Comments "On the Use of the DuFort-Frankel Finite-Difference Approximation for Simulation of Diffusion in Geophysical Fluids"

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Overland (1973) states in a note that the DuFort-Frankel scheme for numerical simulation of diffusion does not necessarily conserve scalar properties. He implies that this depends on the structure of the scheme; instead, it depends on the starting scheme as was pointed out to me by Dušan Djurić. This can be shown analytically. Since the linear heat equation is separable, we need only consider the one-dimensional equation

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}. \quad (1)$$

Taking

$$\gamma = \frac{2K\Delta t}{\Delta x^2},$$

the standard finite-difference scheme for (1) is

$$T_{j+1}^{n+1} = T_j^n + \frac{\gamma}{2}[T_{j+1}^n + T_{j-1}^n - 2T_j^n], \quad (2)$$

while the DuFort-Frankel scheme is

$$T_{j+1}^{n+1} = T_j^{n-1} + \gamma[T_{j+1}^n + T_{j-1}^n - T_j^{n+1} + T_j^{n-1}], \quad (3)$$

where $T(n\Delta t, j\Delta x) = T_j^n$.

Many different boundary conditions may be specified depending on the physical problem. It is expected that the conclusions shown below will be independent of the choice of boundary conditions. In any event, Overland's argument for non-conservation is independent of the boundary conditions. I choose periodic boundary condition for simplicity in the analysis; i.e.,

$$T_0 = T_{J+1}.$$

The initial conditions will be considered later. The analysis is independent of the number of grid points and time steps but, of course, the truncation error is not.

If we define

$$Q_n = \sum_{j=1}^{J} T_j^n,$$

and sum (2) over $j$, we obtain

$$Q_{n+1} = Q_n - Q_0; \quad (4)$$

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which states that the scalar temperature field is conserved.

For the DuFort-Frankel scheme, we sum (3) over \( j \) and obtain

\[
(1 + \gamma)Q_{n+1} = (1 - \gamma)Q_{n-1} + 2\gamma Q_n.
\]  

(5)

Since this is a 3-time-level scheme, the result must depend on the starting scheme. Note that if \( Q_1 = Q_0 \) then (5) implies that \( Q_1 = Q_0 \); repeated application then leads to the general conservation relation (4). There are at least two different ways of assuring that \( Q_1 = Q_e \). The simplest is to take the first two fields of \( \mathbf{T} \) identical, i.e.,

\[ T_j^1 = T_j^0 \]

A less stringent way of assuring that \( Q_1 = Q_0 \) is to employ the forward difference relation (2) for the first step.

Thus, while the DuFort-Frankel scheme does possess certain shortcomings as discussed by Roache (1972), it is shown here that non-conservation is not a problem with proper attention to starting conditions. My results have been verified by computer tests but these were clearly not required. Apparently, Overland (1973) did not ensure that his starting scheme was conservative.

It is of interest to note in passing, for the special case \( \gamma = 1 \), that Eq. (5) shows that conservation of \( Q \) is automatically assured; this must be the case since scheme (3) is identical to (2) under these circumstances.

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REFERENCES
