Diagnosing the Southern Ocean Overturning from Tracer Fields

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ABSTRACT

The strength and structure of the Southern Hemisphere meridional overturning circulation (SMOC) is related to the along-isopycnal and vertical mixing coefficients by analyzing tracer and density fields from a hydrographic climatology. The meridional transport of Upper Circumpolar Deep Water (UCDW) across the Antarctic Circumpolar Current (ACC) is expressed in terms of the along-isopycnal (K) and diapycnal (D) tracer diffusivities and in terms of the along-isopycnal potential vorticity mixing coefficient (KPV). Uniform along-isopycnal (<600 m²s⁻¹) and low vertical mixing (10⁻² < D < 10⁻¹ m²s⁻¹) can maintain a southward transport of less than 60 Sv (Sv = 10⁶ m³s⁻¹) of UCDW across the ACC, which is distributed largely across the South Pacific and east Indian Ocean basins. For vertical mixing rates of O(10⁻⁴ m²s⁻¹) or greater, the inferred transport is significantly enhanced. The transports inferred from both tracer and density distributions suggest a ratio K/D of O(2-3) x 10⁶ particularly on deeper layers of UCDW. Given the range of observed southward transports of UCDW, it is found that K = 300 ± 150 m²s⁻¹ and D = 10⁻⁴ ± 0.5 x 10⁻⁴ m²s⁻¹ in the Southern Ocean interior. A view of the SMOC is revealed where dense waters are converted to lighter waters not only at the ocean surface, but also on depths below that of the mixed layer with vertical mixing playing an important role.

1. Introduction

Oceanographers debate how dense waters, formed at high latitudes, are returned as lighter waters to the ocean surface, completing the meridional overturning circulation. Many argue that dense waters are upwelled through density layers across the abyssal ocean, requiring small-scale mixing processes such as energy dissipation over rough topography. Others argue that dense waters are transported along sloping isopycnals to the outcropping regions of the Antarctic Circumpolar Current (ACC) where vigorous winds of the Southern Hemisphere provide the energy required to convert dense water to light [see Kuhlbrodt et al. (2007), for a comprehensive review].

The Southern Ocean links the three major ocean basins and it is there that many water masses are either formed or modified (Sverdrup et al. 1942). The ACC is a zonal current, circulating around Antarctica, with a transport of 134 ± 13 Sv (Sv = 10⁶ m³s⁻¹) as measured through the Drake Passage (Whitworth 1983; Whitworth and Peterson 1985). The ACC is the dominant dynamical feature of the Southern Ocean. In contrast, the Southern Hemisphere meridional overturning circulation (SMOC) is estimated to involve between 20 and 50 Sv of exchange between density classes over the entire circumpolar extent of the Southern Ocean. This exchange is thought to involve an upper and lower branch. In the upper branch, Upper Circumpolar Deep Water (UCDW) is converted to northward flowing Subantarctic mode and Antarctic Intermediate Waters. In the lower branch, UCDW and Lower Circumpolar Deep Water (LCDW) are converted to northward flowing bottom waters (Sloyan and Rintoul 2001).

Although it is not integral to the analysis, we choose to equate the overturning SMOC to the southward transport of UCDW. It is UCDW that feeds both the upper and lower limbs of the SMOC. We provide evidence that
a southward transport of 20–50 Sv of UCDW into ACC (Sloyan and Rintoul 2001; Lumpkin and Speer 2007) is consistent with observed mixing rates. Both transformations above the mixed layer and vertical mixing in the ocean interior play important roles in determining the strength of the UCDW transport, and hence the SMOC.

The absence of land barrier(s) at latitudes and depths of Drake Passage (around 55°–60°S, and 0–1800 m respectively) denies the possibility of a mean geostrophic velocity across the ACC in depth or pressure coordinates. The SMOC in this region consists of a northward Ekman transport at the surface, as a result of the strong eastward wind stress and an eddy flux resulting from correlations between the thickness of isopycnal layers and the geostrophic flow. Only below 1800 m can the mean geostrophic flow contribute to the SMOC. Considering the overturning in density space, geostrophic flow across the ACC can contribute through both temporal and spatial correlations of the geostrophic velocity with the thickness of isopycnal layers—that is, in density space, both transient and standing eddies contribute to the overturning circulation. It is shown in section 4 that the effect of standing eddies can be neglected only in a small range of densities where a contour of constant potential vorticity (PV), on an isopycnal, runs along the entirety of the ACC (PV = f/h, where f is Coriolis frequency and h is thickness).

Unlike the ACC transport, which can be measured directly, the transport of UCDW can only be estimated using inverse methods and other indirect approaches. Inverse modeling has been used to estimate the southward transport of UCDW across 30°–40°S by Lumpkin and Speer (2007) and Sloyan and Rintoul (2001). They infer 20 and 52 Sv of UCDW, respectively, and find that it feeds both the upper and lower branches of the SMOC. The difference in the estimates lies in the a priori constraints and mixing representations used in the inverse models. Karsten and Marshall (2002) and Speer et al. (2000) estimate the rate of upwelling across the ACC by determining the surface Ekman buoyancy and eddy flux components in a residual-mean framework. They infer a surface divergence, and hence a rate of upwelling of water masses into the mixed layer. Karsten and Marshall (2002) project the inferred upwelling down to depth using a simple vertical advective–diffusive balance (assuming a certain vertical diffusivity D). This method is applied to the Antarctic Intermediate Water layers only (i.e., the upper branch of the SMOC). Assumptions must be made about the upwelling at a particular mean streamline corresponding to a particular density layer, as contours of sea surface density do not follow streamlines along which the divergence is computed. Olbers and Visbeck (2005) investigate the relationship between Ekman transport, eddy fluxes, and vertical mixing in the Southern Ocean. They apply an a priori estimate of the meridional transport of UCDW and Antarctic Intermediate Water and infer a thickness diffusivity. The thickness diffusivity diagnosed accounts for both eddy variability and large-scale standing eddies and their solution is likely to be sensitive to their description of the Ekman velocity and their zonal averaging.

The transport of UCDW can be related to the along-isopycnal and vertical mixing coefficients through the temperature and salinity fields. Along the ACC there exist strong meridional temperature and salinity gradients on isopycnals. More precisely, density layers are cooler and fresher at the outcropping regions to the south and become warmer and saltier to the north. For these gradients to exist in steady state, there must be a balance between advection, transporting heat, and salt up or down the tracer gradient on isopycnals and the effects of both along-isopycnal and vertical mixing. Along-isopycnal mixing (K) acts to mix tracer anomalies on the isopycnal, while vertical mixing destroys or enhances anomalies by transferring temperature and salinity across isopycnals. This advective–diffusive balance is evident from observed tracer distributions.

In this study we determine the transport and spatial structure of UCDW as a function of the vertical and along-isopycnal mixing coefficients using the advective–diffusive balance described earlier (see section 3). As in Zika and McDougall (2008), the advective–diffusive balance is applied by integrating along temperature contours on isopycnal layers.

Using established parameterizations for the bolus flux (i.e., the difference between the mean and thickness-weighted average flow in isopycnal coordinates), we show the dependence of the UCDW transport on the along-isopycnal thickness or potential vorticity mixing coefficient (section 4). The upwelling across isopycnals along the ACC in terms of a vertical advective–diffusive balance is also considered (section 5).

It is shown that below approximately 500 m the ratio of the mean along-isopycnal mixing coefficient K to the mean vertical mixing coefficient D is O(2 × 10^6) (section 6). In section 6, the ratio of K to D is also derived by applying conservation of volume to each layer, reaffirming a value of O(2 × 10^6).

Section 7 contains a comparison and discussion of these results with previous theoretical and numerical studies of the SMOC. The consistency of a low along-isopycnal to vertical diffusivity ratio in the Southern Ocean is discussed in the context of coarse-resolution numerical models.

Here conservative temperature (Θ) is used and is proportional to potential enthalpy, and represents the “heat
content” per unit mass of seawater (McDougall 2003)—that is, where potential temperature (θ) would commonly be used as a conservative variable for heat, we use Θ, as it is equivalent to θ while being far more conservative. Note that the distinction between conservative temperature and potential temperature and neutral and potential density is not central to this paper. We will frequently refer to Θ as temperature and neutral density layers as isopycnals.

2. Water mass equation and cross-contour flow

Consider the idealized scenario where no mixing or diffusive processes occur in the ocean. In such an ocean, the path taken by a parcel of water with salinity S, conservative temperature Θ, and neutral density (γ) is simply the path where S, Θ, and γ are constant. Currents in such an ocean would closely follow contours of constant temperature and salinity on isopycnals. It is clear that the amount by which seawater chooses not to follow such a path—that is, the flow across temperature contours on isopycnals [(∇h · n₀)T], and vertically through density surfaces (wQ)—is determined purely by the magnitude of vertical and along-isopycnal mixing processes. Here, v is the absolute 2D velocity, Vγ is the gradient on the isopycnal, n₀ is the unit vector down the along-isopycnal temperature gradient (n₀ = Vγ/|Vγ|), and h is the vertical distance between closely spaced neutral density γn surfaces. The equation describing the balance between the cross-contour flow (vh · n₀) and mixing process in a steady ocean is

\[
\frac{\nabla h \cdot n_0}{h} = K \left[ \frac{\nabla_\gamma (\nabla \gamma S) \Theta_z - \nabla_\gamma (\nabla_\gamma \Theta) S_z}{h (\nabla \gamma S) \Theta_z - (\nabla_\gamma \Theta) S_z} \right] + \frac{D}{\nabla^2} \left[ \frac{S_z \Theta_z - \Theta_z S_z}{\nabla_\gamma S \Theta_z - (\nabla_\gamma \Theta) S_z} \right] = \frac{K}{\lambda^\perp} + \frac{D}{\lambda^\perp}.
\]

McDougall (1984) first derived (1) (in a slightly different form) and described it as the “Water Mass Transformation” equation (see appendix A for a detailed derivation). Cases where the along-isopycnal gradient of K is a significant term in (1) are not considered in this study.

Here, (1) represents a balance between the advection down a temperature gradient on an isopycnal and both along-isopycnal and vertical mixing. This downgradient advection can be thought of as the “nonadiabatic” component of the along-isopycnal flow. It is important to recognize that (1) does not involve the diapycnal velocity component wQ, vertical differences in D, or individual second derivatives of tracers in z. Instead, vertical mixing appears in (1) through the Θ-S curvature (see appendix A), a quantity less sensitive to noise in hydrographic data than Szz and Θzz individually. In (1), λγ and λγ are diffusive “scale lengths.” Note that although a singularity exists in (1) when VγΘ = 0, no contour exists either.

Ignoring, for a moment, the consequences of the nonlinear equation of state, and ignoring the thickness gradient, the first term on the right-hand side of (1) represents the ratio of along-isopycnal curvature of temperature to the along-isopycnal temperature gradient ∇γ2Θ/|VγΘ| (appendix A). As in the 1D vertical balance of “Abyssal recipes” (Munk 1966) where the ratio of the vertical gradient of temperature to the vertical curvature of temperature (and more accurately density) dictates the ratio of diapycnal advection to vertical mixing, similarly here the ratio for tracers Θ and S along-isopycnals dictates the ratio of cross-contour flow (vh · n₀) to along-isopycnal mixing K. One way of understanding this balance is to consider an isopycnal with an along-isopycnal temperature gradient VγΘ and curvature Vγ2Θ (Fig. 1). Along-isopycnal mixing acts to smooth out the curvature of temperature and if the curvature is to remain in steady state there must be either up or down-gradient advection to maintain it.

The second term on the right-hand side of (1) is proportional to the vertical curvature of temperature and salinity d2S/dθ2. It is not simply Θzz or Szz that affects the balance on the isopycnal, but the vertical curvature that involves both Θzz and Szz. Vertical mixing (D) acts to smooth out the Θ-S curvature. In order for it to be maintained, there must be cross-contour advection (vh · n₀) or along-isopycnal mixing (K) (Fig. 2). Equation (1) allows each of these effects to be quantified. It also includes the effect of a thickness gradient, as well as nonlinear effects as a result of cabbeling and thermobaricity.

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3. The Southern Ocean overturning

By integrating along contours of constant temperature and salinity in layers bounded by density surfaces, we can relate the total isopycnal transport across the ACC to both $K$ and $D$. The total thickness-weighted volume flux across such a contour between a pair of density surfaces provides an estimate of the meridional transport. Integrating (1) yields

$$
\int (\mathbf{v}_h \cdot \mathbf{n}_\Theta) \, dx_\Theta = \int \left( \frac{K}{\Theta^2} \right) \mathbf{h} \, dx_\Theta + \int \left( \frac{D}{\Theta} \right) \mathbf{h} \, dx_\Theta,
$$

(2)

where $x_\Theta$ is oriented along a contour of constant $\Theta$ (which is also a contour of constant salinity as it is on an isopycnal). To apply (2) to the Southern Ocean, we define circumpolar tracer contours of constant temperature from the World Ocean Circulation Experiment (WOCE) Hydrographic Atlas (Orsi and Whitworth 2004) compiled as a gridded climatology on neutral density layers (Gouretski and Koltermann 2004; Jackett and McDougall 1997). Each layer represents an interval of $\gamma_n = 0.1$ kg m$^{-3}$ (i.e., the $\gamma_n = 27.6$ kg m$^{-3}$ layer is between neutral density surfaces $\gamma_n = 27.55$ kg m$^{-3}$ and $\gamma_n = 27.65$ kg m$^{-3}$). In each layer between neutral densities of $\gamma_n = 27$ kg m$^{-3}$ and $\gamma_n = 28$ kg m$^{-3}$ three contours are chosen corresponding to a northern, central, and southern contour of the ACC (Fig. 3). Isopycnals above $\gamma_n = 27$ kg m$^{-3}$ outcrop and temperature contours on isopycnals below $\gamma_n = 28$ kg m$^{-3}$ are interrupted by topography.

UCDW, which is characterized by low oxygen concentration, is sandwiched between overlying fresher and higher oxygen concentrated Antarctic surface water (AASW) and the underlying salinity maximum and higher oxygen concentration of LCDW. Reviewing maps of oxygen and salinity from the WOCE Atlas we define UCDW to be between $\gamma_n = 27.4$ kg m$^{-3}$ and $\gamma_n = 28$ kg m$^{-3}$. Both Lumpkin and Speer (2007) and Sloyan and Rintoul (2001) also use this range to define UCDW.

Fields of the vertical and along-isopycnal tracer gradients and curvatures are determined from the WOCE climatology. The along-isopycnal mixing and vertical mixing terms in (1) are linearly dependent on $K$ and $D$, respectively. Using (1), we estimate the total meridional transport on particular density layers for various values of the along-isopycnal and vertical tracer diffusivities.

![Fig. 2. A $\Theta$–$S$ curvature exists down the water column (solid line). Vertical mixing (curved arrows) acts to smooth this curvature. Temperature and salinity must be advected by $\mathbf{v}_h \cdot \mathbf{n}_\Theta$ (solid arrows) along isopycnals to maintain the curvature in steady state.](image)

![Fig. 3. (a) Color map of conservative temperature (°C) along the ACC on $\gamma_n = 27.7$ kg m$^{-3}$ with positions of the northern, central, and southern contours shown (dashed lines). (b) Temperature and salinity of northern (red), central (green), and southern contours (blue) whose extent is fully circumpolar between neutral densities $\gamma_n = 27.2$ kg m$^{-3}$ and $\gamma_n = 28$ kg m$^{-3}$](image)
We consider the case where $K = 200 \text{ m}^2 \text{s}^{-1}$ and $D = 2 \times 10^{-4} \text{ m}^2 \text{s}^{-1}$. Fluxes across the northernmost temperature contours of the ACC are northward for AASW and mostly southward for UCDW (Fig. 4). The level of zero cross-contour flow (i.e., the level where the flow is neither to the south nor to the north) is at approximately $\gamma_n = 27.5 \text{ kg m}^{-3}$. Transports closer to the center of the ACC show a similar structure to the northern contour, albeit the level of zero cross-contour flow moves to denser layers. Both the vertical mixing and along-isopycnal mixing terms change sign from southward to northward on $\gamma_n = 27.6 \text{ kg m}^{-3}$ across the ACC. This convergence suggests that UCDW feeds the upper and lower limbs of the SMO between these contours.

The cumulative integral of (2) along a circumpolar path of each temperature contour summed over layers from $\gamma_n = 27.4 \text{ kg m}^{-3}$ to $\gamma_n = 28 \text{ kg m}^{-3}$ gives the spatial variation in the meridional transport of UCDW (Fig. 5). Both the along-isopycnal mixing and diapycnal mixing components of the overturning circulation vary smoothly, giving confidence that the use of second derivatives of the tracer fields is not particularly noisy, however, this may also relate to the smoothing applied to hydrographic data in order to produce the Atlas. For the northernmost contour, the two components are mostly negative (southward) and vary in a similar way along the contour, suggesting that warm anomalies are advected southward and both vertical and along-isopycnal mixing act to mix them across and along isopycnals, respectively. At the southernmost contours, the magnitude of the vertical mixing term is much larger than the along-isopycnal mixing term, suggesting that either vertical mixing dominates the balance or $K$ is large relative to $D$.

![FIG. 4. Contributions to layer cross-contour transport from the along-isopycnal mixing term (taking $K$ to be 200 m$^2$ s$^{-1}$; black bars) and the vertical mixing terms (taking $D$ to be $2 \times 10^{-4}$ m$^2$ s$^{-1}$; white bars). Transports are across the (a) southern, (b) central, and (c) northern contours of the ACC. The temperature and salinity of each contour is marked with a circle in Fig. 3b. Positive values are with increasing temperature (northward for the layers shown).](image-url)

![FIG. 5. Terms contributing to the cumulative cross-contour transport between $\gamma_n = 27.4 \text{ kg m}^{-3}$ and $\gamma_n = 28 \text{ kg m}^{-3}$ due to the along-isopycnal mixing term ($K = 200 \text{ m}^2 \text{s}^{-1}$; solid line) and the vertical mixing term ($D = 2 \times 10^{-4} \text{ m}^2 \text{s}^{-1}$; dot-dashed line). Transport across the (a) southern, (b) central, and (c) northern contours of the ACC. Positive values are with increasing temperature (northward for the layers shown).](image-url)
We sum the vertical and along isopycnal mixing terms in (2), again from $\gamma_n = 27.4$ kg m$^{-3}$ to $\gamma_n = 28$ kg m$^{-3}$ into the northern side of the ACC from the sum of terms in (2) for various values of $K$ and $D$. Gray shading across the center of the figure represents abyssal estimates of $D$ from both Munk (1966) and Ledwell et al. (1993). Below the $x$ axis, colored bars show relevant estimates of lateral or along-isopycnal diffusivities from McKeague et al. (2005) (black bar — $K_y$; gray bar — $K_x$), Phillips and Rintoul (2000) (red bar), Gille (2003) (green bar), and Naveira-Garabato et al. (2007) (blue bar). UCDW transport estimates across hydrographic sections at 30°—40°S are shown from Lumpkin and Speer (2007) (20 Sv; green contour) and Sloyan and Rintoul (2001) (52 Sv; red contour). Blue lines represent estimates of the ratio $K/D$ below 27.7 $\gamma_n$, made in section 6, plus or minus one standard deviation. Taking the spread between the green and red contours to be a reasonable error range for the UCDW transport, the predicted range for the circumpolarly averaged mixing coefficients is crosshatched in light blue.

In the Southern Ocean, estimates exist for a surface eddy diffusivity from satellite observations (Marshall et al. 2006) and float measurements have been used to calculate eddy kinetic energy and eddy diffusivity. Recognizing the various estimates that range from less than $O(100$ m$^2$ s$^{-1}$) to $O(10,000$ m$^2$ s$^{-1}$) is difficult, as there are likely to be large differences between buoyancy diffusivities and tracer or potential vorticity diffusivities (see Smith and Marshall 2009). In addition, the grid spacing of inverse models and coarse-resolution ocean models can play a large role in determining the estimated or required diffusivity. Phillips and Rintoul (2000) estimate the lateral diffusivity ($K_{xy} = \nabla^2 \theta_i / \nabla \theta_i$; $z$ being a constant depth surface) of temperature from a mooring array time series of velocity and temperature placed within the ACC near 50°S, 143°E. Their estimates are in the broad range 100—1000 m$^2$ s$^{-1}$ for this one geographical location.
(500–1000 m$^2$ s$^{-1}$ above 500-m depth and 100–500 m$^2$ s$^{-1}$ below; Dr. H. Phillips 2008, personal communication). Gille (2003) estimated eddy heat fluxes in the Southern Ocean using Autonomous Lagrangian Circulation Explorer (ALACE) floats and found the lateral mixing coefficient to be between 300 and 600 m$^2$ s$^{-1}$ (at around 900-m depth).

Estimates from McKeague et al. (2005), based on an inverse model of the ocean circulation on $\gamma_n = 28$ kg m$^{-3}$ in the South Atlantic, are relevant to our study as they considered the along-isopycnal mixing of tracers, including temperature and salinity, assuming steady state. They find a meridional diffusivity $K^\nu = 100 \pm 50$ m$^2$ s$^{-1}$ and zonal $K^\zeta = 750 \pm 100$ m$^2$ s$^{-1}$. As temperature contours are close to lines of constant latitude, the meridional diffusivity is perhaps the most relevant here. However, as the authors suggest, the difference in magnitude may relate to the difference in grid sizing, which again makes interpretation of the eddy diffusivity difficult. Naveira-Garabato et al. (2007) were able to estimate the along-isopycnal mixing coefficient for a passive tracer in the southeast Pacific and southwest Atlantic Oceans along $\gamma_n = 27.98$ kg m$^{-3}$. They measure an along-isopycnal diffusivity of $360 \pm 330$ m$^2$ s$^{-1}$ in the frontal regions of the ACC and an area average of $1860 \pm 440$ m$^2$ s$^{-1}$, which is thought to be associated with intensification of eddy-driven mixing in the Scotia Sea relative to ACC-mean conditions. The range 100–1000 m$^2$ s$^{-1}$ is consistent with that used by coarse-resolution models, higher diffusivities leading to unrealistic ACC transports.

In this study, estimates of vertical and along-isopycnal mixing may be used to infer the southward transport of UCDW. Lumpkin and Speer (2007) and Sloyan and Rintoul (2001), both determine the southward transport of UCDW with an inverse model, inferring 20 and 52 Sv, respectively. Direct comparison of our estimates with those of Lumpkin and Speer (2007) and Sloyan and Rintoul (2001) is not exact, as the transport diagnosed from (2) is across the northern flank of the ACC meandering close to 52.5°S, whereas the inverse estimates are calculated for hydrographic sections between 30° and 40°S.

At the limit where vertical mixing $D$ is zero, an overturning circulation of $O(20–50$ Sv) would require an along-isopycnal diffusivity of about 200–500 m$^2$ s$^{-1}$. At this limit, the overturning circulation is driven by Ekman and eddy transport close to the surface—that is, UCDW flows to the south in the presence of along-isopycnal mixing only until it reaches the mixed layer. It is worth noting, however, that the zero vertical mixing case is only possible for the upper branch of the SMOC where UCDW is transformed into Antarctic Intermediate, Subantarctic Mode, and surface waters. The lower branch of the SMOC involves conversion of UCDW and LCDW to Antarctic Bottom Waters (AABW). The lower branch requires abyssal diapycnal mixing in the Southern Ocean or other ocean basins to close the overturning circulation.

For an UCDW transport of $O(20–50$ Sv), there must be either strong vertical or strong along-isopycnal mixing or some combination thereof. An overturning circulation of less than 5 Sv would require small along-isopycnal diffusivities ($K < 50$ m$^2$ s$^{-1}$), and vertical diffusivities between 0 and $10^{-4}$ m$^2$ s$^{-1}$ would make little difference to the size of the overturning circulation (Fig. 6). If a limit of 60 Sv where placed on the transport of UCDW at the northern side of the ACC this would imply an upper bound on $K$ of 600 m$^2$ s$^{-1}$ and on $D$ of $10^{-3}$ m$^2$ s$^{-1}$, as both have a positive contribution to the southward transport.

Any distribution of $K$ and $D$ can be applied to (2) to diagnose a transport of UCDW. Various potential distributions of mixing strengths may be considered by reviewing Figs. 4 and 5. Here, if a lateral diffusivity of 400 m$^2$ s$^{-1}$ is assumed on a specific layer, the strength of the transport on that layer, as a result of the $K$ term, would be double that what was shown in Fig. 4. The longitudinal variation in the transport can be considered in the same way for various distributions of $K$ and $D$ (Fig. 5). Cases such as stronger along-isopycnal mixing resulting from added kinetic energy provided by the winds close to the surface, or a steering level of baroclinically unstable waves (Smith and Marshall 2009), or stronger vertical mixing on deeper layers, perhaps a result of interaction with topography, could all be considered.

4. The residual-mean overturning and bolus velocity

In depth coordinates, and at constant latitude above the depth of the shallowest topographic feature, the circumpolar integral of the geostrophic velocity is zero. Here, we consider the circumpolar integral in isopycnal coordinates, rather than depth coordinates, and integrate along a contour of constant potential vorticity $PV = f\theta$, rather than latitude. In this case also, the geostrophic component is zero and only an ageostrophic flow remains.

To a good approximation, the mean circulation on an isopycnal satisfies geostrophy and so the mean velocity can be represented by a geostrophic streamfunction $\Psi$ and an Ekman velocity $v^{Ek}$ such that

$$v = -\frac{1}{f} \nabla \times \Psi + v^{Ek},$$

where $\nabla$ is the lateral velocity vector (overbars represent temporal averages), $f$ is the Coriolis frequency, and $k$ is
the unit vector in the vertical direction. The actual nature of the streamfunction is not relevant here, merely that one approximately exists. Following from (3), we define the transport within a density layer across a contour of constant potential vorticity PV and decompose the thickness flux (velocity times layer thickness correlated) into a temporal mean and perturbation component

$$\vec{v}_h = \vec{v}_h + \vec{v}_h', \tag{4}$$

and considering the components down the PV gradient for $\vec{n}_{PV} = \nabla_{PV}/|\nabla_{PV}|$ we have

$$\vec{v}_h \cdot \vec{n}_{PV} = \nabla \cdot \vec{n}_{PV} \vec{h} + \nabla \cdot \vec{n}_{PV} \cdot \nabla h. \tag{5}$$

We wish to find a relationship between the total transport across the ACC and mixing. Integrating $\vec{v}_h \cdot \vec{n}_{PV}$ circumpolarly along a PV contour in an isopycnal layer allows the mean geostrophic component to be eliminated:

$$\int_0 \vec{v}_h \cdot \vec{n}_{PV} \, dx_{PV} = \int_0 \vec{v}_h \cdot \vec{n}_{PV} \, dx_{PV} + \int_0 \vec{v}_h' \cdot \vec{n}_{PV} \, dx_{PV}. \tag{6}$$

Hence, the meridional transport is purely an ageostrophic one involving eddy and Ekman transports. Here, the PV contours need not remain within the latitude band of Drake Passage. The PV contours only need to be fully circumpolar, as Eq. (6) shows that in isopycnal layers for PV contours, which run along the entirety of the ACC, there is a barrier to the mean geostrophic transport (Fig. 7). The transport in these layers can only come from the Ekman and transient eddy components. In layers where PV contours do not run along the entire ACC there can be a net southward transport because of the mean geostrophic current (i.e., $\int \nabla h \, dx \neq 0$ where $dx$ runs along the ACC). The mean geostrophic flow may add significantly to a net along-isopycnal transport across the ACC, even if the net transport is zero on any given pressure level (Fig. 7). The SMOC is defined in density

![Fig. 7](image-url)
space (Hallberg and Gnanadesikan 2006), therefore mean
geostrophic flows can contribute to the overturning in
layers where PV contours do not follow the entirety of
the ACC—that is, large-scale standing eddies. Olbers and
Visbeck (2005) consider the circumpolarly integrated
transport of Antarctic Intermediate Water and UCDW in
terms of an Ekman and eddy flux. For layers where PV
contours do not sufficiently coincide with the streamwise
averaging they use, the along-isopycnal mixing coefficient
they impose must account for both eddy variability and
large-scale correlations between thickness and the geo-

The eddy transport \( \nabla h^\gamma \), may be parameterized as
a downgradient flux of thickness or bolus flux (with
a corresponding bolus velocity \( v^\gamma \)) such that (following
McDouggall 1991)

\[
\frac{\nabla h^\gamma}{\gamma} = v^\gamma = -K_{PV} V_\gamma \log(h) + K_{PV} \left( \frac{\beta}{f} \right) j
\]

(7)

\[
\int \nabla h^\gamma dx_{PV} = \int K_{PV} h \left[ -V_\gamma \log(h) + \frac{\beta}{f} j \right] \cdot n dx_{PV},
\]

(8)

where in (8), \( K_{PV} \) is the along-isopycnal potential vorticity mixing coefficient, \( v^\gamma \) is the bolus velocity, \( \beta = \partial f/\partial y \), and \( j \) is the unit vector in the meridional direction. Here,

along-isopycnal mixing \( (K_{PV}) \) acts to evenly distribute
PV. Cast in terms of a velocity, the first component
represents the effect of gradients of \( h \) and the second is
due to the meridional gradient of \( f \). A different approach
is taken by Gent et al. (1995) to parameterize the quasi-
Stokes velocity \( v^\gamma \) (the counterpart to the bolus velocity in
Eulerian coordinates):

\[
v^\gamma = -K_{PV} V_\gamma \log \left( \frac{\partial h}{\partial \rho} \right).
\]

(9)

This parameterization ignores the \( \beta \) effect and is com-
monly used in ocean circulation models and is the same as
the first term on the right-hand side of (7) for small \( \Delta \rho \).

Given (8), we quantify the meridional transport of
UCDW in terms of the along-isopycnal potential vor-
ticity diffusivity \( (K_{PV}) \). This is done in layers where
contours of constant PV are continuous along the cir-

cumpolar path of the ACC and are sufficiently distant, in
the vertical, from the Ekman layer. We define contours
of constant PV in \( \gamma_n \) layers from the WOCE cli-

matology, where \( h \) is the thickness of the layer. Contours
may only be defined for northern, central, and southern
pathways of the ACC in layers \( \gamma_n = 27.5 \) kg m\(^{-3} \) to \( \gamma_n = 27.8 \) kg m\(^{-3} \) (\( \gamma_n = 27.5 \) kg m\(^{-3} \) to \( \gamma_n = 27.7 \) kg m\(^{-3} \) in
the case of the northern contour). We assume a uniform
\( K_{PV} \) of 200 m\(^2\) s\(^{-1}\) and quantify both the thickness and \( \beta \)
terms in (8).

If \( K_{PV} \) is assumed to be the same as the along-isopycnal
mixing coefficient for tracers \( K \), it is expected that the
meridional transports derived from (2) and (8) should
be similar for contours with similar paths and depths
(away from Ekman effects). This is the case for the
deepest of layers (Figs. 4 and 8). Northward transports
suggested from (2) for \( \gamma_n < 26.7 \) kg m\(^{-3} \) may be ex-
plained by Ekman effects as isopycnals shoal in the
southward direction and perhaps a strong gradient of \( K \)
at the bottom of the mixed layer.

5. Diapycnal flow and the 1D balance

Since the work of Munk (1966), it has been common to
consider a 1D balance where a vertical mixing “scale
length” relates a rate of upwelling to a vertical mixing
coefficient.

Following a similar approach as the one that led to
(1) an equation may be derived for the diapycnal ve-

locity \( \gamma \):

\[
w^\gamma = K \left[ \nabla^\gamma S |V_\gamma| - \nabla^\gamma S \frac{|V_\gamma|}{|V_\gamma|} \right] + D \left( \nabla^\gamma S \frac{|V_\gamma|}{|V_\gamma|} - \nabla^\gamma \frac{|V_\gamma|}{|V_\gamma|} \right)
\]

\[
= \frac{K}{\eta^\gamma} + \frac{D}{\eta^\gamma},
\]

(10)

where \( \eta^\gamma \) and \( \eta^\gamma \) are diffusive “scale heights” (see
appendix B). Cases where \( D_{\gamma} \) is a significant term in (10)
are not considered in this study. If a linear equation of
state is assumed, the along-isopycnal mixing term is zero
and (10) reduces to \( w^\gamma = D_{\gamma z} /\gamma_z \) (appendix B). This
simplification of (10) is commonly used in studies at-
tempting to infer large-scale dynamics in the Southern
Ocean from observations (Karsten and Marshall 2002;
Naveira-Garabato et al. 2007). Here, we shall retain
both the linear and nonlinear components and deter-
mine their relative role in cross isopycnal transport.

The diapycnal velocity is computed for neutral density
surfaces from tracer fields of the WOCE climatology and
integrated circumpolarly between temperature con-
tours corresponding to the southern and northern sides of
the ACC. With a vertical diffusivity of \( 2 \times 10^{-4} \) m\(^2\) s\(^{-1}\),
the vertical velocity through \( \gamma_n = 27.7 \) kg m\(^3\), a result
of the \( D \) term in (10), is determined (Fig. 9). The velocity
is of \( O(10^{-7} \) m s\(^{-1}\) to the north of the ACC
and is of \( O(10^{-6} \) m s\(^{-1}\) to the south (Fig. 9). The
accumulated diapycnal transport between the tempera-
ture contours is approximately 8 Sv for the \( D \) term in
(10). The diapycnal transport is spread reasonably even
across the entirety of the ACC, with the net effect being
When a value for $K$ of 200 m$^2$ s$^{-1}$ is used, the downwelling resulting from this term is negligible ($\approx 1$ Sv). If $K$ were $O(1000$ m$^2$ s$^{-1}$) the downwelling would be first order [see Iudicone et al. (2008), for a comprehensive discussion]. The lateral mixing term is often neglected, even when large along-isopycnal or lateral diffusivities are observed (Karsten and Marshall 2002; Naveira-Garabato et al. 2007).

6. The relative role of vertical and along-isopycnal mixing

In this section we present evidence that the ratio of the along-isopycnal and vertical mixing coefficients is $O(2 \times 10^6)$ along the ACC in the density layers of UCDW. This is done by combining the concept of a cross-contour flow $\nabla h \cdot \mathbf{n}_\psi$, which was discussed in section 2, with the PV parameterizations of section 4. This combination yields a balance of vertical and along-isopycnal mixing, independent of the mean velocity. We next combine the cross-contour transports derived in section 3, with the diapycnal transports in section 5. We then apply continuity to finite volumes around the entire ACC, arriving at a balance of along-isopycnal and vertical mixing.

From (4) we have

$$\mathbf{v} = \frac{\nabla h}{h} - \frac{\nabla h'}{h}.$$  

We take the component of the mean geostrophic velocity in the direction of the along-isopycnal temperature gradient ($\mathbf{n}_\psi$). Substituting for (1) we have

$$\nabla \cdot \mathbf{n}_\psi = \frac{K}{\lambda^2} + \frac{D}{\lambda V} + K_{\text{PV}} \left( \frac{\beta}{f} \right) \mathbf{j} \cdot \nabla \log(h) \cdot \mathbf{n}_\psi,$$  

(12)

where $\mathbf{v} \cdot \mathbf{n}_\psi$ is the mean velocity down the along-isopycnal temperature gradient. Equation (12) differs from (1) as the left-hand side is only the mean velocity and not the residual one. Away from the Ekman layer the mean velocity is approximately geostrophic, hence

$$\mathbf{v} \cdot \mathbf{n}_\psi = -\frac{1}{f} \mathbf{v} \cdot \nabla \Psi \times \mathbf{k} \cdot \mathbf{n}_\psi,$$  

(13)

where the circumpolar integral of $f\mathbf{v} \cdot \mathbf{n}_\psi$ along circumpolar temperature contours is zero. Hence, integrating $f$ times Eq. (12) gives

Fig. 8. Contributions to bolus transport for $K = 200$ m$^2$ s$^{-1}$ from the thickness gradient term (black bars) and the beta gradient term (white bars) in (8). (a) Southern contour, (b) central contour, and (c) northern contours of the ACC. Positive values are with increasing PV (northward for the layers shown).


Here, Eq. (14) relates the along-isopycnal, vertical, and potential vorticity mixing coefficients and is independent of the mean velocity. It is thought that the along-isopycnal diffusivity is approximately the same for both PV and passive tracers (Smith and Marshall 2009). On isopycnals, $S$ and $Q$ are approximately passive tracers, as the act of stirring them locally does not change the stratification (although there may be some effects that result from the nonlinear equation of state).

We assume $K = K_{PV}$ and from (14) the ratio of $K$ to $D$ is

$$
\frac{K}{D} = \frac{\int f/\lambda^\gamma \, dx_\theta}{\int [\beta j - f \nabla_{\gamma} \log(h)] \cdot n \, dx_\theta - \int f/\lambda^\gamma \, dx_\theta}.
$$

(15)

Here, (15) holds for any enclosed temperature contour on an isopycnal that does not interact with the Ekman layer. With above-average depths of 500 m, the ratio of $K$ to $D$ is zero or negative, suggesting these contours interact with the Ekman layer or lateral gradients of $K$, which becomes important somewhere along their circumpolar path. Below 500 m the ratio is consistently positive and of $O(2 \times 10^6)$ (Fig. 10).

We have established equations to determine both the along-isopycnal transport across temperature contours (2) and the transport through isopycnals (10) in terms of the vertical and along-isopycnal mixing coefficients $K$ and $D$. We may apply continuity of volume in order to ascertain the ratio of these two coefficients. Writing the steady continuity equation for a particular volume on a density layer bound by contours of constant temperature to the south and north ($\Theta_1$ and $\Theta_2$ respectively), we have

$$
\left[ \int \left( \frac{K}{\lambda^\gamma} + \frac{D}{\lambda^\gamma} \right) h \, dx_\theta \right]_{\Theta_2}^{\Theta_1} + \left[ \int \left( \frac{K}{\eta^\gamma} + \frac{D}{\eta^\gamma} \right) dA \right]_l^u = 0,
$$

(16)

where $\int dA$ are integrals over the upper $u$ and lower $l$ bounding neutral density surfaces between temperature contours ($\Theta_1$ and $\Theta_2$ respectively). Assuming again that the mixing coefficients are constant in space we may write

$$
\frac{K}{D} = -\frac{\left( \int 1/\lambda^\gamma h \, dx_\theta \right)_{\Theta_2}^{\Theta_1}}{\left( \int 1/\lambda^\gamma h \, dx_\theta \right)_{\Theta_1}^{\Theta_2} + \left( \int 1/\eta^\gamma \, dA \right)_{l}^{u}}.
$$

(17)

Here, Eq. (17) is applied to volumes between contours on the northern side of the ACC below an average depth of 500 m (Fig. 10). We find that the ratio of $K$ to $D$ is again of $O(2 \times 10^6)$ there.

7. Discussion and conclusions

This study has investigated the relationship of along-isopycnal mixing ($K$) and diapycnal mixing ($D$) to the strength of the Southern Ocean meridional overturning circulation (SMOC) as quantified by the southward...
transport of Upper Circumpolar Deep Water (UCDW). The total transport of UCDW, within isopycnal layers, has been diagnosed from tracer distributions. The transports are inferred directly from observations through a linear relationship with the mixing coefficients. The sensitivity of the overturning transport and spatial characteristics to a range of possible diffusivities is a direct result of the analysis (section 3).

An important aspect of this study is the discussion of cross-contour transport in density-temperature space. The thickness-weighted velocity down the temperature gradient on an isopycnal $v_h n_C Q_{1n}$, discussed in section 2, is dependent on $K$ and $D$ locally, and thus the spatial structure of the transport of specific water masses can be analyzed as well as their circumpolar integral.

The diffusive scale lengths, $\lambda^1$ and $\lambda^\gamma$, integrated circumpolarly show that UCDW is transported southward where it feeds the upper and lower cells of the SMOC. Careful comparison of the fluxes inferred from tracer gradients, the bolus transport, and conservation considerations for along-isopycnal and vertical transports suggests a ratio of $K$ to $D$ of $O(2 \times 10^6)$. The implications of such a ratio for the overturning, the possible range of mixing coefficients, and in turn for numerical modeling of the Southern Ocean are discussed below.

The southward transport of UCDW inferred from inverse studies (Sloyan and Rintoul 2001; Lumpkin and Speer 2007) suggests a range of 20–52 Sv. We have derived a linear relationship between the southward transport of UCDW and $K$ and $D$ (2) and we have estimated the ratio of $K$ to $D$ in the UCDW layers to be $2 \times 10^6 \pm 10^6$ (Fig. 10). Thus, we estimate $K$ and $D$ individually and find $K = 300 \pm 150$ m$^2$ s$^{-1}$ and $D = 10^{-4} \pm 0.5 \times 10^{-4}$ m$^2$ (blue cross-hatching in Fig. 6). Such rates of diapycnal mixing are considered to be large in the mid-latitude oceans, but are supported by observations of diapycnal mixing ($D$) in the ACC such as Sloyan (2005), Naveira-Garabato et al. (2004), and Kunze et al. (2006). Our results suggest $D$ is $O(10^{-4}$ m$^2$ s$^{-1}$) beneath the mixed layer in the Southern Ocean, supporting the hypothesis that vertical mixing, in the ocean interior along the ACC, makes a significant contribution to water mass conversion. The diagnosed along-isopycnal mixing coefficient ($K$) is also within the range estimated by both Phillips and Rintoul (2000) and McKeague et al. (2005).

For along-isopycnal mixing coefficients of $O(200$ m$^2$ s$^{-1}$), the nonlinear terms contributing to the diapycnal transport below the mixed layer in the Southern Ocean are small (section 5). However, for values of $O(10^7$ m$^2$ s$^{-1}$) as used in Naveira-Garabato et al. (2007), downwelling resulting from cabbeling and thermobaricity is significant. This study has not considered cases where $V_h K$ is a significant term in (1). Although this term is not necessarily negligible in the Southern Ocean,
sufficient observations of $K$ do not exist to reasonably quantify it.

Although there is mounting evidence that there is intense vertical mixing along the ACC in the Southern Ocean, global climate simulations are yet to include spatially and vertically varying diffusivities. This study presents further evidence that in the Southern Ocean $D$ is indeed of $O(10^{-4}$ m$^2$ s$^{-1}$) or greater, and that a significant fraction of upwelling UCDW is transported across isopycnals below the mixed layer. This study also suggests that the along-isopycnal mixing coefficient of $O(10^2$ m$^2$ s$^{-1}$), used in the majority of such simulations, is appropriate. Isopycnal mixing of $O(10^3$–$10^4$ m$^2$ s$^{-1}$) as measured at the surface (Karsten and Marshall 2002; Sallée et al. 2008) is inappropriate for deep layers as they would give extremely large values for the southward transport of UCDW (Fig. 6).

Along-isopycnal and vertical mixing coefficients inferred from observations are often difficult to compare with those required by coarse-resolution ocean models. Observational studies using moorings, Lagrangian tracers, and satellite altimetry diagnose an effective diffusivity, whereas numerical models require a diffusivity to represent the velocity to property correlations (i.e., $\overline{\nu^{\gamma} \theta}$) not accounted for by the temporal and spatial resolution of the model. Numerical models also require a diffusivity to remain numerically stable. The conservation equations used to derive (1) involve a vertical and along-isopycnal diffusivity, which represents the long-term effect of temporal correlations between perturbations of the mean velocity and tracer. It is expected that the major contribution to $K$ is a result of submesoscale eddies, and hence the $K$ discussed in this paper is equivalent to that desired in a coarse-resolution ocean model. In continuing work we have developed an inverse technique that diagnoses the downgradient transport and the mixing coefficients $K$ and $D$ (Zika et al. 2009). This is done by using the integral of (1) along a tracer contour and a form of the thermal wind equation cast in terms of differences in geostrophic streamfunction $\Psi$ along such a contour. The technique is validated against the output of a 20-yr average of a 100-yr climate simulation of the Hallberg Isopycnal Model at $1^\circ \times 1^\circ$ resolution. In so doing, we show that the $K$ and $D$ used in (1) is close to that when explicitly applied to a coarse-resolution ocean model.

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APPENDIX A

Derivation of the Water Mass Equation

The conservation equations for salinity $S$ and conservative temperature $\Theta$ in steady state are

$$\frac{\nabla h}{h} \cdot \nabla S + \nu^{\gamma} S_z = h^{-1} \nabla \cdot (hK\nabla S) + (D S_z)$$

(A1)

$$\frac{\nabla h}{h} \cdot \nabla \Theta + \nu^{\gamma} \Theta_z = h^{-1} \nabla \cdot (hK\nabla \Theta) + (D \Theta_z).$$

(A2)

These equations have been written in the advective form, and with respect to neutral density ($\gamma_n$) layers (Jackett and McDougall 1997), so that $\nu^{\gamma}$ is the vertical velocity through neutral density surfaces (i.e., the diapycnal velocity component of the vertical velocity) and $(\nabla h/\overline{h})$ is the thickness-weighted horizontal velocity obtained by temporally averaging the horizontal volume transport between closely spaced neutral density surfaces. Similarly, the salinity $S$ and conservative temperature $\Theta$ in (A1) and (A2) are the thickness-weighted values obtained by averaging between closely spaced pairs of neutral density surfaces (McDougall and McIntosh 2001). In these equations $\overline{h}$ is the mean thickness between two closely spaced neutral density surfaces. The mixing processes that appear on the right-hand sides are simply along-isopycnal mixing of passive tracers (with coefficient $K$) along the density layer and vertical small-scale turbulent mixing (with coefficient $D$). We have not included double-diffusive convection or double-diffusive interleaving. We define the cross-contour direction as $n_\gamma = \nabla \gamma/|\nabla \gamma|$. Notably, $\nabla \gamma / |\nabla \gamma| = \nabla \Theta / |\nabla \Theta|$ as we are considering gradients on a neutral density layer. Multiplying (A1) by $\Theta$ and (A2) by $S$ gives

$$\frac{\nabla h}{h} \cdot n_\gamma |\nabla \gamma S| \Theta_z + \nu^{\gamma} S_z \Theta_z = h^{-1} \nabla \cdot (hK\nabla \gamma S) \Theta_z + DS_z \Theta_z + D_z S_z \Theta_z.$$  

(A3)

$$\frac{\nabla h}{h} \cdot n_\gamma |\nabla \gamma \Theta| S_z + \nu^{\gamma} \Theta_z S_z = h^{-1} \nabla \cdot (hK\nabla \gamma \Theta) S_z + D \Theta_z S_z + D_z \Theta S_z.$$  

(A4)

Subtracting (A4) from (A3), the $D_z$ and $w^{\gamma}$ terms are eliminated and we find
where
\[
\frac{1}{\lambda_z} = \frac{\Theta \nabla_y^2 S - S \nabla_y^2 \Theta}{\Theta \nabla_y^2 S - S \nabla_y^2 \Theta} + \nabla_y \log(h) \cdot \mathbf{n}_\Theta. 
\]

In the case of a linear equation of state \(\nabla_y \alpha = \nabla_y \beta = 0\), hence
\[
\left(\frac{\sqrt{h}}{h}\right) \cdot \mathbf{n}_\Theta |\nabla_y S| |\nabla_y \Theta| + w^\gamma \nabla_z |\nabla_y S| |\nabla_y \Theta| = h^{-1} \nabla_y (\nabla_y K) \cdot \mathbf{n} |\nabla_y S| |\nabla_y \Theta| + K \nabla_y^2 S |\nabla_y \Theta| + (D S_z) |\nabla_y \Theta| \quad \text{and} \quad \text{(B1)}
\]
\[
\left(\frac{\sqrt{h}}{h}\right) \cdot \mathbf{n}_\Theta |\nabla_y \Theta| |\nabla_y S| + w^\eta \nabla_z |\nabla_y S| = h^{-1} \nabla_y (\nabla_y K) \cdot \mathbf{n} |\nabla_y \Theta| |\nabla_y S| + K \nabla_y^2 \Theta |\nabla_y S| + (D \nabla_x) |\nabla_y S| \quad \text{B2)}
\]

Subtracting (B1) from (B2), the \(\nabla_y (hK)\) and \(\left(\frac{\sqrt{h}}{h}\right) \cdot \mathbf{n}_\Theta\) terms are eliminated and we find
\[
\frac{w^\gamma = \frac{K}{\eta^\gamma} + \frac{D}{\eta^\gamma} + D_z}{\eta^\gamma}, \quad \text{B3)}
\]
where
\[
\frac{1}{\eta^\gamma} = \frac{\nabla_y^2 \Theta |\nabla_y S| - \nabla_y^2 S |\nabla_y \Theta|}{\Theta |\nabla_y S| - S |\nabla_y \Theta|} \quad \text{B4)}
\]
and
\[
\frac{1}{\eta^\gamma} = \frac{\Theta |\nabla_y S| - S |\nabla_y \Theta|}{\Theta |\nabla_y S| - S |\nabla_y \Theta|} \quad \text{B5)}
\]

Here, neither \(1/\eta^\gamma\) nor \(1/\eta^\gamma\) is singular if the water column is stably stratified. If a linear equation of state is assumed such that \(\rho_z \propto -\alpha \Theta_z + \beta S_z\), for \(\alpha\) and \(\beta\) constant and again \(\alpha/\beta = |\nabla_y S|/|\nabla_y \Theta|\), the vertical mixing term \(1/\eta^\gamma\) becomes
\[
\frac{1}{\eta^\gamma} = \frac{\alpha \Theta_z^2 - \beta S_z^2}{\alpha \Theta_z - \beta S_z} = \frac{\rho_z^2}{\rho_z^2}, \quad \text{B6)}
\]

Also, as \(\nabla_y \alpha = \nabla_y \beta = 0\) the along-isopycnal mixing term \(1/\eta^\gamma\) becomes
\[
\frac{1}{\eta^\gamma} = \frac{\nabla_y^2 \Theta |\nabla_y S| - (\alpha/\beta) \nabla_y^2 \Theta |\nabla_y \Theta|}{\Theta |\nabla_y S| - S |\nabla_y \Theta|} = 0. \quad \text{B7)}
\]

Hence, for a linear equation of state, the vertical velocity through isopycnal surfaces simplifies to \(w^\gamma = D \rho_{zz}/\rho_z + D_z\).
REFERENCES


