Wind Spatial Variability and Topographic Wave Frequency

ELAD SHILO
Department of Soil and Water Sciences, Hebrew University of Jerusalem, Rehovot, Israel

YOSEF ASHKENAZY
Department of Solar Energy and Environmental Physics, Jacob Blaustein Institutes for Desert Research, Ben-Gurion University of the Negev, Sede Boqer Campus, Sede Boqer, Israel

ALON RIMMER
Yigal Alon Kinneret Limnological Laboratory, Oceanographic and Limnological Research, Migdal, Israel

SHMUEL ASSOULINE
Department of Environmental Physics and Irrigation, Institute of Soils, Water and Environment Sciences, Agricultural Research Organization, Volcani Center, Bet Dagan, Israel

YITZHAQ MAHRER
Department of Soil and Water Sciences, Hebrew University of Jerusalem, Rehovot, Israel

(Manuscript received 26 July 2007, in final form 25 January 2008)

ABSTRACT

The association of topographic waves with wind action has been documented in several natural lakes throughout the world. However, the influence of the wind’s spatial variability (wind stress curl) on the frequency of topographic waves has only been partially investigated. Here the role of wind stress curl on the frequency of topographic waves in an idealized elliptic paraboloid basin has been studied both analytically and numerically. It is shown that the analytical solution is the sum of an elliptic rotation determined by the wind stress curl and two counterrotating circulation cells, which propagate cyclonically after the wind ceases. Furthermore, it is shown that cyclonic elliptical rotation (associated with positive wind stress curl) increases the rotation frequency of the double-gyre pattern while anticyclonic elliptical rotation (associated with negative wind stress curl) decreases the oscillatory mode frequency. It is also shown that bottom friction has some effect on the structure of the double-gyre pattern but hardly affects the oscillatory frequency. Numerical solutions of the depth-integrated nonlinear shallow-water equations confirmed that the frequency of the topographic wave increases (decreases) when forcing the model with cyclonic (anticyclonic) wind curl.

1. Introduction

A sudden addition of momentum to a variable-depth lake might result in the generation of topographic (vorticity) waves, also referred to as second-class motions. The underlying mechanism of these waves is the conservation of potential vorticity associated with the presence of variable water depth. This kind of motion does not significantly disturb the free surface, and typical frequencies are considerably lower than the local Coriolis parameter \( f \) (the time period is on the order of several days). The lower modes of the second-class motion of a liquid contained in an elliptic paraboloid were investigated by Ball (1965): the first nontrivial solution was steady elliptical rotation in which the current runs along the depth contours (isobaths) of the basin. The first oscillatory mode consisted of two opposite-
circulation cells and rotated in a cyclonic manner with a time period of several days for the special case of a circular basin. Another, higher-order oscillatory mode consisted of four circulation cells separated by the principal axes of the elliptical basin (Ball 1965). Among the oscillatory modes of the second-class motion, only the first oscillatory mode has been associated with the low-frequency current oscillations observed in lakes and enclosed basins around the world, such as Lake Ontario (Csanady 1976), southern Lake Michigan (Saylor et al. 1980), the Gulf of Riga, and the Baltic Sea (Raudsepp et al. 2003).

The frequency of the first oscillatory mode of topographic waves has been studied in detail for the last four decades. For example, Csanady (1976) used an idealized lake model to link several flow-reversal episodes observed in Lake Ontario to the appearances of barotropic topographic waves. The estimated time period for the free wave propagation was 12–16 days. Saylor et al. (1980) observed low-frequency oscillatory motions in the southern basin of Lake Michigan. The current vector rotated cyclonically at the center of the basin and anticyclonically elsewhere, and had a time period of about 4 days. Solutions of a homogeneous linear vorticity equation indicated that the wave frequency is sensitive to the lake’s shape and bathymetry. More specifically, using either a circular or an elliptical basin with a parabolic depth profile yielded a time period of about 123 h, which did not agree well with that observed (90 h). Changing the depth profile to a conical shape yielded a time period of 88 h, which is much closer to the observed value. Mysak (1985a,b) developed analytical methods to calculate the topographic wave frequency. The description of the effect of depth profile on the topographic wave frequency was more detailed than those developed by Ball (1965) and Saylor et al. (1980); Mysak (1985a,b) demonstrated good agreement between the predicted and observed low-frequency currents in lakes such as Lake Lugano, Lake Zurich, and Lake Michigan.

Despite the detailed investigations of topographic waves (e.g., Csanady 1976; Saylor et al. 1980; Mysak 1985a,b), the effect of external forcing on their characteristics has received little attention. The effects of wind stress and bottom friction on topographic waves were addressed by Huang and Saylor (1982, hereafter HSS2), in their analysis of the topographic (vorticity) wave dynamics in the southern basin of Lake Michigan. Interaction of a free vortex mode with a forced mode was suggested as the mechanism for the generation of topographic waves with a time period of close to 4 days, as previously reported by Saylor et al. (1980). By assuming that the scale of atmospheric systems is generally much larger than the lake basin’s dimensions, the effect of the wind stress curl was considered to be small relative to the terms affected by the wind stress itself (HS82); the frequency of the forced mode, therefore, depended on the wind frequency, and resonance coupling between the wind field and the basin response could only be achieved if the wind happened to force the lake at or near the natural wave frequency. Bottom friction was found to have a negligible effect on the frequency of the basin-scale topographic wave.

Numerical experiments have also been performed to simulate the low-frequency barotropic oscillations in lakes (Schwab 1983) and enclosed seas [e.g., Gulf of Riga, Baltic Sea; see Raudsepp et al. (2003)]. The results of the numerical experiments performed by Shilo et al. (2007) indicated that the presence of a curl in the wind field during the forcing period prior to the free oscillatory period might modify the frequency of the free-topographic wave in a closed barotropic lake. The physical basis behind the observation of modified topographic wave frequency in the presence of a wind stress curl was that the curl of the wind field leads to the formation of a single circulation cell in the water body (constant vorticity, elliptic rotation). After the relaxation of the wind, a basin-scale free-topographic wave (with two circulating cells) is initiated, the frequency of which is affected by the steady elliptical rotation. The main goal of this study was to investigate the role of wind stress curl on the frequency of topographic waves. We show that positive wind stress curl leads to an increase in the rotation frequency of the two circulation cells while a negative wind stress curl leads to a decrease in their rotation frequency.

In section 2, we present the linear vorticity equation used to study the topographic waves. Analytical solutions for this equation are provided and discussed in section 3. We also examine the effect of bottom friction on the topographic wave’s magnitude and frequency. Then, we investigate the interaction between the elliptical rotation and the oscillatory mode in section 4. Numerical solutions of the nonlinear shallow-water model are provided and discussed in section 5, and section 6 concludes.

2. Formulation

We consider a basin with an elliptical periphery and parabolic depth profile filled with homogeneous fluid; the Coriolis parameter is assumed to be constant (f...
plane). By neglecting the horizontal friction terms, the motion of the fluid is governed by the depth-integrated shallow-water equations [for more details see Ball (1963, 1965) and HS82]:

$$\begin{align*}
\frac{DU}{Dt} - fu &= -g \frac{\partial}{\partial x} (H + Z) + \frac{1}{H} \left( \tau_x^* - ru \right), \\
\frac{DV}{Dt} + fu &= -g \frac{\partial}{\partial y} (H + Z) + \frac{1}{H} \left( \tau_y^* - rv \right), \\
\frac{\partial H}{\partial t} + \frac{\partial}{\partial x} (Hu) + \frac{\partial}{\partial y} (Hv) &= 0,
\end{align*}$$

where $u$ and $v$ are the eastward and northward horizontal velocities and $f = 2\Omega \sin \phi$ is the local Coriolis parameter, with $\Omega$ being the angular velocity of the earth and $\phi$ the latitude. Here, $H$ is the depth of the fluid (from the free surface to the bottom, $H > 0$), $Z$ is the basin bathymetry usually measured from $Z = 0$ ($Z < 0$), $Z + H$ is the free surface height (see Fig. 1), $g$ is the gravity acceleration, $\tau_x^*$ and $\tau_y^*$ are the wind stress components in the $x$ (eastward) and $y$ (northward) directions, respectively, and $\rho_w$ is the constant water density. The bottom stress components $ru$ and $rv$ describe the bottom friction, where $r$ is the bottom drag coefficient with dimensions of velocity.

Following Ball (1965), we switch to dimensionless quantities; for this purpose, we use a horizontal length scale $L$, a vertical depth scale $M$, and a time scale $f^{-1}$. The new dimensionless quantities $u^*, v^*, x^*, y^*, z^*, t^*, \tau^*, r^*$ are given by

$$t = tf^{-1},$$

$$x, y = L(x^*, y^*),$$

$$u, v = Lf(u^*, v^*),$$

$$H, Z = M(h^*, z^*),$$

$$\tau = \rho_w Lf^2 M \tau^*,$$

$$r = fMr^*.$$  \hspace{1cm} (4)

By defining

$$\varepsilon = \frac{f^2 L^2}{gM^2},$$

Eqs. (1)–(3) become

$$\begin{align*}
\varepsilon \left[ \frac{DU}{Dt} - v \right] &= -\frac{\partial}{\partial x} (h + z) + \frac{1}{h} (\tau_x^* - ru), \\
\varepsilon \left[ \frac{DV}{Dt} + u \right] &= -\frac{\partial}{\partial y} (h + z) + \frac{1}{h} (\tau_y^* - rv), \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (hu) + \frac{\partial}{\partial y} (hv) &= 0.
\end{align*}$$

For convenience, the asterisks have been omitted.

For most of world’s lakes, $\varepsilon \ll 1$ (e.g., Lake Kinneret in northern Israel has $L = 2 \times 10^4$ m, $f = 10^{-17} s^{-1}$, $M = 40$, leading to $\varepsilon \approx 0.01$), such that the variables $u, v,$ and $h$ can be expanded in terms of $\varepsilon$. By using the scaling [Eq. (4)], we were able to write the equation of motion using a single parameter $\varepsilon$ that depends on known constants; this scaling approach was introduced by Ball (1965). Note that we assume here that the wind stress and bottom friction terms are sufficiently small compared to the pressure term.

By setting $\varepsilon = 0$ in Eqs. (6)–(8), we obtain the zero-order approximation for which the free surface is a rigid lid

$$h_0 + z = 0.$$  \hspace{1cm} (9)

In the continuity equation [Eq. (8)], we now replace $h_0$ by $-z$ and get

$$\frac{\partial}{\partial x} (zu_0) + \frac{\partial}{\partial y} (zu_0) = 0.$$  \hspace{1cm} (10)

Using Eq. (9), it is possible to find $u_0$ and $v_0$ by considering terms of the order of $\varepsilon$ in Eqs. (6) and (7). After some manipulations, we obtain the following vorticity equation:

$$\begin{align*}
\frac{D}{Dt} &\left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + 1 \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \\
&= - \left[ \frac{\partial}{\partial x} \left( \tau_x^* - \frac{\tau_y^*}{z} \right) \right] + \frac{\partial}{\partial y} \left( \frac{\tau_y^*}{z} \right) + \frac{\partial}{\partial x} \left( \frac{\tau_x^*}{z} \right) + \frac{\partial}{\partial y} \left( \frac{\tau_y^*}{z} \right).
\end{align*}$$

where the zero subscript has been omitted. In Eq. (11) we assume that there is no horizontal friction and the bottom friction is linear. In addition, there is no normal flow at the edges of the basin. Wind stresses $\tau_x^*$, $\tau_y^*$ are applied on the water surface.

The velocities in Eq. (11) may be expressed in terms of the modified “streamfunction” suggested by Ball (1965):
where \( z \) is the depth configuration of the basin measured from the free surface at rest, not the vertical coordinate.

We now define \( z \) to describe the shape of an elliptical paraboloid (following Ball, 1965; HS82) where \( a \) is the shape parameter:

\[
z = 0.5[(1 - a)x^2 + (1 + a)y^2] - 1 = \delta - 1 - 1 < a < 1, \tag{13}
\]

The streamfunction defined in Eq. (12) automatically satisfies the continuity equation [Eq. (10)]. The normal flow boundary condition at the coastline is also satisfied because \( \Psi = 0 \) when \( z = 0 \). In addition, \( zu \) and \( zv \) are the depth-integrated currents and thus \( \Psi \) stands for the depth-integrated streamfunction.

Although Eq. (14) is separable, analytical solutions might be complicated because of the presence of the depth profile \( z \) in the denominator of several terms. Here, we restrict the solution to be close to the center of the basin where \( z = -1 \) (and thus \( 1/z \sim -1 \) and \( 1/z^2 \sim -1 \)); HS82 reached the same approximation using first-order terms of the binomial expansion of \( z^{-1} \). Under this assumption, Eq. (14) becomes

\[
\left( \frac{\partial}{\partial t} + r \right) \left\{ 4\psi + z^2\psi + 3 \left[ (1 - a)x \frac{\partial \psi}{\partial x} + (1 + a)y \frac{\partial \psi}{\partial y} \right] \right\} + (1 - a)x \frac{\partial \psi}{\partial y} - (1 + a)y \frac{\partial \psi}{\partial x} = \frac{1}{z^2} \left[(1 - a)x^2\right]. \tag{14a}
\]

Because we are focusing our attention on the dynamics near the basin center, solutions of Eq. (14a) may be not valid in shallow areas near the shore; nevertheless, it has been shown that topographic wave activity, which is our main concern here, is absent in shallow areas (Raudsepp et al. 2003).

### 3. Solutions of the forced vorticity equation

There are two terms in Eq. (14a) that are related to the wind stress forcing. The first term on the right-hand side is proportional to the wind stress while the second term is the wind stress curl. Because we are dealing with a linear problem, the streamfunction can be expressed as

\[
\psi = \psi_{\text{curl}} + \psi_{\text{stress}} + \psi_f,
\]

where \( \psi_{\text{curl}} \) represents the part of the solution that is associated with the wind stress curl, \( \psi_{\text{stress}} \) represents the part of the solution that is associated with the effect of the wind stress itself, and \( \psi_f \) represents the solution to the homogeneous part of Eq. (14), with only bottom friction.

We now return to Eq. (14a) and for simplicity assume that the wind stress acts along the east–west axis, inde-
where for \( z \) the boundary condition \( \Psi_{\text{curl}} = 0 \) is satisfied.

Solution (15a) is a particular solution of Eq. (14a), indicating that a curl in the wind field results in an elliptical rotation (zero mode) as a forced response. The streamfunction (15b) for the zero mode is depicted in Fig. 2. It is a single elliptical circulation cell with constant (cyclonic) vorticity. The currents follow the depth contours and the direction of the flow is solely determined by the sign of the wind stress curl: a positive wind stress curl leads to cyclonic (elliptical) rotation while a negative wind stress curl leads to anticyclonic rotation. The magnitude of the flow is determined by the ratio between the magnitude of the wind stress curl and the bottom friction coefficient. Obviously, without friction \( (r = 0) \), there is no steady state and a solution does not exist.

Note that solution (15a) also solves the time-dependent form of Eq. (14a) under the conditions discussed here.

b. Response to the action of steady wind stress

We now seek a (particular) solution for Eq. (14a), under the action of wind stress alone, when there is no wind shear (i.e., \( \tau^x = \tau_0^x + k_1 y; \tau^y = 0; k_1 \ll \tau_0^x \)).

Similar to the separation method chosen by Ball (1965), we assume a simple form of the solution \( \psi = C_1 x + C_2 y \), where \( C_1 \) and \( C_2 \) are constants given by

\[
C_1 = \frac{(1 - a^2)}{4r^2(9 - a^2) + (1 - a^2)^2} \tau_0^x,
\]

\[
C_2 = -\frac{2r(1 + a)(3 - a)}{4r^2(9 - a^2) + (1 - a^2)^2} \tau_0^x,
\]

and the streamfunction is

\[
\Psi_{\text{stress}} = z^2 \psi = z^2(C_1 x + C_2 y).
\]

This streamfunction is depicted in Fig. 3a; wind stress forcing resulted in the formation of two counterrotating circulation cells (cyclonic and anticyclonic). This is essentially the topographic double-gyre mode, which is well known in the relevant scientific literature as the steady barotropic response of a variable-depth basin driven by uniform wind (e.g., Rao and Murty 1970; Csanady 1973). The return flow generally opposes the direction of the wind stress, and tends in a cyclonic manner, which is determined by the effect of bottom friction on the flow. Reducing the bottom friction coefficient will cause the return flow to tend more in a cyclonic manner, as part of balancing the vorticity input by the wind field (Birchfield 1972). In the case of negligible bottom friction, the return flow will be 90° (cyclonically) relative to the wind direction, while in the case of very strong friction the return flow will be in the direction opposite to that of the wind.

As noted above, both particular solutions [Eqs. (15b) and (16b)] also satisfy the time-dependent vorticity equation. Figure 3b depicts the streamlines received from the sum of those solutions (\( \Psi = \Psi_{\text{curl}} + \Psi_{\text{stress}} \)). The effect of the wind stress curl (cyclonic in this spe-
specific example) on the double-gyre pattern appears to enhance the gyre that rotates in the same direction as the wind curl, a result that is also well documented in limnological studies. Rao and Murty (1970) demonstrated this effect by numerically solving a steady circulation model. A similar model was applied by Serruya et al. (1984), where it was demonstrated that realistic values of wind stress curl may be sufficient to generate a single (cyclonic) circulation cell in both Lake Kinneret and Lake Constance. Endoh et al. (1995) showed that the presence of a negative wind stress curl leads to the formation of double-gyre circulation in Lake Biwa, with a dominant anticyclonic gyre. In Lake Michigan, the effect of positive wind stress curl was found to be primarily responsible for the dominant cyclonic winter circulation (Schwab and Beletsky 2003).

c. Time-dependent solution of the homogeneous vorticity equation

Let us now seek the solution for Eq. (14a) without the presence of external forcing ($r_0$, $k_1 = 0$).

Following Ball (1965), we consider a solution of the form

$$\psi = \psi_{10}(t)x + \psi_{01}(t)y.$$  \hspace{1cm} (17)

Substituting Eq. (17) into Eq. (14) we obtain two linear homogeneous ordinary equations; solving for the time-dependent coefficients we get

$$\psi_{10}(t) = e^{-t/9} \left[ A \cos(\omega t) - \frac{4arA + (1-a)(7+3a)B}{\omega(49 - 9a^2)} \sin(\omega t) \right].$$

$$\psi_{01}(t) = e^{-t/9} \left[ \frac{4arB + (1+a)(7-3a)A}{\omega(49 - 9a^2)} \sin(\omega t) + B \cos(\omega t) \right],$$

where the time decay constant $\gamma$, and the topographic wave frequency $\omega$ are

$$\gamma = \frac{6r(7-a^2)}{(49 - 9a^2)},$$

$$\omega^2 = \frac{49 + 9a^4 - 2a^2(29 + 8r^2)}{(49 - 9a^2)^2}.$$  \hspace{1cm} (17a)

When $a = 0$ and $r = 0$, we obtain the homogeneous topographic wave frequency (circular basin) derived by Ball (1965): $\omega = 1/7$.

Reasonable values for the bottom drag coefficient $r$ were provided by Simons (1980); $r$ does not exceed $10^{-3}$ m s$^{-1}$. Thus, the values of the nondimensional drag coefficient $r$ are less than or equal to 1/4. Moreover, the bottom friction affects the frequency only when the basin is far from being circular (e.g., $a > 0.5$). For example, for $r = 1/4$ and $a = 0.5$, $\omega = 0.126$ 21, while for $r = 0$ and $a = 0.5$, $\omega = 0.126$ 66, which is a change of less than half of a percent. Thus, we can safely claim that the effect of bottom friction on the topographic wave frequency is negligible.

The time-dependent streamfunction is

$$\Psi(t) = z^2[\psi_{10}(t)x + \psi_{01}(t)y].$$  \hspace{1cm} (17b)

In Figs. 4a–d, we present the streamfunction evolution at $t = 0$, $t = T/8$, $t = T/4$, and $t = 3T/8$, where $T$ is the time period of the topographic wave. The countercirculating cells rotate cyclonically around the basin perimeter with frequency $\omega$, which is given in Eq. (17a).

The solution to the nonhomogeneous Eq. (14a) is the
sum of the particular solutions of Eqs. (15b) and (16b), and the homogeneous solution of Eq. (17b):

\[ \Psi(x, y, t) = \Psi_{\text{curl}}(x, y) + \Psi_{\text{stress}}(x, y) + \Psi_t(x, y, t). \]

Higher-order modes may be computed by considering that the solution \( \psi \) is a higher-order polynomial. Ball (1965) demonstrated that when \( \psi \) is a quadratic function of \( x \) and \( y \), the streamlines form a four-cell rotational current pattern. This resulted in zero flow at the center of the basin. As mentioned above (see the introduction), only the first oscillatory mode was observed in several enclosed basins.

4. The effect of wind stress curl on the lowest-mode topographic wave

We showed above that the wind stress curl generates a single circulation cell (an elliptical rotation with steady flow pattern along the depth contours, as depicted in Fig. 2) with constant vorticity, also called the zero-mode solution. We also discussed the cases of uniform wind stress and the case of topographic wave solution when no wind forcing is applied. We shall now examine the interaction between the zero mode (steady elliptical rotation) associated with the wind stress curl and the lowest oscillatory topographic wave mode; for this purpose, we will follow Ball’s (1965) analysis. Let us consider the adjustment process of water contained in an elliptic paraboloid basin immediately after the relaxation of an east-to-west wind field associated with linear north-to-south horizontal shear. The wind stress curl action will result in the formation of a single circulation cell, while the action of the wind stress itself will lead to the formation of a double-gyre pattern; the topographic wave will start to rotate after the relaxation of the wind when only the free dynamic is present. The elliptical rotation solution [Eqs. (15a) and (15b)] is an exact solution of the homogeneous form of the nonlinear equation [Eq. (11)] without the effect of bottom friction (Ball 1965), and we now consider perturbation from a state of elliptical rotation with constant vorticity. The solution for the steady elliptical rotation is

Fig. 4. Streamlines of the time-dependent homogenous solution (oscillatory mode). Two counterrotating circulation cells propagating cyclonically as a free-topographic wave are depicted at (a) \( t = 0 \), (b) \( t = T/8 \), (c) \( t = T/4 \), and (d) \( t = 3T/8 \); \( T \) is the time period.
\[ u = -2\psi_{00}(1 + a)y, \]
\[ v = 2\psi_{00}(1 - a)x, \]

where \( \psi_{00} \) is the constant associated with the steady rotation.

Accordingly, we define the velocities to be
\[ u = \bar{u}(y) + u(x, y, t'), \]
\[ v = \bar{v}(x) + v(x, y, t'), \]
and substitute the velocities [Eq. (18)] into Eq. (11). The presence of the zero-mode rotation in the basin is represented by the time-independent part, which is an elliptical rotation with a constant vorticity, say, \( \Lambda \):

\[
\frac{\partial}{\partial t} \left\{ 4\psi + 3 \left[ (1 - a)x \frac{\partial \psi}{\partial x} + (1 + a)y \frac{\partial \psi}{\partial y} \right] \right\} + \bar{u}(7 - 3a) \frac{\partial \psi}{\partial x} + \bar{v}(7 + 3a) \frac{\partial \psi}{\partial y} + (1 + \Lambda) \left[ (1 - a)x \frac{\partial \psi}{\partial y} - (1 + a)y \frac{\partial \psi}{\partial x} \right] = 0.
\]  

(19a)

The solution for this homogeneous equation can be obtained by assuming the same form of Eq. (17), namely, \( \psi = \psi_{00}(t)x + \psi_{0i}(t)y \) [for this reason, spatial derivatives higher than the first derivative were excluded in Eq. (19a)]. After solving Eq. (19a), we obtain a modified frequency \( \omega_0^2 \) given by

\[
\omega_0^2 = \omega_0^2 \left[ \frac{2}{3} \frac{\Lambda(3 + a) + 1}{\Lambda(3 - a) + 1} \right]^{3/2}. \]

(19b)

Equation (19b) provides a direct relationship between the wind stress curl and the frequency of the lowest-mode topographic wave. From this particular solution [Eqs. (15a) and (15b)], we see that the amount of constant vorticity \( \Lambda \) is directly determined by the presence of a curl in the wind stress field. For those values of \( \Lambda \) that lead to a positive right-hand side of Eq. (19b), the solution exhibits the same properties as the oscillatory mode [Eq. (17b)]. Therefore, during the forcing period, the wind stress itself generates a steady double-gyre pattern while the wind stress curl leads to a steady single elliptical circulating cell, which together with the double-gyre pattern causes one cell to be bigger than the other. After the wind relaxes, a free wave is developed in which the elliptical circulation interacts with the evolution of the double-gyre pattern and changes its propagation frequency. Figure 5 shows that a constant vorticity (elliptical rotation), within the range of stability, causes a shift in the frequency of the freely propagating topographic wave to either higher or lower values, depending on the sign of the wind stress curl.

It is possible to gain some information regarding the effect of wind stress curl on topographic wave dynamics from the studies of Saylor et al. (1980), HS82, and Schwab (1983), all of whom analyzed the same dataset (i.e., Lake Michigan’s southern basin, from April to November 1976). Saylor et al.’s (1980) observations showed low-frequency current oscillations in Lake Michigan’s southern basin with a time period of 90 h, corresponding to the basin-scale topographic wave. However, the analysis using an elliptical paraboloid yielded a time period of 123 h. HS82 suggested two possible mechanisms for closing this gap. In addition to the one described earlier (the resonant forced mode), it was suggested that there is an existing amount of constant cyclonic vorticity in the water body in the zero-mode form that is increasing the frequency and thus reducing the time period of the wave to one more closely approximating that observed. However, the effect of the wind stress curl was neglected in their calculations (see the introduction), preventing a direct relation between the wind curl and the shift in frequency. Schwab (1983) solved the barotropic vorticity equation numerically to study the large-scale circulation in Lake Michigan for the period of April–November 1976. The model was forced with an interpolated wind stress field, which was found to maintain a net cyclonic curl. The low-frequency part of the average kinetics energy spectra showed peaks between 102 and 89 h, in much better agreement than the results from using the analytical solution derived from the homogeneous vorticity equation. Additionally, it was indicated that the average response of the lake to periodic low frequency is nonresonant. It is worth noting that in the presence of density stratification, the (subinertial) frequency of barotropic shelf waves increases (Huthnance 1978). This may pro-
provide an alternative explanation for the decreasing time period of the wave. However, Schwab (1983) achieved good agreement with observations while excluding baroclinic effects from his simulation. In a recent paper (Shilo et al. 2007), we demonstrated two cases of the negative and positive shift in topographic wave frequency in homogenous Lake Kinneret resulting from the presence of a shear in the wind field.

5. Numerical solution of the nonlinear shallow-water model

Thus far, we have demonstrated the impact of the wind stress curl on the frequency of topographic waves in an idealized basin using approximated analytical solutions. Solving the nonlinear shallow-water equations [Eqs. (1)–(3)] numerically will allow us to validate our results in the absence of the previous restrictions, that is, (a) the free surface is not necessarily a rigid lid, (b) the Rossby number is not required to be small, and (c) the idealized basin bathymetry now conforms to the case of a real lake (e.g., the depth approaches zero at the coast). Furthermore, it allows us to bridge the gap between the analytical solutions and the realistic simulations performed by Shilo et al. (2007). Equations (1)–(3) were solved numerically using a state-of-the-art oceanic circulation model [the Regional Ocean Model System (ROMS); see Shchepetkin and McWilliams (2005)]. The response of homogeneous water in an elliptic paraboloid was investigated under the action of the following three theoretical wind-forcing fields: I, a spatially uniform easterly wind; II, an easterly wind with anticyclonic curl (i.e., negative wind stress curl where the wind magnitude decreases from south to north); and III, an easterly wind with cyclonic curl (i.e., positive wind curl where the wind magnitude decreases from north to south).

ROMS is a free-surface, hydrostatic, finite-difference, primitive-equation model that uses orthogonal curvilinear coordinates in the horizontal direction. In the vertical direction, the primitive equations are discretized over variable topography using stretched terrain-following coordinates (Song and Haidvogel 1994). An analytical grid was used for this application, composed of 36 × 58 grid cells, each 400 × 400 m² in size, covering an area of about 14 km × 23 km. The analytical expression [Eq. (13)] describing the elliptic paraboloid basin was used (a = −0.5) to provide the bathymetry data for the model grid.

The minimum nearshore depth was 1 m, and the maximal depth was 40 m, which was representative of Lake Kinneret, in northern Israel (see Shilo et al. 2007). This model was configured to solve the barotropic response on an elliptic paraboloid using the nonlinear, depth-integrated equations. In contrast to the analytical calculations, the horizontal diffusion terms were added.

![Fig. 5. Approximate time period (days) of the topographic wave propagation as the function of the magnitude and the direction of a present elliptic rotation.](image)
to provide numerical stability. The background value of the horizontal diffusion was $5 \text{ m}^2 \text{s}^{-1}$. For the depth-integrated currents, a bottom linear friction law was assumed with a drag coefficient of $10^{-3}$ (a value of $10^{-2}$ was also tested). The components of the currents normal to the boundary were set to zero, and free-slip conditions were imposed for the lateral boundaries. In each simulation, the forcing field was applied during the first 24 h and then the model was allowed to run to 192 h (8 days). The model was forced using wind stress values that were calculated analytically using a linear relation of the form $\tau_x(y) = a(y/L) + b$, $\tau_y = 0$, where $\tau_x$, $\tau_y$ are the east–west and north–south components of the wind stress, respectively, and $a$ and $b$ are constants. For the case of positive wind curl, $a = -0.043 \text{ N m}^{-2}$, $b = -0.02 \text{ N m}^{-2}$, while for the case of negative curl, $a = -0.02 \text{ N m}^{-2}$, $b = -0.043 \text{ N m}^{-2}$. The resultant average wind magnitude is $4.3 \text{ m s}^{-1}$, and the wind stress curl is $\pm 1 \times 10^{-6} \text{ N m}^{-3}$, similar to the values estimated for Lake Kinneret (northern Israel) by Shilo et al. (2007). The magnitude of $L (~10^4 \text{ m})$ was taken as being representative of Lake Kinneret.

The depth-integrated currents for cases I, II, and III are compared in Fig. 6. The formation of a double-gyre pattern and its cyclonic propagation as the lowest-mode topographic wave is present in all cases. The addition of anticyclonic wind curl leads to a slight enhancement of the anticyclonic gyre located in the southern portion of the basin (see top of Fig. 6). In addition, the propagation rate of the free-topographic wave is clearly slower than in the spatially uniform wind-driven case (case I), as demonstrated in Fig. 6. After 5 days of simulation, the case I pattern rotated about 1/8 cycle ahead relative to the anticyclonic wind-formed gyres. In contrast, the presence of a cyclonic curl in the wind field (case III) resulted in a faster propagation of the double-gyre pattern relative to the homogeneous case (case I). The difference is clearly depicted in Fig. 6 (center and bottom, respectively); the double-gyre pattern that was formed by a cyclonic wind curl propagates about 1/8–1/4 cycle ahead relative to the homogeneous one (see Fig. 6, right-most parts of the center and bottom).

To demonstrate the cyclonic rotation of the current vectors more clearly, the simulated currents for all
three cases were taken from the basin center and progressive vector diagrams were plotted for the entire simulated period of 8 days (Fig. 7). For the reference case (no curl), the current vector completed about 1-1/8 cycles. For the case imposing a cyclonic rotation (using positive wind stress curl) the current vector completed roughly 1-1/2 cycles, while for the anticyclonic rotation case (negative wind stress curl) the current vector rotated about one cycle during the entire period (close to 8 days). The effect of bottom friction is also indicated as a decrease in the curvature radii via a reduction in current magnitude for all three cases.

To gain a more quantitative evaluation of the simulations results, the frequency and time period of the current vector rotation (at the center of the basin) were calculated for cases I–III above. The resultant time periods were 7.7, 7.3, and 5.6 days for the anticyclonic, uniform, and cyclonic wind stress fields, respectively. Below we compare these periods to the ones predicted by Eq. (19b). For this purpose we had to estimate the vorticity $\Lambda$ “injected” by the wind as follows: We first extended solution (15) to be time dependent but spatially independent $\psi_{00} = \psi_{00}(t)$ and obtained $\psi_{00}(t) = (-k/4r)(1 - e^{-rt})$. We then converted the values of $r$, $k$, and $t$ of the numerical experiment to their nondimensional values and used the fact that $\Lambda = 4\psi_{00}$. The estimated nondimensional value for $\Lambda$ was $\pm 0.026$ for the cyclonic and anticyclonic cases. Accordingly, the time periods (days) were 7.8, 7, and 6.2 days for the anticyclonic, uniform, and cyclonic wind curl, respectively [in Eq. (19b), $a = -0.5$]. Using larger wind stress curl values in the numerical simulations (e.g., $2 \times 10^{-6}$ N m$^{-3}$, $3 \times 10^{-6}$ N m$^{-3}$) resulted in an increase of the time period of the anticyclonic case, and reduced the time period of the cyclonic case (results not shown) in agreement with Eq. (19b) (see Fig. 5).

Although these numerical simulations bear similarity to the analytical results, there are still significant differences between the two. It is clear, for example, that in all three cases of the numerical simulations, one of the gyres was bigger than the other. Moreover, the double-gyre pattern was not completely maintained throughout the simulation. These differences may be attributed to the series of assumptions that were made in order to enable the analytical solution. For example, in the analytical solution we restricted ourselves to the dynamics close to the center of the lake, and also neglected the nonlinear terms and those involving higher-order derivatives.

6. Summary and conclusions

We studied the effect of wind field on the dynamics of topographic waves in an enclosed basin with a variable-depth profile. An approximated (linear) vorticity equation was developed and solved analytically, and the complete solution consisted of three terms. The first particular solution was the steady elliptical rotation caused by a constant wind stress curl. The direction of the rotation (cyclonic/anticyclonic) was determined solely by the (positive/negative) sign of the curl of the wind stress. The magnitude of the flow was approximated by the ratio between the wind curl and the bottom drag coefficient. The second particular solution was the formation of two steady counterrotating circulation cells resulting from the effect of the wind stress field. A counterclockwise gyre is formed to the right of the wind and a clockwise gyre is formed to the left of the wind. Both of these solutions also solve the time-dependent vorticity equation. The homogeneous solution was found to be the lowest oscillatory mode topographic wave—a double-gyre pattern that cyclonically propagates around the basin perimeter.

The formation of an elliptical rotation and its interaction with the higher-mode double-gyre pattern resulted in several aspects of the flow. The first was the enhancement of the gyre that rotates in the same manner as the elliptical rotation. The second was the shifting in frequency of the topographic wave to higher or lower values depending on the sign of the wind curl.

Instabilities in the flow field might occur for a certain range of anticyclonic elliptical rotation magnitudes. This issue, however, is not discussed in detail in the present paper. Numerical simulations, solving the depth-integrated, nonlinear shallow-water model...
showed similar results as follows: a cyclonic wind stress curl causes faster propagation (shorter time period) of the free-topographic wave, while the imposition of an anticyclonic curl causes slower propagation (longer time period) of the free wave than in the case of a uniform wind. Time periods were calculated from the simple analytical model and were in reasonable agreement with those calculated from the numerical simulations. The present study is based on the findings of Shilo et al. (2007) who studied the dynamics of a shallow homogenous lake (Lake Kinneret, northern Israel). A combination of both observations and model results (wind fields, water currents) indicated that the wind curl affect the topographic wave frequency. However, the lake is shallow and the wave does not complete even one full cycle per event [the same is reported for the Gulf of Riga, Baltic Sea; see Raudsepp et al. (2003)]. Hence, we cannot provide a complete set of observational evidence to strengthen our findings here. Observations that span several topographic wave periods can be obtained from an enclosed, relatively deep basin such as Lake Michigan (see Saylor et al. 1980) or Lake Ontario, which is also covered with enough regular monitoring observations to provide wind properties. Our theoretical predictions are compared successfully with state of the art numerical model.

The bottom line of the present study is intuitive: a positive wind stress curl “assists” the cyclonic rotation of the double-gyre pattern and thus increases the rotation frequency, while a negative wind stress curl “opposes” the cyclonic rotation of the double-gyre pattern and slows down its rotation frequency. Despite this simple and intuitive mechanism, changes in topographic wave frequency resulting from wind stress curl have received only scant attention.

Acknowledgments. The authors thank Hezi Gildor, Eyal Heifetz, Zvi Källbermann, and Yair Zarmi for fruitful discussions. We also thank David Bromwich for kindly providing us with computer facilities and two anonymous reviewers for helpful comments. YA is supported by the Israeli Center for Complexity Sciences. This research was supported by The Israel Science Foundation (Grant 1319/05) and by the Admiral Yohay Ben-Nun Foundation for Marine and Freshwater Sciences, Israel Oceanographic and Limnological Research, Ltd.

REFERENCES