Simulation of the Transformation of Internal Solitary Waves on Oceanic Shelves

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(Manuscript received 17 October 2003, in final form 15 June 2004)

ABSTRACT

Internal solitary waves transform as they propagate shoreward over the continental shelf into the coastal zone, from a combination of the horizontal variability of the oceanic hydrology (density and current stratification) and the variable depth. If this background environment varies sufficiently slowly in comparison with an individual solitary wave, then that wave possesses a soliton-like form with varying amplitude and phase. This stage is studied in detail in the framework of the variable-coefficient extended Korteweg–de Vries equation where the variation of the solitary wave parameters can be described analytically through an asymptotic description as a slowly varying solitary wave. Direct numerical simulation of the variable-coefficient extended Korteweg–de Vries equation is performed for several oceanic shelves (North West shelf of Australia, Malin shelf edge, and Arctic shelf) to demonstrate the applicability of the asymptotic theory. It is shown that the solitary wave may maintain its soliton-like form for large distances (up to 100 km), and this fact helps to explain why internal solitons are widely observed in the world’s oceans. In some cases the background stratification contains critical points (where the coefficients of the nonlinear terms in the extended Korteweg–de Vries equation change sign), or does not vary sufficiently slowly; in such cases the solitary wave deforms into a group of secondary waves. This stage is studied numerically.

1. Introduction

Nonlinear internal waves observed in the coastal ocean frequently occur as solitary waves, often simply described as solitons. Examples have been reported from many regions in the world’s oceans, including the Sulu Sea, the North West shelf (NWS) of Australia, the Bay of Biscay, the Sea of Okhotsk, East and West China Seas, and the east coast of the Canadian shelf. Ostrovsky and Stepanyants (1989) and Jeans (1995) have provided recent reviews of observations of internal solitary waves in the ocean. Some observations made from space have demonstrated the influence of a variable background (e.g., variable bottom topography, an inhomogeneous thermocline) on the evolution of internal solitons (Liu and Chang 1998; Zheng et al. 2001).

The theoretical description of weakly nonlinear long internal solitary waves in a fluid with continuous stratification of density and current is based on the Korteweg–de Vries (KdV) and the extended Korteweg–de Vries (eKdV) equations (see, e.g., Benney 1966; Lee and Beardsley 1974; Lamb and Yan 1996; Grimshaw 2001; Holloway et al. 2001; Grimshaw et al. 2002b). The oceanic stratification, as well as the total water depth, usually varies slowly (on the scale of the internal solitary waves) in space, and possibly also in time. This variability can be included in theoretical models leading to various variable-coefficient nonlinear evolution equations (see, e.g., Grimshaw 2001). The variability of the coefficients of the Korteweg–de Vries equations has been studied for some areas of the world’s oceans (Poloukhin et al. 2003). The extended Korteweg–de Vries...
equation, as well as the more familiar Korteweg–de Vries equation, is often used in oceanographic practice to investigate nonlinear wave transformation in the coastal zone, taking into account both the variable depth and horizontal variability of the hydrology (Holloway et al. 1997, 1999; Liu and Chang 1998; Small 2001a,b; Cai et al. 2002). The results of these calculations demonstrate the transformation of internal tides (or long-scale disturbances) into large-amplitude short-scale pulses of solitary-like form. Similar results have been obtained in numerical simulations using fully nonlinear models in which usually only the variable bottom is taken into account (Vlasenko and Hutter 2002; Lamb 2002, 2003). In many cases the number of such pulses is large and they interact between themselves and with the background flow. Consequently, it is difficult to determine from observations to what extent an individual solitary wave may propagate for a long distance preserving its form. Hence, in this paper, we address this issue using as a model equation, the variable-coefficient extended Korteweg–de Vries equation. The determination of the “lifetime” of an internal soliton propagating in the ocean, with real variability of hydrology and depth is our main goal. In this paper, we focus solely on the effects of the variable hydrology and depth, and the effects of dissipation and the earth’s rotation are left for further study.

The paper is organized as follows. The theoretical background

2. Theoretical background

The Korteweg–de Vries equation for long weakly nonlinear internal waves was first derived by Benney (1966), taking into account the continuous stratification of the fluid for both background density and shear flow. Subsequently, this equation has been widely used for an explanation of the properties of the solitary-like disturbances observed in the pycnocline, and for the interpretation of synthetic aperture radar (SAR) images of internal solitary waves. For many coastal seas the coefficient of the nonlinear term in the Korteweg–de Vries equation is small (e.g., when the pycnocline lies near the middle depth of the fluid), and high-order correction terms should be taken into account. The extended Korteweg–de Vries equation was first derived for a two-layer fluid (Kakutani and Yamasaki 1978), and then for the more general case of a continuous stratification in density and shear flow (Lamb and Yan 1996; Grimshaw et al. 2002b). It can be written as

\[ \frac{\partial \eta}{\partial t} + (c + \alpha \eta + \alpha_i \eta^3) \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} = 0, \]

where \( \eta(x,t) \) is the vertical displacement of the pycnocline (see below); \( x \) is a horizontal coordinate; \( t \) is time; and the coefficients \( c, \alpha, \alpha_i \), and \( \beta \) are the wave speed, quadratic and cubic nonlinear coefficients, and dispersion parameter, respectively. The wave speed \( c \) is determined from the eigenvalue problem for the modal structure function \( \Phi(z) \) of the vertical displacement in the linear long-wave limit (the Boussinesq approximation and the rigid-lid approximation are used here for simplicity):

\[ \frac{d}{dz} \left[ (c - U(z)) \frac{d\Phi}{dz} \right] + N^2(z) \Phi = 0 \quad \text{and} \quad \Phi(-H) = \Phi(0) = 0, \]

where \( N(z) \) is the Brunt–Väisälä frequency, \( U(z) \) is the background shear flow, and \( H \) is the undisturbed water depth. We assume that modal function \( \Phi(z) \) is normalized, and its maximum \( \Phi_{\max} = 1 \). The dispersion parameter \( \beta \) is given by the integral

\[ \beta = \frac{1}{2} \frac{\int_{-H}^{0} (c - U)^2 \Phi^2 \ dz}{\int_{-H}^{0} (c - U) \left( \frac{d\Phi}{dz} \right)^2 \ dz}, \]

and the nonlinear coefficients \( \alpha \) and \( \alpha_i \) are given by

\[ \alpha = \left( \frac{3}{2} \right) \int_{-H}^{0} (c - U)^2 \left( \frac{d\Phi}{dz} \right)^3 \ dz \quad \text{and} \quad \alpha_i = \int_{-H}^{0} (c - U) \left( \frac{d\Phi}{dz} \right)^2 \ dz. \]
\[ \alpha_1 = \frac{\int dz \left\{ 3(c-U)^2 \left[ \frac{d\Phi}{dz} \right]^2 - 2 \left( \frac{d\Phi}{dz} \right)^2 - \alpha^2 \left( \frac{d\Phi}{dz} \right)^2 + \Pi \right\} }{\int (c-U) \left( \frac{d\Phi}{dz} \right)^2 dz}, \]

where

\[ \Pi = \alpha(c-U) \left[ 5 \left( \frac{d\Phi}{dz} \right)^2 - 4 \frac{d\Phi}{dz} \frac{d\Phi}{dz} \right]. \]

Here \( T(z) \) is the nonlinear correction to the modal structure, and it is found from the inhomogeneous eigenvalue problem

\[ \frac{d}{dz} \left[ (c-U)^2 \frac{dT}{dz} \right] + N^2 T = -\frac{d}{dz} \left[ (c-U) \frac{d\Phi}{dz} \right] + \frac{3}{2} \frac{d}{dz} \left[ (c-U)^2 \left( \frac{d\Phi}{dz} \right)^2 \right], \]

with zero boundary conditions on the sea bottom and free surface. Also we normalize the function \( T(z) \) so that \( T(z_{max}) = 0 \), where \( z_{max} \) is found from \( \Phi(z_{max}) = 1 \). The vertical displacement of the isopycnal surface, \( \xi(x, z, t) \) is given by, with the same accuracy,

\[ \xi(x, z, t) = \eta(x, t)\Phi(z) + \eta^2(x, t)T(z). \]

Taking into account the normalization of the functions, \( \Phi \) and \( T \), the wave function \( \eta(x, t) \) represents the vertical isopycnal displacement at the depth \( z_{max} \).

The extended Korteweg–de Vries equation has been well studied from the mathematical point of view. It is a fully integrable model with the availability of an inverse scattering method to find solutions. The Cauchy problem and multisoliton solutions are discussed in Slyunyaev and Pelinovsky (1999), Slyunyaev (2001), and Grimshaw et al. (2002a).

When the water depth and the oceanic stratification vary slowly in the horizontal direction, the modal structure of the internal wave also varies slowly, but to the first approximation can be calculated from Eq. (2), as for constant conditions, with the horizontal variability appearing parametrically. Likewise, all the coefficients of the extended Korteweg–de Vries equation can be calculated as for a constant background, but they will now also be functions of the horizontal variability. Using an asymptotic method, the variable-coefficient extended Korteweg–de Vries equation can be derived (Zhou and Grimshaw 1989; Holloway et al. 1999; Grimshaw 2001)

\[ \frac{\partial \eta}{\partial t} + c \frac{\partial \eta}{\partial x} + \alpha \frac{\partial \eta}{\partial x} + \alpha \eta \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} + \frac{c}{2Q} \frac{dQ}{dx} \eta = 0, \]

where

\[ Q = \frac{c_0^2 \int_{-H}^{0} (c_0 - U_0) \left( \frac{d\Phi}{dz} \right)^2 dz}{c^2 \int_{-H}^{0} (c-U) \left( \frac{d\Phi}{dz} \right)^2 dz}. \]

The term \( Q \) represents the amplification factor in the linear long-wave theory due to variable depth and a horizontally variable stratification and current. The index “0” corresponds to a fixed horizontal location \( x_0 \), which may be taken to be the initial position of a solitary wave. After the change of variables

\[ s = \int \frac{dx}{c(x)} = t \quad \text{and} \quad \xi(x, s) = \frac{\eta(x, s)}{Q(x)}, \]

Eq. (8) can be reduced in the same approximation to

\[ \frac{\partial^2 \xi}{\partial x^2} + \left( \frac{\alpha Q}{c^2} \xi + \frac{\alpha_1 Q^2}{c^2} \xi^2 \right) \frac{\partial \xi}{\partial s} + \beta \frac{\partial^3 \xi}{\partial s^3} = 0. \]

This is the spatial version of the extended Korteweg–de Vries equation, and the “initial condition” for it corresponds to a time series of the wave displacement at the fixed point \( x_0 \). Equation (11) in various modifications (with/without cubic nonlinear term, with additional dissipative terms, and with/without taking into account the earth’s rotation) has been applied to the study of internal wave transformation in the coastal zone (Djordjevic and Redekopp 1978; Zhou and Grimshaw 1989; Holloway et al. 1997, 1999, 2001; Grimshaw et al. 1998, 1999; Liu and Chang 1998; Small 2001a,b; Pelinovsky et al. 2002; Cai et al. 2002).

3. Adiabatic transformation of a solitary wave

The main goal of this paper is to investigate the dynamics of internal solitary waves in a horizontally inhomogeneous ocean taking into account the real variability of the oceanic parameters. Before considering our numerical simulations of Eq. (11), we will discuss possible scenarios for wave transformation, using the asymptotic theory of a slowly varying solitary wave, in which the solitary wave locally maintains its soliton-like form.

First, let us briefly reproduce the well-known formula for the steady-state solitary wave solutions of Eq. (11)
for “frozen” coefficients (see, e.g., Grimshaw 2001). In its most general form, the soliton is described by
\[ \zeta = \frac{A}{1 + B \cosh[\gamma(s - \kappa z)]}, \]
where the parameters \( A, B, \kappa, \) and \( \gamma \) are given by
\[ A = \frac{6\beta \gamma^2}{\alpha c^2}, \quad \gamma = \frac{\alpha^2 c^2 \beta}{6 \alpha \kappa}, \quad \kappa = \frac{\beta \gamma^2}{c^3}. \]
The amplitude of the solitary wave is
\[ a = \frac{A}{1 + B} = \frac{\alpha}{Q \alpha_i} (B - 1). \]
Only one of the parameters in Eq. (12) is arbitrary, and the other parameters then depend on this one and the values of the coefficients of Eq. (11). The family of solitary waves described by Eq. (12) depends on the sign of the coefficient \( \alpha_i \) of the cubic nonlinear term in Eq. (11).

\[ a = \frac{\alpha}{Q \alpha_i} (B - 1). \]

\[ a_a = -\frac{2\alpha}{Q \alpha_i}. \]

Both branches of solitary waves in dimensionless form \( (c = \alpha = \alpha_i = \beta = 1) \) are presented in Fig. 1b. The algebraic soliton is marked by a dashed line.

For a variable oceanic background, the wave shape changes, and the wave parameters (amplitude, width, speed) vary. If the oceanic parameters vary sufficiently slowly (the actual condition will be discussed later), the wave is locally close to the solitary wave Eq. (12) but with a variable amplitude, width, and speed. The variation of the solitary wave parameters can be found from the conservation of wave action flux, sometimes referred to as momentum (or energy) conservation. This approach may be justified by the use of a multiscale asymptotic expansion (see, e.g., Grimshaw and Mitsudera 1993). The relevant conservation law can be easily derived from Eq. (11).
\[ \int_{-\infty}^{\infty} \xi^2 \, ds = \text{const.} \quad (17) \]

Using Eq. (12) it becomes

\[ \frac{A^2}{\gamma} \int_{-\infty}^{\infty} \frac{d\tau}{(1 + B \cosh \tau)^2} = \text{const}, \quad (18) \]

or, in terms of just one independent parameter \( B \),

\[ \frac{1}{Q^2 c} \sqrt{\frac{\alpha^2 \beta}{\alpha^*}} F(B)|B^2 - 1|^{3/2} = \text{const}, \quad (19) \]

where \( F(B) \) is the value of the integral in Eq. (18) and depends on the choice of the family of the soliton solutions.

For \( \alpha_1 < 0 \), the value of \( B \) lies between 0 and 1, and Eq. (19) reduces to

\[ \frac{1}{Q^2 c} \sqrt{\frac{\alpha^2 \beta}{\alpha^*}} \left( 2 \tanh^{-1} \frac{1 - B}{1 + B} - \sqrt{1 - B^2} \right) = \text{const}. \quad (20) \]

For \( \alpha_1 > 0 \) there are two branches of solitary waves. If the solitary wave polarity has the same sign as the coefficient of the quadratic nonlinear term, \( \alpha_1 \), the value of \( B > 1 \), and Eq. (19) takes the form

\[ \frac{1}{Q^2 c} \sqrt{\frac{\alpha^2 \beta}{\alpha^*}} \left( \sqrt{B^2 - 1} - 2 \tanh^{-1} \frac{B - 1}{B + 1} \right) = \text{const}. \quad (21) \]

If the solitary wave has the opposite polarity to the sign of \( \alpha_1 \), the value of \( B < -1 \), and Eq. (19) reduces to

\[ \frac{1}{Q^2 c} \sqrt{\frac{\alpha^2 \beta}{\alpha^*}} \left( \sqrt{B^2 - 1} + 2 \tanh^{-1} \frac{|B| + 1}{|B| - 1} \right) = \text{const}. \quad (22) \]
Using Eqs. (20)–(22) and (14), the solitary wave amplitude $a$ can be determined as a function of the oceanic parameters through the variation of $c$, $\alpha$, $\alpha_1$, $\beta$, and $Q$ calculated from the local hydrology and depth at each point of the wave path. In fact, the solitary wave changes adiabatically (retaining its soliton-like shape) only at the leading order of the asymptotic theory. At the next order the solitary wave radiates, generating a tail behind the solitary wave. A detailed investigation of the solitary wave tail structure in the framework of the variable-coefficient Korteweg–de Vries equation can be found in Grimshaw and Mitsudera (1993) and El and Grimshaw (2002). The nature of the shelves appearing behind the solitary wave (in particular, its polarity) can be found from the mass conservation law, derived from Eq. (11):

$$\int_{-\infty}^{\infty} \zeta \, ds = \text{const.}$$

(23)

It is easy to show that the calculation of the mass integral with the use of only the soliton solution Eq. (12) leads to a variable solitary wave mass, and, therefore, the corresponding mass deficit is associated with formation of a tail behind this variable solitary wave.

The slow variation of the relevant oceanic parameters (depth, shear flow, den stratification) is usually understood as slowness on the scale of the internal solitary wave.

Fig. 4. Transformation of a solitary wave with initial amplitude of 5 m across the NWS.
waves (cf. geometrical optics). It is used to neglect wave reflection and to derive a variable-coefficient nonlinear evolution equation for unidirectional waves. For an adiabatic transformation of a solitary wave this condition by itself is not enough, and the smallness of the last term in Eq. (8) is also required here to provide a weak perturbation to the solitary wave. Physically, this condition requires that the characteristic scale for the horizontal variability of the oceanic parameters should be large as compared with the characteristic scales of both nonlinearity and dispersion. Because of the present case of weak nonlinearity and dispersion, this condition is harder to fulfill than the usual conditions for the calculations of geometrical optics and similar systems. For the determination of the applicability of these adiabatic formulas to the processes of internal soliton transformation on real oceanic shelves, it is important to select all possible “nonadiabatic” factors even when the horizontal variability of the oceanic parameters may seem slow enough. First, this approximation fails for the table solitons that exist for negative values of the coefficient of the cubic nonlinear term. As one begins to come close
to the limiting amplitude, the soliton width greatly increases and becomes comparable to the characteristic scale of the oceanic horizontal variability. Thus the adiabatic approximation begins to break down as the solitary wave amplitude tends to the limiting value in Eq. (15). The same situation is realized for the opposite case of a very weak amplitude solitary wave (for either sign of the coefficient of the cubic nonlinear term); its width is large and comparable to the characteristic scale of the oceanic horizontal variability. The third situation appears for solitary waves with an amplitude close to that of the algebraic soliton Eq. (16); such a situation may be realized for positive values of the coefficient of the cubic nonlinear term. The algebraic soliton in the framework of the extended Korteweg–de Vries equation is unstable (Pelinovsky and Grimshaw 1997), and the soliton breaks down if its amplitude passes through the algebraic value in Eq. (16). Further, there will be possible failures in the adiabatic theory if the coefficients of nonlinearity or dispersion in the extended Korteweg–de Vries equation pass through a zero value. The dispersion parameter $\beta$ is always positive for stable flows [see Eq. (3)], but the coefficients of the quadratic and cubic nonlinear terms can be zero (independently) for certain conditions of the oceanic environment. In the unlikely event when both nonlinear coefficients become zero simultaneously, the soliton cannot exist (at least in the present model) and any disturbance evolves into a dispersive wave packet; in this case the soliton-like structure is destroyed. If the coefficient of the quadratic nonlinear term passes through zero, but not that of the cubic nonlinear term, the solitary wave is destroyed at the critical point if the cubic nonlinear term is negative (Grimshaw et al. 1999). Formally, the soliton amplitude tends to zero at the critical point according to the adiabatic Eq. (20). Secondary solitary waves of the opposite polarity can appear from the tail of the primary soliton, but the primary soliton itself disappears. The impossibility of an adiabatic transformation of a solitary wave in this case is illustrated in Fig. 2, where there are no connecting arrows between solitary waves in the lower half-plane (between quadrants III and IV). If the cubic nonlinear term is positive, an adiabatic transfer of the solitary wave is possible between quadrants I and II (upper plane in Fig. 2), and formally the soliton amplitude is determined at the critical point according to the adiabatic Eqs. (21) and (22). But if the soliton amplitude is close to the algebraic value, the adiabatic transformation is impossible, and the soliton-like structure is destroyed. If the coefficient of the cubic nonlinear term changes its sign, but not the coefficient of the quadratic nonlinear term (Nakoulima et al. 2004), the transformation of the solitary wave of the same polarity (negative for $\alpha < 0$ and positive for $\alpha > 0$) can be adiabatic (between quadrants I and III on the left half-plane or II and IV on the right half), except for the case in which the solitary wave amplitude is close to the amplitude of the table soliton. If the soliton in the upper half-plane (Fig. 2) has opposite polarity (positive for $\alpha > 0$ and negative for $\alpha < 0$), it is destroyed at the
critical point, when the cubic nonlinear term becomes zero. This simple classification will be used to analyze the data from our numerical simulations of internal solitary wave transformation on oceanic shelves.

4. Solitary wave transformation across ocean shelves

The formulas given above describe the adiabatic transformation of an internal solitary wave, when the wave maintains its soliton-like shape. However, oceanic shelves may contain wave paths along which the parameters do not vary sufficiently slowly, and they may also include several critical points. In this case an internal solitary wave transforms with loss of its identity as a soliton, and so we may say the soliton-like wave has a finite lifetime. To consider nonadiabatic effects of wave transformation, and to estimate the soliton lifetime, direct numerical simulations are performed using the variable-coefficient extended Korteweg–de Vries Eq. (11), with the coefficients calculated for three oceanic shelves: the NWS of Australia, the MSE (west of Scotland), and the Arctic shelf (in the Laptev Sea). Equation (11) is solved numerically using a finite-difference scheme with periodic boundary conditions (see Holloway et al. 1999) with the initial condition being a typical solitary wave for each shelf. In all simulations we use only the density stratification of the coastal zone and ignore any effect of background currents.

a. North West shelf of Australia

Nonlinear internal waves on the NWS of Australia have been investigated by Holloway over many years.
(Holloway 1987; Holloway et al. 1997, 1999). He made several measurements of the background stratification for this area. Here the observations obtained from a CTD survey carried out on the NWS in January 1995 (summer) are used. There were 11 CTD station data from the point 19.2°S, 115.7°E to the point 19.8°S, 116.5°E. At each of these 11 locations, ranging in depth from 416 to 66 m, a sequence of repeated profiles were measured every 30, 60, or 90 min, depending on the water depth, over a 13-h cycle. These profiles have been averaged at each location to remove the variability induced from the internal waves themselves, and the resulting CTD cross section is used to define the coefficients. The results from calculating the coefficients of the extended Korteweg–de Vries equation are presented in Fig. 3 (Holloway et al. 1999, 2001). It is necessary to note that the characteristic scale of the horizontal variability of the oceanic parameters (more than 10 km) exceeds the characteristic soliton wavelength (about 1–2 km), and the hydrology can indeed be considered as slowly varying. It is also clearly seen from Fig. 3 that both nonlinear coefficients change sign several times along the wave path.

Two simulations have been done for NWS conditions. The first run was for a solitary wave of initial amplitude 5 m, and the second run for an initial wave amplitude of 15 m. The initial solitary wave has negative polarity (because the coefficients of quadratic and cubic nonlinear terms are negative), and its amplitude is significantly less than the limiting value (250 m); hence the coefficient of cubic nonlinearity is negligible in the initial stage. The wave evolution at several distances from the initial point is presented in Figs. 4–5 (the time in all figures is the time in a shifted system of coordinates). The solitary wave maintains a soliton-like shape for a distance of about 40 km during which the influence of

![Fig. 9. Transformation of a solitary wave with initial amplitude of 21 m across the MSE.](image-url)
the cubic nonlinear term is weak. Consequently the two critical points where the coefficient of the cubic nonlinear term passes through zero do not lead to any change in the wave dynamics. At larger distances both nonlinear coefficients change sign and have non-monotonic behavior. At a distance 45 km the coefficient of the quadratic nonlinear term changes its sign and becomes positive. According to Fig. 2 the negative soliton from quadrant I in the vicinity of this location transfers to a negative soliton in quadrant II, and this process can be described by the adiabatic formulas (i.e., no change in the wave dynamics). However, the solitary wave that initially has a weak amplitude is close to the algebraic soliton wave in this quadrant, and their amplitudes are equal at a distance of 46 km. After this point the wave amplitude is less than the value for the algebraic soliton, and so such a wave cannot be a soliton and has to transform into a breather. This process is slow because the difference in amplitudes is weak; in particular, at a distance of 55 km they are equal again. Then the coefficient of the quadratic nonlinear term tends to zero (at about 60 km) and becomes negative for a short distance. In this case, the wave amplitude again exceeds the value for an algebraic soliton, thus supporting a soliton-like structure. As a result, there is no visible deformation in the soliton-like shape. At a distance of 68 km the coefficient of the cubic nonlinear term changes its sign and becomes negative. A soliton of negative polarity cannot transform from quadrant II to quadrant IV in Fig. 2, and so the soliton now disappears transforming into a dispersive wave packet. This process is clearly seen in Fig. 4.

The evolution of a solitary wave with an initially large amplitude follows the same scenario for a distance up to 45 km (where the sign of the quadratic nonlinear term changes from negative to positive). Now the wave amplitude significantly exceeds the algebraic value, and the solitary wave can develop adiabatically as a solitary wave of the modified Korteweg–de Vries equation [i.e., Eq. (11) with the quadratic nonlinear term omitted]. For a relatively short distance (from 42 to 48 km) the coefficient of the cubic nonlinear term grows approximately by a factor of 10, inducing a large amplification in the wave amplitude (a factor of 2–3). The tail behind the solitary wave forms mainly at this stage. It has a large mass and transforms into an isolated pulse (see Fig. 5). Such a transformation is typical for a variable large-amplitude solitary wave and was demonstrated previously in the case of soliton damping (Grimshaw et al. 2003). For greater distances, the coefficient of the cubic nonlinear term becomes negative, and both negative solitary-like waves disappear, forming dispersive wave packets (transfer from quadrant II to quadrant IV in Fig. 2). Because the dispersive tail is energetic enough, a group of solitary waves of positive polarity is generated from the tail; it is visible at a distance of 72 km (Fig. 5).

Thus a solitary wave of negative polarity for an initial
The transformation of a solitary wave with initial amplitude of 4 m across the Arctic shelf.

amplitude which may be either weak or strong, transforms at a distance after 68 km into a nonlinear dispersive tail and a group of secondary solitons (this latter case only for large-amplitude solitons). The appearance of a group of secondary waves at the depth of about 50 m in our simulations corresponds to the observations collected by Holloway (1987) and Holloway et al. (1999, 2001) at this depth; one or two single solitons were not observed on the NWS at this depth.

A comparison of the computed wave amplitude with the prediction of the adiabatic theory is illustrated by Fig. 6 for distances less than 50 km. At distances up to 42 km (the first critical point) the difference between the adiabatic theory and the numerical simulation is relatively small. The adiabatic theory for the soliton whose initial amplitude is weak is valid only for a distance of 46 km (this is related to transition to an algebraic soliton). For the soliton whose initial amplitude is large, there is no formal breakdown of the asymptotic theory; but the wave amplitude grows relative rapidly at distances about 45–50 km (predicted in the adiabatic theory, as well as in the numerical simulation). As is usual the wave amplitude in the direct numerical simulations grows more slowly than in the adiabatic theory. The wave amplitude at distances of more than 50 km becomes comparable with the total depth (130 m). For such a wave of great amplitude the applicability of the extended Korteweg-de Vries equation becomes questionable. Overall, we may conclude that internal solitary wave propagation on this selected region of the North West Australian Shelf (depth varies from 400 to 160 m) for a distance of about 50 km may be well described by the adiabatic theory in the framework of the variable-coefficient extended Korteweg-de Vries equation. The characteristic lifetime is estimated to be approximately 14 h. Of course, this time would possibly be increased if we extended our analysis to include propagation from the generation region through a homogeneous environment to the shoaling region, but data for this are not so readily available, and it is beyond the scope of this present work. This lifetime is comparable with the characteristic tidal cycle (12 h), and therefore groups of internal solitary waves observed on SAR images cannot be interpreted as absolutely independent groups.

It is pertinent to mention here the issue of the range of applicability of this weakly nonlinear theory, based
on the extended Korteweg–de Vries equation, since the observed internal solitary waves can have quite large amplitudes, particularly in the shallower water. A definitive answer to this question would need a detailed comparison of this theory with direct numerical simulations of the full Euler equations, or a detailed comparison with observations; this is beyond the scope of this present study [but see, e.g., Lamb and Yan (1996) for a numerical study of this issue, Michallet and Barthelemy (1998) for laboratory experiments, and Stanton and Ostrovsky (1998) for an observational study]. However, what we can say is that asymptotic consistency of the theory presented here requires, inter alia, that the nonlinear correction to the wave speed should be small. In our computations, the parameter $a_a/c$ varies from 0.05 (for an initial wave of amplitude 15 m) to 0.5 (at the farthest distance), while the parameter $a_1/a^2/c$ varies from 0.003 to 0.5 and hence can be considered as relatively small.

b. Malin shelf edge

Nonlinear internal waves on the MSE were the subject of an intense study during the Shelf Edge Study Acoustic Measurement Experiment (SESAME) of 1995 and 1996 (Small et al. 1999a,b; Hallock et al. 2000). Hydrological measurements were taken along 56.5°N latitude from the point 9.56°W (depth 1276 m) to the point 8.81°W (depth 149 m). The computed coefficients of the extended Korteweg–de Vries equation are shown in Fig. 7 (Pelinovsky et al. 1999). The coefficient of the quadratic nonlinear term is everywhere negative. There is only one critical point due to a change in the sign of the coefficient of the cubic nonlinear term at a distance of about 8 km. The initial solitary wave has negative polarity, as observed. According to the asymptotic theory this negative solitary wave can transform through this critical point adiabatically (from quadrant I to quadrant III in Fig. 2). The characteristic scale of the oceanic parameter variability (6 km) is larger then the solitary wavelength (1–2 km), and the horizontal variability can be considered slow. Nevertheless, the ratio between the wavelength and the characteristic scale of background variability for the MSE is not as small as for the NWS of Australia, and the relatively steep slopes of the parameter variability after the first 6 km may lead to non-adiabatic wave deformation across the shelf.

**Fig. 13.** The transformation of a solitary wave with initial amplitude of 13 m across the Arctic shelf.
Two runs were done for MSE, the first for a solitary wave with an initial amplitude of 9 m and the second one for an initial wave amplitude of 21 m. The wave profiles across the shelf are shown in Figs. 8 and 9. For a distance of 10–15 km the influence of the cubic nonlinear term is weak, and the solitary wave transformation can be described in the framework of the Korteweg–de Vries equation. The coefficient of the quadratic nonlinear term is approximately constant for this distance, but the dispersion parameter decreases significantly. The adiabatic theory gives a good prediction for the wave amplitude for a distance up to 6 km (Fig. 10), and then the balance between nonlinearity and dispersion is destroyed because the oceanic parameters do not vary slowly enough. This leads to the transformation of the soliton-like wave into a shocklike wave with a steep front. After 25 km the depth is constant and the shock-like wave evolves into a set of solitary waves. This process is visible for both initial wave amplitudes. The coefficient of the cubic nonlinear term reaches its maximum (in modulus), and the limiting value of the soliton amplitude at this stage is about 32 m. If the initial amplitude is 9 m, the wave amplitude does not reach this limiting value, and the solitons can be considered as Korteweg–de Vries solitons (Fig. 8). However, the soliton with an initially large amplitude transforms and reaches the limiting value, where the influence of the cubic nonlinear term becomes important. In this case the group of secondary solitons appears quite early.

The results presented here demonstrate that the soliton-like shape seems is maintained for a distance of about 20 km. This distance corresponds to a lifetime of the solitary wave of approximately 10 h. As discussed above, the significant decrease of the dispersion parameter leads to the formation of a shock wave, and such waves were observed during the SESAME (Small et al. 1999b). The evolution of such borelike waves for distances more than 25 km can be described in the framework of the extended Korteweg–de Vries equation with constant coefficients. This equation was used to explain the observed transformation of the borelike disturbance into solitary waves, and the results of simulations and observations are in reasonable agreement (Small et al. 1999b; Pelinovsky et al. 1999). As for the NWS, here the nonlinearity of the computed wave can also be considered as weak, since the parameter $a_0/c$ varies from 0.15 (for an initial wave amplitude 21 m) to 0.5 (at the end of the distance), and the parameter $a_1 a_0^2/c$ varies from 0.05 to 0.5.

c. Arctic shelf

Internal waves in the Arctic shelves are now being intensively studied (Sandven and Johannessen 1987; Konyaev et al. 1996; Dokken et al. 2001; Pelinovsky et al. 2002). Figure 11 demonstrates the variability of the coefficients of the extended Korteweg–de Vries equation for the shelf of the Laptev Sea (Talipova et al. 2003). They are calculated using hydrological data obtained from the cruise of R/V Ivan Kireev (the meridian 127.5°E from latitude 75.4°–74°N) in the summer of 1993. The depth decreases by almost a factor of 2 over a distance of 160 km and is quite slow. The linear speed of wave propagation decreases only slightly from 0.84 to 0.7 m s$^{-1}$, and this leads to only a small variation of the linear amplification ratio $Q$ (10%). Meanwhile, the nonlinear coefficients, as well as the dispersion parameter, vary significantly, but also quite slowly. The coefficient of the cubic nonlinear term is everywhere negative, and increases by a factor of 3. The characteristic wavelength is only about 20–30 m, and this shelf is thus an excellent example for the application of the adiabatic theory; hence, we expect that the wave should maintain its soliton-like shape. Taking into account that the coefficient of the quadratic term is negative everywhere (except in the last 5 km) only a solitary wave of negative polarity may exist. The critical point (a zero value of the coefficient of the quadratic nonlinear term) appears at the end of the wave path at a distance of 155 km. Adiabatic transformation of the wave at this critical point (from quadrant III to quadrant IV in Fig. 2) is not allowed, and so the solitary wave should be destroyed. Hence, here we may expect that the adiabatic approximation will be valid for the transformation of a solitary wave for a large distance, excepting only the last 20 km.

Two solitary waves with initial amplitudes of 4 and 13 m are chosen for study. The results of our numerical simulation are shown in Figs. 12 and 13. It is evident that solitary waves of both amplitudes hold the soliton-
like shape up to the critical point. The adiabatic formulas for the solitary wave amplitude agree well with the computed results (Fig. 14). It is interesting to note that here the relative nonlinear correction to the wave speed \( \alpha_a/c \) decreases with distance, and at the end of the distance is 0.1 (for initial amplitude 4 m) and 0.4 (for initial amplitude 13 m). The cubic nonlinear correction \( \alpha_a a^2/c \) under the same conditions is about 0.5. This indicates that the solitary wave may maintain its soliton-like shape for very long distances (140 km) if the background varies very sufficiently slowly and does not include any critical points. The characteristic predicted lifetime of a solitary wave on the Arctic shelf is about 50 h. However, in this case the lifetime may be affected by other factors—for instance, by energy dissipation (see, e.g., Grimshaw et al. 2003).

It is interesting to note that the structure of the wave amplitude curve is different here for different amplitudes. For instance, a solitary wave of small amplitude grows slightly for a distance of 79 km, but a solitary wave of large amplitude falls in this range to an amplitude of 10 m. This may be explained by the significant contribution of the cubic nonlinear term. Thus the limiting value for the soliton amplitude varies with distance; it is 30 m at the beginning of the wave path and only 11.8 m at the point 79 km. The large-amplitude solitary wave must vary between these limits. In the vicinity of the critical point the solitary wave amplitude decreases as is predicted by the asymptotic theory.

5. Transformation and interaction of two solitary waves

The results described so far demonstrate the complicated dynamics of a single solitary wave in a horizontally inhomogeneous ocean containing critical points. An important feature of the adiabatic deformation of an internal solitary wave is the inevitable generation of a
trailing tail. If this tail is long enough and large enough, it may influence the propagation of any solitary waves behind the main wave. A full treatment of this issue would require us to consider the propagation of a modulated nonlinear periodic wave packet through a horizontally inhomogeneous ocean. This is beyond the scope of the present study. Instead, we briefly describe here the transformation of two solitary waves across two typical oceanic shelves, taken here to be the NWS of Australia and the Arctic shelf. Thus we replace the previous initial condition of a single solitary wave by an initial condition that consists of two widely separated solitary waves. In the case when the background is homogeneous so that all the coefficients in Eq. (11) are constants, the separation distance is sufficient to ensure that each solitary wave propagate essentially independently of the other wave. Of course, since in this case the governing equation is integrable there would be an elastic collision between the waves, albeit very small, in which there would be a slight adjustment in amplitudes and positions.

The first example for the NWS demonstrates the transformation of two solitary waves of the same amplitude of 15 m generated with an initial shift of almost 2 h (Fig. 15). Up to the first critical point, there is no observable effect of the first soliton on the second soliton. But, after the first critical point (at a distance of 45 km), when the coefficient of the quadratic nonlinear term becomes positive, large tails form behind both solitary waves. Between the distances of 50 and 55 km, the second solitary wave interacts with the tail behind the first solitary wave. Because of the positive polarity of the tail, the amplitude of the second wave decreases, and the time series at a distance of 52 km can be interpreted as a group of three solitons typical for the evolution of a borelike disturbance. Here it is due to the unsteady process of a solitary wave interaction with the tail of the preceding solitary wave, and this ambiguity of wave images may lead to difficulties in interpretation of observed wave data. After this interaction, the solitary waves again have the almost the same amplitudes, and the result of any inelastic interaction is rather weak.

The second example is for the Arctic shelf (Fig. 16). The solitary waves now propagate for a long distance up to 100 km with no interaction because of the very slow variation of the oceanic parameters in this case. However, the wave evolution is not described by the adiabatic scenarios because as the waves pass the critical point (155 km) the solitary wave structure is destroyed. The tail formed just before the critical point has opposite polarity to the wave, as predicted by theory (see, e.g., Grimshaw et al. 1998, 1999). The second solitary wave, interacting with the positive tail of the first wave, now increases in amplitude.

6. Conclusions

In this paper, we have studied the dynamics of internal solitary wave dynamics on oceanic shelves using the framework of the variable-coefficient extended Korteweg-de Vries equation. For a slowly varying background (water depth, horizontal variability of the density, and current stratification) theoretical formulas for the wave amplitude are derived in explicit form, based on the adiabatic theory of a slowly varying solitary wave. The possible breakdown of the asymptotic approach related to the presence of critical points (i.e., zero values of the
coefficients of quadratic or cubic nonlinear terms) is discussed. Comprehensive numerical simulations have been described for the NWS of Australia, the MSE, and the Arctic shelf (Laptev Sea). They demonstrate the regimes of applicability of the adiabatic theory, the wave dynamics at the critical points, and also the situation when the oceanic parameters do not vary sufficiently slowly. The results of our numerical simulations show that an internal solitary wave maintains a soliton like shape for various distances, ranging from 20 km for the MSE to 140 km for the Laptev Sea in the Arctic Ocean. The corresponding characteristic lifetime of internal solitary waves due to the effect of a horizontally inhomogeneous background may vary from 10 to 50 h, which is comparable with the internal tide cycle (12 h).

Acknowledgments. This paper is dedicated to the memory of Peter Holloway, whose influential work on internal solitary waves through theory, numerical simulations, and observations has stimulated our own work in this area. This work is supported particularly by the London Mathematical Society through Loughborough University for EP and TT, INTAS (01-0025) and RFBR (03-05-64978) for TT, RFBR (03-05-64426) for AK, and an ONR grant for RG.

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