theta}$ differs from $\theta^{\#}$ (and of course $\overline{\theta}$) by $O(\alpha)$ (Killworth 2001).

Note that in our model run, by inspection (e.g., at 350-m depth, 300 km north in Part I, Fig. 4a) $\alpha \sim 1/2(\theta_{max} - \theta_{min})/(\theta_{surface} - \theta_{bottom}) \sim 1/7$ is indeed of the same order as Z'/D. Also $D' \sim 500$ m: smaller, but of the same order as D.

The approximations (A.3)–(A.6) hold equally well for the temporal mean.

b. Approximation of θ^*

We can perform a similar expansion on $\theta(x, y_a + y'_a)$, where now y_a is the mean meridional displacement of the θ_a isotherm and y'_a is the perturbation. As before, $\theta_a = \theta(x, y_a + y'_a) = \theta^*(y_a)$. Kushner and Held (1999) used such a horizontal expansion for potential vorticity in a barotropic model.

The following additional scalings are needed:

- Y' a scaling for the isothermal meridional displacement y'_a,
- L_0 the length scale over which $\overline{\theta}$ varies significantly, and
- L' the length scale over which θ' varies.

Formally, a similar expansion gives

$$\theta^*(y_a) - \overline{\theta} = \underbrace{\theta'}_{O(\alpha)} + \underbrace{\frac{\partial\overline{\theta}}{\partial y}y'_a}_{O(Y'/L_0)} + \underbrace{\frac{\partial\theta'}{\partial y}y'_a}_{O(\alpha\cdot Y'/L')} + \underbrace{\frac{1}{2}\frac{\partial^2\overline{\theta}}{\partial y^2}(x, y_a)y'^2_a}_{O(Y'^2/L_0^2)} + O\left(\alpha\frac{Y'^2}{L'^2}\right) + O\left(\frac{Y'^3}{L_0^3}\right). \tag{A.7}$$

Here, all terms have been scaled by the large-scale θ change, Θ. The approximation is only valid if both $Y'/L_0 \ll 1$ and $Y'/L' \ll 1$. The latter is unlikely to be satisfied since for fully nonlinear eddies the scale of variation of the perturbations is the same as the meridional displacement of the isotherms, $Y'/L' \sim 1$ (see Part I, Fig. 4a). Note also that Y' is not much smaller than L_0 in this particular run. All of the terms involving θ' in the expansion are likely to be the same order. So, the whole approximation fails and, in particular,

$$\theta' \nsim -\frac{\partial\overline{\theta}}{\partial y}y'_a.$$

Indeed, Fig. 5b in Part I shows that θ^* differs from $\overline{\theta}$ by at least 0.5°C in most of the channel above 500 m. In this case, the third term, $(\partial\theta'/\partial y)y'_a$, in (A.7) must be of similar order to $(\partial\overline{\theta}/\partial y)y'_a$.

Although formally we can still write down

$$\theta^* = \theta^\dagger + O\left(\alpha\frac{Y'^2}{L'^2}\right) + O(\alpha^3) \quad \text{and}$$

$$\theta^\dagger = \overline{\theta} - \frac{\partial}{\partial y}\left(\frac{\phi}{\overline{\theta}_y}\right), \tag{A.8}$$

it will not be a good approximation to θ^* because $Y'/L' \sim 1$. There can be good agreement between $\theta^\dagger$ and θ^* only when the perturbation to $\overline{\theta}$ is so small that broken off eddies do not form. We will see that a similar problem also occurs in the approximation for the transport streamfunction.

The above approximations, (A.7) and (A.8), again hold equally well for the temporal and zonal mean.

c. Approximation of the transport streamfunctions $\tilde{\Psi}$ and Ψ^*

We start with the zonally integrated meridional overturning streamfunction $\tilde{\Psi}$. This is defined as the zonal and depth integral of the velocity v, integrated below the θ_a isotherm, whose height is $z_a + z'_a$. That is,

$$\tilde{\Psi} = \iint_{\theta\le\theta_a} v\, dz\, dx = \iint_{z_{\text{cold}}}^{z_a} \overline{v}\, dz\, dx + \iint_{z_a}^{z_a+z'_a} v\, dz\, dx. \tag{A.9}$$

By the mean value theorem,

$$\int_{z_a}^{z_a+z'_a} v\, dz = v(x, z_a)z'_a + \frac{1}{2}\frac{\partial v}{\partial z}(x, z_a)z'^2_a + \frac{1}{6}\frac{\partial^2 v}{\partial z^2}(x, z_a + z'_r)z'^3_r, \tag{A.10}$$

where $0 < z'_r < z'_a$. We break up v into zonal-mean and perturbation parts, $\overline{v}$ and v', and scale these by a notional velocity (perhaps the zonal velocity) $V \sim gf^{-1}\alpha_E\Theta DL_0^{-1}$ in balance with the large-scale thermal wind. Here g is gravity and α_E the coefficient of thermal expansion. Assuming that the perturbation velocity is in thermal wind balance on the eddy depth scales D' and length scale L', it then follows that

$$v' \sim \mu V,$$

where

$$\mu = \alpha D'L_0/(L'D) \sim \alpha L_0/L', \tag{A.11}$$

if the eddy and large-scale depth scales are similar, $D' \sim D$. At depths and latitudes that encounter meridional boundaries, there is a zonal-mean geostrophic shear $\overline{v}_z \sim L(y, z)^{-1}gf^{-1}\alpha_E[\theta]^E_W$, where $[\theta]^E_W$ is the difference in temperature between the eastern and western boundaries. Hence (McIntosh and McDougall 1996) $\overline{v} \sim \alpha L_0L^{-1}(y, z)V$ is a perturbation quantity. Where the channel is zonally reentrant, only the ageostrophic velocity remains, insignificant except within the Ekman layer (within which the Taylor expansion fails in any case). Hence

$$\overline{v} \sim \nu V,$$

where

$$\nu \le \alpha. \tag{A.12}$$

Zonally averaging (A.10) and implicitly scaling all terms by VD, we have

$$\overline{\int_{z_a}^{z_a+z_a'} v\,dz} = \underbrace{\overline{v'(x, z_a)z_z'}}_{O(\mu \cdot Z'/D)} + \underbrace{\frac{1}{2}\frac{\partial \overline{v}}{\partial z}\overline{z_a'^2}}_{O(v \cdot Z'^2/D^2)} + O\left(\mu\frac{Z'^2}{DD'}\right) + O\left(\frac{Z'^3}{D^3}\right). \quad \text{(A.13)}$$

Here we have assumed that $\overline{v}$ varies on the same large-scale depth D as does $\overline{\theta}$ and that v' varies on the eddy depth scale D' (which may be smaller). We now assume as before that $Z'/D \sim \alpha \ll 1$ and $Z'/D' \ll 1$; then, again using (A.4) for z_a' gives

$$\overline{\int_{z_a}^{z_a+z_a'} v\,dz} = \underbrace{-\frac{\overline{v'(x, z_a)\theta'}}{\overline{\theta}_z}}_{O(\mu\alpha)} + \underbrace{\frac{\phi\overline{v}_z}{(\overline{\theta}_z)^2}}_{O(v\alpha^2)} + O\left(\mu\alpha\frac{Z'}{D'}\right) + O(\alpha^3). \quad \text{(A.14)}$$

By (A.12) the $\overline{v}_z$ term is $O(\alpha^3)$, and we have

$$\tilde{\Psi} \sim \Psi^{\#} + O\left(\mu\alpha\frac{Z'}{D'}\right) + O(\alpha^3),$$

where the zonally integrated meridional streamfunction

$$\Psi^{\#} = \overline{\Psi} - \frac{\overline{v'\theta'}}{\overline{\theta}_z}L(y, z). \quad \text{(A.15)}$$

In the temporal mean (MM01), on the other hand, since the time-mean velocity is now $O(V)$, it follows that the $\overline{v}_z$ term in (A.14) is now $O(\alpha^2)$. The question is whether this term is now the same order as the $O(\mu\alpha)\overline{v'\theta'}$ term. By (A.11) $\mu \sim \alpha L_0/L'$ and since L' is the same order as the Rossby radius (~30 km), we might expect that $L_0/L' \gg 1$, and so $\mu \gg \alpha$. This would suggest that the first $\overline{v'\theta'}$ term should be larger than $O(\alpha^2)$[A1] and so still dominate. On the other hand, the correlation coefficient between v' and θ' in $\overline{v'\theta'}$ may be small, and the mean flow may vary on relatively small lateral scales of $O(L')$. Hence the first $\overline{v'\theta'}$ term may not in practice always dominate, and for the temporal mean both terms are retained, with, for example, the time-mean residual meridional overturning streamfunction given by

$$\Psi^{\#}_{[y]} = \overline{\Psi} - \frac{\overline{v'\theta'}}{\overline{\theta}_z} + \frac{\phi\overline{v}_z}{(\overline{\theta}_z)^2}. \quad \text{(A.16)}$$

This $\Psi^{\#}$ is called the TRM streamfunction by MM01.

Formally, a similar expansion and scaling analysis can be applied to Ψ^*, as was done by Kushner and Held (1999) in a barotropic model, to obtain, for the zonally integrated vertically isopycnally averaged streamfunction,

$$\Psi^* \sim \Psi^{\dagger} = \overline{\Psi} + \frac{\overline{w'\theta'}}{\overline{\theta}_y}L(y, z). \quad \text{(A.17)}$$

This approximation is likely to be poor for the same reason as that for θ^*—the eddies are finite amplitude and so $Y'/L' \sim 1$. The sign of the perturbation in $\Psi^{\dagger}$ is different to that in $\Psi^{\#}$ because $\Psi^{\dagger}(y_a, z)$ is the upward flow integrated over $y > y_a$, whereas $\Psi^{\#}(y, z_a)$ is the northward flow integrated over $z < z_a$.

Again, the time-mean vertical velocity is relatively more important; so, for example, the time-mean vertical meridional overturning streamfunction

$$\Psi^{\dagger}_{[z]} = \overline{\Psi} + \frac{\overline{w'\theta'}}{\overline{\theta}_y} - \frac{\overline{w}_y\phi}{(\overline{\theta}_y)^2}. \quad \text{(A.18)}$$

d. Approximating $\tilde{Q}$ and Q^*

In the same way as the mean velocity $\tilde{v}$ is simply the differential of $\tilde{\Psi}$, the mean forcing can be thought of as a differential of the depth-integrated forcing (equivalent to the streamfunction). We approximate the thickness weighted forcing by applying a Taylor expansion to this depth-integrated forcing. Implicitly scaling all terms by $\mathbf{Q}D$, where $\mathbf{Q}$ is a scaling for $\overline{Q}$, we have

$$\int^{z_a+z_a'} Q\,dz = \int^{z_a} \overline{Q}\,dz - \underbrace{\frac{\overline{Q'(x, z_a)\theta'}}{\overline{\theta}_z}}_{O(\xi\alpha)} + \underbrace{\frac{\phi\overline{Q}_z}{(\overline{\theta}_z)^2}}_{O(\alpha^2)} + O(\alpha^3) + O(\xi \cdot \alpha^2).$$

Here we assume $Q' \sim \xi\mathbf{Q}$, and that Q and Q' both vary on the depth scale D (this last assumption may be of dubious validity). Thus $\tilde{Q} = Q^{\#} +$ third-order terms, where

$$Q^{\#} = \overline{Q} - \frac{\partial}{\partial z}\left(\frac{\overline{Q'\theta'}}{\overline{\theta}_z}\right) + \frac{\partial}{\partial z}\left[\frac{\phi\overline{Q}_z}{(\overline{\theta}_z)^2}\right]. \quad \text{(A.19)}$$

Here the extra terms in $Q^{\#}$ represent an attempt to consider the forcing following the isopycnal. Both terms are important, both in the zonal mean and the temporal mean. Again, this approximation will fail near the surface.

The horizontally isopycnally averaged heating Q^* is similarly approximated, but as a meridional derivative, by

$$Q^{\dagger} = \overline{Q} - \frac{\partial}{\partial y}\left(\frac{\overline{Q'\theta'}}{\overline{\theta}_y}\right) + \frac{\partial}{\partial y}\left[\frac{\phi\overline{Q}_y}{(\overline{\theta}_y)^2}\right]. \quad \text{(A.20)}$$

Again, this approximation is likely to be poor where eddies are closed.

[A1] In fact for $\mu \sim 1$, the $O(\mu\alpha Z'/D')$ term may be as large as the $\overline{v}\phi O(\alpha^2)$ term.

REFERENCES

Andrews, D. G., and M. E. McIntyre, 1976: Planetary waves in horizontal and vertical shear: The generalized Eliassen–Palm relation and the zonal mean acceleration. *J. Atmos. Sci.,* **33,** 2031–2048.

——, and ——, 1978: Generalized Eliassen–Palm and Charney–Drazin theorems for waves on axisymmetric mean flows in compressible atmosphere. *J. Atmos. Sci.,* **35,** 175–185.

——, J. R. Holton, and C. B. Leovy, 1987: *Middle Atmosphere Dynamics.* Academic Press, 489 pp.

de Szoeke, R. A., and A. F. Bennett, 1993: Microstructure fluxes across density surfaces. *J. Phys. Oceanogr.,* **23,** 2254–2264.

Gille, S. T., and R. E. Davis, 1999: The influence of mesoscale eddies on coarsely resolved density: An examination of subgrid-scale parameterization. *J. Phys. Oceanogr.,* **29,** 1109–1123.

Held, I. M., and T. Schneider, 1999: The surface branch of the zonally averaged mass transport circulation in the troposphere. *J. Atmos. Sci.,* **56,** 1688–1697.

Killworth, P. D., 2001: Boundary conditions on quasi–Stokes velocities in parameterization. *J. Phys. Oceanogr.,* **31,** 1132–1155.

Kushner, P. J., and I. M. Held, 1999: Potential vorticity fluxes and wave–mean flow interactions. *J. Atmos. Sci.,* **56,** 948–958.

McDougall, T. J., 1998: Three-dimensional residual-mean theory. *Ocean Modelling and Parameterization,* E. P. Chassignet and J. Verron, Eds., NATO Science Series, Vol. 516, Kluwer, 269–302.

——, and P. C. McIntosh, 2001: The temporal-residual-mean velocity. Part II: Isopycnal interpretation and the tracer and momentum equations. *J. Phys. Oceanogr.,* **31,** 1222–1246.

McIntosh, P. C., and T. J. McDougall, 1996: Isopycnal averaging and the residual mean circulation. *J. Phys. Oceanogr.,* **26,** 1655–1660.

Nurser, A. J. G., and M.-M. Lee, 2004: Isopycnal averaging at constant height. Part I: The formulation and a case study. *J. Phys. Oceanogr.,* **34,** 2721–2739.