The Linear Stability of Time-Dependent Baroclinic Shear

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ABSTRACT

This article aims to advance the understanding of inherent randomness in geophysical fluids by considering the particular example of baroclinic shear flows that are spatially uniform in the horizontal directions and aperiodic in time. The time variability of the shear is chosen to be the Kubo oscillator, which is a family of time-dependent bounded noise that is oscillatory in nature with various degrees of stochasticity.

The author analyzed the linear stability of a wide range of temporally periodic and aperiodic shears with a zero and nonzero mean to get a more complete understanding of the effect of oscillations in shear flows in the context of the two-layer quasigeostrophic Phillips model. It is determined that the parametric mode, which exists in the periodic limit, also exists in the range of small and moderate stochasticities but vanishes in highly erratic flows. Moreover, random variations weaken the effects of periodicity and yield growth rates more similar to that of the time-averaged steady-state analog. In the limit of an $f$ plane, the linear stability problem is solved exactly to reveal that individual solutions to the linear dynamics with time-dependent baroclinic shear have growth rates that are equal to that of the time-averaged steady state. This implies that baroclinic shear flows with zero means are linearly stable in that they do not grow exponentially in time. This means that the stochastic mode that was found to exist in the Mathieu equation does not arise in this model. However, because the perturbations grow algebraically, the aperiodic baroclinic shear on an $f$ plane can give rise to nonlinear instabilities.

1. Introduction

There is a vast amount of literature devoted to understanding the stability of steady shear flows and a much more limited body of work exploring the stability of time-periodic shear flows. In contrast, nature has few if any flows that are truly steady or periodic because variations are ubiquitous in physical systems. The fact that almost all geophysical fluids are aperiodic motivates the need to better understand the dynamics of aperiodic flows in general and shear flows in particular. To date, there has not been much attention paid to studying aperiodic flows in an oceanographic context, and this article attempts to remedy this gap. Recent notable exceptions are Durski et al. (2008), who studied normal mode instabilities in an aperiodic coastal upwelling jet, and Inoue and Smyth (2009), who investigated the mixing efficiency in unsteady shear flow.

To begin, we classify motion in roughly three different regimes depending on the degrees of aperiodicity: flows that have only a slight degree of aperiodicity, flows that are highly erratic, and flows with moderate aperiodicities in between the previous two. The first class is the simplest by far and has received the most attention in the literature. If we approximate the flow as periodic, we can use Floquet analysis to determine the stability of the system. This is advantageous because it only requires integrating the system of equations for one period and then analyzing the eigenvalues of the resulting Floquet matrix. Because the integration is usually done numerically, this is by far the least expensive problem to study.

The great amount of irregularity in the second class typically allows it to be approximated with a white noise process, the most common of which is Gaussian white noise. Monte Carlo simulations can be performed to find the statistics of the solution. The presence of white noise allows us to separate the fast and slow time scales to approximate the moments of the solution. Some of the disadvantages of white noise are that it is discontinuous, has no memory, and is unbounded. Moreover, it does
not have a unique calculus, which raises the question of which to use (Gardiner 2004).

The third class, the intermediate regime, is perhaps the most complex because few simplifying approximations can be made. We argue that it is also the most interesting class because it describes the broadest range of motions. One example of this class is an Ornstein–Uhlenbeck process, also known as Gaussian-colored noise (Gardiner 2004; Van Kampen 2001); however, this is not ideal for our purposes because it is unbounded and not oscillatory. Instead, we chose to mainly focus on the dynamics of slightly aperiodic systems where the noise is bounded, oscillatory, and tends to a sinusoidal function in one limit. A particular nonautonomous function that satisfies these criteria is the Kubo oscillator (Risken 1984; Gardiner 2004).

This choice follows Poulin and Flierl (2008), who used this oscillator as the internal noise of a stochastic generalization to the classical Mathieu equation. They found two modes of instability in their aperiodic system. The first is the parametric mode that exists in both the periodic and aperiodic limit as long as the stochasticity is moderate. The second is the stochastic mode that is only present in the case of moderate stochasticities. It was speculated in Poulin and Flierl (2008) that most systems with a similar internal noise should have common stability characteristics. In this present article, we investigate baroclinic shear that is uniform in space and has a temporal variation determined by the Kubo oscillator. This is done in the context of the Phillips model.

The Phillips model is a two-layer quasigeostrophic model on a $\beta$ plane, usually in the confines of a zonal channel (Phillips 1951, 1954; Pedlosky 1964a,b, 1987). This simple geometry is ideally suited for studying atmospheric flow. The applicability of the Phillips model to the ocean is complicated by the presence of coastal boundaries that can alter the dynamics. The Antarctic Circumpolar Current (ACC) is one part of the ocean where the channel geometry is most applicable (Nowlin and Klinck 1986). The winds in the Southern Ocean provide forcing that can alter the baroclinic shear. The seasonal variations in the winds do not significantly alter the long-term phase of the shear, but they do cause the strength of the shear to wander in amplitude. Even though the Kubo oscillator depicts a state that has a random phase, it yields behavior that is similar to one that has a varying amplitude. It is for this reason that our analysis is applicable to the ACC and other examples of baroclinic shear in the ocean.

Two phenomena that are certainly important in the ACC are the nonzero mean flow and topography. Here, we investigate the first of these but not the second. In the case of steady baroclinic shear, these have previously been studied in both the Phillips model (Benlov 2001) and the $n$-layer analog (Treguier and McWilliams 1990). The inclusion of topography is beyond the scope of this research and will be addressed in future work.

The spatial and temporal variability in the ACC was studied by Gille and Kelly (1996) using Geosat altimeter data. The objective analysis revealed that the ACC had a strong spectral peak at 0.33 cycles yr$^{-1}$ but there was also a lot of other significant variability at much higher frequencies. This suggests that it is of interest to understand the dynamics of baroclinic shear with varied spectral peak frequencies. Here, we do not aim to study the ACC per se, but instead we look at the types of motions that arise as a result of aperiodic shear flows. In particular, our focus will be on baroclinic shears that have spectral peaks at higher frequencies than the peak that Gille and Kelly (1996) observed.

In the Phillips model, it is well known that a necessary condition for instability in a steady flow is that the shear velocity violates the Charney–Stern criteria for stability (Gill 1982; Pedlosky 1987). One of the recent applications of the Phillips model was to study temporally periodic baroclinic shear. Pedlosky and Thomson (2003) determined, as has been found in other models (Benjamin and Ursell 1954; Kelly 1967; Farrell and Ioannou 1996b, 1999a; Poulin et al. 2003; Poulin and Scott 2005), that parametric instability can exist when the shear is periodic even though the Charney–Stern criteria is not violated at any instant in time. Subsequently, Flierl and Pedlosky (2007) analyzed the stability of periodic baroclinic shears to determine the nonlinear manifestation of parametric instability.

Here, we build upon previous works and study the stability of time-dependent baroclinic shear flows to include more realistic effects. First, we linearize around the nonautonomous background state and numerically integrate the governing equations for a wide range of parameters. Second, we present the Kubo oscillator and illustrate how it is a good choice for looking at some general time-dependent variations. Third, we discuss the important notions required to investigate the stability of a time-dependent dynamical system. Fourth, we study the parametric mode in the case of an aperiodic baroclinic shear to observe what effect the aperiodicity has on the growth rates of the system. Fifth, we find an analytical solution for the waves on an $f$ plane, and determine that their growth rate for an aperiodic baroclinic shear with zero mean is zero in the long time limit.

2. The model equations

Following Flierl and Pedlosky (2007), we focus on the two-layer Phillips model with Rayleigh damping and
external forcing on a $\beta$ plane confined within a meridional channel of width $L$: the Rayleigh term damps the potential vorticity and the forcing maintains the time-varying basic state. The subindices 1 and 2 denote the upper and lower layers, respectively. Furthermore, for simplicity we restrict our attention to the case of equal-layer depths $D$. The fully nonlinear equations are (Pedlosky 1987)

$$\frac{\partial q_n}{\partial t} + J(\psi_n, q_n) + \beta \frac{\partial \psi_n}{\partial x} = -\mu q_n + G_n(y, t) \quad \text{for} \quad n = 1, 2, \quad (1)$$

where the potential vorticities $q_n$ and the streamfunctions $\psi_n$ are related as follows:

$$q_n = \nabla^2 \psi_n + (-1)^n F(\psi_1 - \psi_2), \quad \text{for} \quad n = 1, 2 \quad (2)$$

and the velocity is

$$(u_n, v_n) = \left( -\frac{\partial \psi_n}{\partial y}, \frac{\partial \psi_n}{\partial x} \right). \quad (3)$$

The two nondimensional parameters for this system are the Froude $F$ and nondimensional $\beta$ parameters,

$$F = \frac{\beta L}{g D}, \quad \beta = \frac{\beta \dim L^2}{U_{\text{scale}}} \quad (4)$$

Furthermore, $\mu$ is the Rayleigh-damping parameter and $G_n(y, t)$ denotes the forcing that maintains the time-varying shear.

The zonal velocity of the basic state is spatially uniform and defined to be $U_m(t)$. Thus, the vertical shear between the two horizontal layers is $U_m(t) = U_1(t) - U_2(t)$ and the mean velocity is defined to be $U_m = [U_1(t) + U_2(t)]/2$, which we assume is constant. This allows us to decompose the solution into a time-dependent basic state and the perturbation from that state,

$$\psi_n = -U_m y + (-1)^n y \frac{U_1(t)}{2} + \phi_n \quad (5)$$

We use $\phi_n$ and $q_n'$ to denote the streamfunction and potential vorticities of the perturbation. To ensure that the basic state is an exact nonlinear solution we require the forcing to be

$$G_n(y, t) = \frac{\partial q_n}{\partial t} + \mu q_n = (-1)^n F y \left( \frac{dU}{dt} + \mu U_1 \right) \quad (6)$$

This forcing could be interpreted as wind forcing that maintains the time-varying basic state.

We substitute Eq. (5) into Eq. (1) to acquire the nonlinear equations that govern the evolution of the perturbation,

$$\left\{ \frac{\partial}{\partial t} + \left[ U_m + (-1)^{n+1} U_2 \frac{\partial}{\partial x} + \mu \right] q_n' + \frac{\partial \phi_n}{\partial x} \right\} = \left[ \beta + (-1)^{n+1} F U_s \right] + J(\phi_n, q_n') = 0 \quad (7)$$

The potential vorticity–streamfunction relation for the perturbation is

$$q_n' = \nabla^2 \phi_n + (-1)^n F(\phi_1 - \phi_2) \quad (8)$$

Following convention, we define the barotropic and baroclinic perturbation streamfunctions $\phi_i$ and $\phi_c$, respectively, and potential vortices $q_i'$ and $q_c'$ as

$$\phi_i = \frac{\phi_1 + \phi_2}{2}, \quad q_i' = \frac{q_1' + q_2'}{2} = \nabla^2 \phi_i \quad (9)$$

$$\phi_c = \frac{\phi_1 - \phi_2}{2}, \quad q_c' = \frac{q_1' - q_2'}{2} = \nabla^2 \phi_c - 2F \phi_c \quad (10)$$

By adding and subtracting Eq. (7), we obtain the following evolution equations for the barotropic and baroclinic potential vortices:

$$\left\{ \frac{\partial}{\partial t} + U_m \frac{\partial}{\partial x} + \mu \right\} q_i' + \frac{U_1}{2} q_i' + \frac{U_2}{2} (q_i' + 2F \phi_i) + \frac{\partial \phi_i}{\partial x} + J(\phi_i, q_i') + J(\phi_i', q_i') = 0 \quad (11)$$

$$\left\{ \frac{\partial}{\partial t} + U_m \frac{\partial}{\partial x} + \mu \right\} q_c' + \frac{U_1}{2} q_c' + \frac{U_2}{2} (q_c' + 2F \phi_c) + \frac{\partial \phi_c}{\partial x} + J(\phi_c, q_c') + J(\phi_c', q_c') = 0 \quad (12)$$

The associated enstrophy for the perturbation $V$ is

$$V = \frac{1}{2} \langle q_i'^2 + q_c'^2 \rangle_{(x,y)} = \langle q_i'^2 + q_c'^2 \rangle_{(x,y)} \quad (13)$$

where $\langle \cdot \rangle_{(x,y)}$ denotes the spatial average over the width and zonal extent of the channel. This is the norm that we use to measure the growth of the perturbations, but other norms would yield qualitatively similar results.

To analyze the stability of the basic state to infinitesimal perturbations we linearize the governing Eqs. (11) and (12). Because the resulting equations have coefficients that are spatially uniform, we can decompose the perturbation into normal modes in the horizontal plane,

$$q_i' = \hat{q}_i(t) e^{ikx} \sin(ly) + \text{c.c.}, \quad \phi_i = \hat{\phi}_i(t) e^{ikx} \sin(ly) + \text{c.c.,} \quad \hat{q}_i' = \hat{q}_i(t) e^{ikx} \sin(ly) + \text{c.c.,} \quad \phi_c = \hat{\phi}_c(t) e^{ikx} \sin(ly) + \text{c.c.}$$
The structure of the modes is chosen to ensure that the perturbations satisfy the no-normal flow boundary conditions at the channel walls. Substituting this decomposition into the linearized problem yields the following equations in terms of the barotropic and baroclinic streamfunctions that govern the linear stability:

\[ \frac{d}{dt} + \mu + ik \left( U_m - \frac{\beta}{K^2} \right) \phi_i = -ik \frac{U_m}{2} \phi_i \quad \text{and} \quad (14) \]

\[ \frac{d}{dt} + \mu + ik \left( U_m - \frac{\beta}{K^2 + 2F} \right) \phi_c = -ik \frac{U_s}{2} \left( \frac{K^2 - 2F}{K^2 + 2F} \right) \phi_i, \quad (15) \]

These two equations illustrate the following properties: first, the decay parameter \( \mu \) acts equally strongly on both the barotropic and baroclinic modes; second, in the absence of any coupling of the modes the two waves propagate at the barotropic and baroclinic phase speeds \( c_i = \beta/K^2 \) and \( c_c = \beta/(K^2 + 2F) \), respectively; third, the effect of the constant mean flow \( U_m \) is simply to alter these phase speeds, and without loss of generality we can take \( U_m = 0 \); and fourth, and most importantly, it is the baroclinic shear \( U_s \) that couples the interactions between the barotropic and baroclinic modes.

The two evolution equations, (14) and (15), can be combined to yield one equation in terms of the barotropic streamfunction,

\[ U_s \frac{d^2 \phi_i}{dt^2} + \left[ ((c_i + c_c)ik - 2\mu)U_s - \frac{dU_s}{dt} \right] \frac{d\phi_i}{dt} \\
+ \left\{ ((c_i + c_c)ik\mu - c_ic_c + \mu^2)U_s + \frac{U_s^3k^2}{4} \left( \frac{K^2 - 2F}{K^2 + 2F} \right) \right\} \phi_i = 0. \quad (16) \]

To factor out the mean flow and the Rayleigh-damping term we substitute the following decomposition into the above equation:

\[ \phi_i = e^{ik(c_i + c_c)t + \mu t} z(t), \quad (17) \]

and to obtain the following evolution equation for a modified barotropic streamfunction \( z(t) \):

\[ U_s \frac{d^2 z}{dt^2} - \frac{dU_s}{dt} \frac{dz}{dt} + \frac{U_s^3k^2}{4} \left( \frac{K^2 - 2F}{K^2 + 2F} \right) + (c_i - c_c)^2 k^2 \\
+ 2i(k(c_i - c_c) \frac{dU_s}{dt}) \right] z = 0. \quad (18) \]

Combining Eqs. (2.9a) and (2.9b) of Pedlosky and Thomson (2003) yields a special case of this equation that determines the evolution of the linearized dynamics evaluated near the marginal stability curve. Equation (18) is valid for any time-dependent baroclinic shear that is uniform in the horizontal direction. Instead of our decomposition in Eq. (17), it is possible to choose another one that would completely eliminate the term proportional to \( dz/dt \); however, the equation is not anymore insightful and is slightly more complicated.

3. The Kubo oscillator

Physical, biological, and chemical processes naturally have time-dependent variations that arise because of the various deterministic forces at work. This can result in highly irregular behavior that is difficult, if not impossible, to predict using a deterministic approach. In such cases, it is advantageous to approximate the irregularity using a stochastic process. Within this methodology there is no attempt to reproduce the behavior exactly. Instead, attempts are made to find qualitatively similar realizations. This approach of parameterizing complex flows can be very useful in understanding the nature of a particular system, without knowing the finer details of the irregularities.

There are many different ways to incorporate stochasticity into a physical model, and care should be taken in determining which process is appropriate. The most common choice is Gaussian white noise (Gardiner 2004). This is unphysical in some ways because it has zero correlation time and is unbounded. Because we are interested in motions that are aperiodic, but still wander between positive and negative values, we choose a noise that is bounded, oscillatory, and tends to a sinusoid in one limit. In particular, we pick the baroclinic shear in our model to be represented by the Kubo oscillator. This implicitly sets the winds that thereby generate aperiodic baroclinic shear. In Poulin and Flierl (2008), we studied the stochastic Mathieu equation with four different types of stochasticity, all of which gave qualitatively similar results. Here, we restrict our attention to the Kubo oscillator, but many other types of noise would yield similar results.

One way to define the Kubo oscillator (Kubo et al. 1985) is that it is the real part of the solution to the following stochastic differential equation:

\[ \frac{d\xi}{dt} = i[\Omega + \sigma \eta(t)]\xi, \quad (19) \]

where \( \eta(t) \) is Gaussian white noise with the following statistics:
\[ (\eta(t)) = 0, \quad \text{and} \quad (\eta(t) \eta(t - \tau)) = \delta(\tau). \] (20)

The real constant \( \Omega \) is the mean frequency while \( \sigma \) is a parameter that determines the strength of the stochasticity. From Eq. (19) it is clear that \( \sigma \) controls the amount of phase diffusion that exists in the Kubo oscillator (Talkner et al. 2005; Gleeson 2006); when \( \sigma = 0 \) the phase is fixed, but for nonzero values the phase can and will wander in time. An alternative interpretation is that \( \sigma \) is inversely proportional to the correlation time. [Equation (2.24) of Poulin and Flierl (2008) gives an expression for the correlation time of the Kubo oscillator but it is not possible to evaluate it explicitly.]

Formally integrating Eq. (19) yields a weak form of the equation and an alternative definition of the Kubo oscillator,

\[ \xi(t) = \cos\left[\Omega t + \sigma \int_0^t \eta(s) \, ds\right]. \] (21)

where the integral of \( \eta(t) \) is a Wiener process (Gardiner 2004). We use this expression to generate the stochastic process in our numerical simulations because it is easier to evaluate. Equation (21) indicates that in the limit of vanishing \( \sigma \) we have a simple cosine function; otherwise, we have a stochastic process that has a mean frequency of \( \Omega \). Figure 1 plots three aperiodic realizations of the Kubo oscillator in comparison to the periodic limit. It is clear that as \( \sigma \) increases the solution becomes more irregular. In the limit, as \( \sigma \rightarrow \infty \) we obtain a bounded white noise with a nonuniform distribution. By computing the power spectrum of the Kubo oscillator, we observe that for \( \sigma = 0 \) the spectrum is a delta function, whereas for positive values of \( \sigma \) the spectrum has a finite width that broadens with increasing \( \sigma \). Sample plots of the spectrum can be found in Poulin and Flierl (2008).

Thus, we take the nondimensional form of the time-dependent basic state to be

\[ U_s(t) = \beta F \left[ g_0 + h_0 \cos(\Omega t + \sigma \int_0^t \eta(s) \, ds) \right]. \] (22)

We have defined \( g_0 \) and \( h_0 \) to be the mean component of the shear and the amplitude of the variations, respectively. This nondimensionalization does not apply to the \( f \)-plane limit because there \( \beta = 0 \). In that regime, there are a multitude of scales that could be used. In the final section when we look at baroclinic shear on the \( f \) plane, we will assume that we use the previous equation, except where \( \beta/F \) is replaced by the maximal velocity of the shear.

4. The stability of aperiodic systems

It is a challenging task to understand the evolution of an aperiodic, or stochastic, system. In the two extreme cases of periodic and white noise variations, it sometimes is possible to use analytical methods to determine approximate behavior. However, in the more interesting case, with moderate levels of stochasticity, numerical methods are almost always the only option available. In later sections, we explain what particular methods we have used to numerically integrate the system of equations. In this section, we discuss how we define the stability of a system. First, we review the Lyapunov exponent and how we use it to determine the growth rates of a numerical solution. Second, we compare finding the stability of a particular solution and the ensemble of many solutions.

Lyapunov exponent

To determine the stability of a system it is necessary to choose a metric by which to quantify the growth in the system. One metric that can be applied to virtually any dynamical system is the Lyapunov exponent (Coddington and Levinson 1955). Farrell and Ioannou (1996a,b) explain how this is the ideal measure for determining the stability of any system, whether it is autonomous or nonautonomous. In particular, we compute the Lyapunov exponent of a solution for a given initial condition and a particular realization of the Kubo oscillator. The precise formula is

\[ \lambda = \lim_{t \to \infty} \frac{\log \|q(t)\|}{t}, \] (23)

where \( \|q(t)\| \) is the norm of the solution, in our case the enstrophy, at a given time. When numerically integrating
a system of differential equations, as we will do, one cannot compute the Lyapunov exponent, because we cannot achieve the limit of time tending to infinity. Instead, we compute the quantity in Eq. (23) at finite time. This is referred to as a finite-time Lyapunov exponent (Shadden et al. 2005).

To get an accurate approximation for the Lyapunov exponent it is necessary to integrate our equations for very large time. One difficulty that can occur in the case of unstable systems is that the solutions grow too large to be represented easily on the computer. To resolve this issue we found it useful to renormalize the solution periodically. We do this by finding the magnitude that the solution grows in a given time, say the mean period $T$, and renormalize the solution by that amount. This ensures that at the start of the next interval the solution has a norm of one. If we integrate the solution for $N$ periods we have a sequence of growth rates, that is, $\gamma_1, \gamma_2, \ldots, \gamma_N$. To avoid calculating the product of these numbers directly, we use the following equivalent expression for the Lyapunov exponent:

$$\lambda = \frac{1}{N} \sum_{i=1}^{N} \frac{\log \gamma_i}{T} = \frac{\langle \log \gamma \rangle}{T}.$$ 

This states that the Lyapunov exponent is exactly the average of the Lyapunov exponent for each subinterval.

5. The parametric mode in baroclinic shear

Parametric instability (stability) is the name coined to describe instabilities (stabilities) that arise because of periodic variations in the basic state. Benjamin and Ursell (1954) first observed this phenomenon manifested in interesting spatial patterns that appear on the free surface of water, which is vertically oscillated up and down. Since then it has been observed in many different scenarios, one example of which is barotropic and baroclinic shear flows (Poulin et al. 2003; Pedlosky and Thomson 2003).

Farrell and Ioannou (1996b) have explained that the cause of parametric instability is due to a coupling of the nonnormal structure of the basic state, which causes transient growth, and time dependence. Most studies of parametric instabilities have focused on periodic time variations because of their simplicity. However, this type of instability can also arise in aperiodic systems (Farrell and Ioannou 1996b, 1999b; Poulin and Flierl 2008). In this section, we investigate the linear stability properties of aperiodic baroclinic shear flows to understand the stability characteristics of the parametric mode.

### a. The periodic limit

Pedlosky and Thomson (2003) studied the linear stability of periodic baroclinic shear and, furthermore, they performed a weakly nonlinear analysis to focus on basic states near marginal stability. They presented three figures that show how the qualitative nature of the solution changes with the amplitude and frequency of the shear. Also, Flierl and Pedlosky (2007) plotted the linear growth rates for $\beta = 20, F = 50$ to compare the stability of the different zonal wavenumbers. They found that the mode one wave is the most unstable.

To complement previous findings, we plot the growth rates of periodic baroclinic shear as a function of the amplitude of the oscillation and the period of the shear for different values of the mean baroclinic shear. We focus on the wavenumbers $k = \pi, l = \pi$ because the calculations that we did with respect to the other modes showed that they had lower growth rates if they were, in fact, unstable. This is not surprising considering the result on the $f$ plane stated that $K^2 < 2F$ is a necessary criteria for linear instability (Pedlosky 1987; Pedlosky and Thomson 2003).

In the case of steady shear flow, Eqs. (14) and (15) can be used to obtain the dispersion relation for the frequency $\omega$ of the allowable waves in the system,

$$\omega = k \left(\frac{\alpha + \frac{c}{2}}{2} \right) \pm k \sqrt{\left(\frac{\alpha - \frac{c}{2}}{2}\right)^2 - \frac{U_s^4}{4} \left(\frac{2F - K^2}{2F + K^2}\right)},$$

as is predicted from the classical theory (Pedlosky 1987). The plus (minus) sign corresponds to the barotropic (baroclinic) wave. Parametric instability, or resonance as it is often called, is a result of a resonant triad between two waves in the system with the same wavenumbers and the oscillatory basic state (Poulin et al. 2003). Clearly, if the wavenumbers are equal the three components satisfy the criteria that the wavenumbers sum to zero. The other requirement is that the frequencies of the three components also sum to zero. Equation (24) denotes the frequencies of the two waves. The plus and minus sign correspond to the barotropic and baroclinic waves, respectively. For there to be a resonance between the barotropic wave, the baroclinic wave, and the mean flow, we require that the frequency of the shear $\sigma$ satisfies the following resonance condition:

$$n\sigma = k \sqrt{(c_i - c_o)^2 - 4g_0^2 \left(\frac{2F - K^2}{2F + K^2}\right)},$$

where $n$ is a positive integer. In the previous equation we substituted in with the mean baroclinic shear $g_0$ for $U_s/2$. 

and shear velocities; the $x$ axis is the mean period $T = 2\pi/\Omega$, and the $y$ axis is $h_0$, which is essentially the magnitude of how much the baroclinic shear can vary from the mean. In all six plots, we have $\beta = 20$, $A = 20$, and $\mu = 0$ and the different values of the mean baroclinic shear are $g_0 = 0.0, 0.2, 0.4, 0.5, 0.6$, and $0.7$. The first subplot corresponds to the case of a baroclinic shear with zero mean and there are two Arnold instability tongues that are present (Arnold 1989). The first is the first subharmonic, which is the most unstable region and thus of the greatest interest. By looking at the location of the transition points in Fig. 2, we observe that the second unstable region, which is only apparent at $h_0 = 0.45$ because of the contour intervals chosen, is not the first harmonic but is actually the second subharmonic. There is no evidence that the first harmonic produces an instability tongue. This is consistent with the theory since the instability regions must stem from transition points; however, it is not necessary that each transition point gives rise to an instability tongue. There are of course an infinite number of other unstable modes, but their growth rates decay with the period and thus they are of much less interest than the first instability tongue.

These regions of instability are examples of a parametric instability; therefore, we refer to the unstable modes as parametric modes. As the amplitude of the oscillation increases the unstable region widens and the magnitude of the growth rates increase (Nayfeh and Mook 1995; Stoker 1950). This plot differs from those in Pedlosky and Thomson (2003) and Flierl and Pedlosky (2007) in that we have chosen the $x$ axis to be the period and not the frequency of the shear.

The second, third, and fourth subplots in Fig. 3 have $g_0 = 0.2, 0.4,$ and $0.5$, respectively. All are subcritical of the Charney–Stern criteria for baroclinic stability in the Phillips model. The second subplot denotes the first three subharmonics along with the first two harmonics. The third subplot consists of only the first two subharmonics and harmonics. The fourth subplot yields, as predicted in Fig. 2, that there are only three instability tongues that exist in this range of periods. One interesting feature in this case is that the growth rates of the instability tongues are all very similar, and they do not seem to have nearly as much of a tilt with increasing mean baroclinic shear $h_0$. These examples demonstrate that the effect of increasing $g_0$ is twofold: it increases the period of the instability tongues, as suggested in Fig. 2, and it also increases their growth rates. It is clear that as we move toward the right the growth rates of the tongues decreases; however, the third subplot shows that the growth rates in the different instability regions are more comparable than in the case of zero mean flow.
The fifth and sixth subplots have \( g_0 = 0.6 \) and 0.7 and are supercritical of the classical criteria for instability. This means that in the absence of any oscillations, the growth rate is nonzero and is invariant of the period (the \( x \) axis). These two plots indicate that the effect of periodic oscillations can be either to destabilize the flow, as we have already seen, or to stabilize the flow. This is apparent because in each of these plots there are regions

**Fig. 3.** Contour plots of the growth rate as a function of \( T \) (\( x \) axis) and \( h_0 \) (\( y \) axis), with \( \beta = 20, F = 20, \mu = 0, k = \pi, \) and \( l = \pi \). The plots are for six different values of the mean shear: \( g = 0.0, 0.2, 0.4, 0.5, 0.6, \) and 0.7. They demonstrate that periodic oscillations can either stabilize or destabilize the flow depending on the particular parameters.
where the growth rates are lower than the associated growth rate of the time-averaged state. The regions where the growth rates are essentially zero are illustrated in white in the two plots. These could be referred to as Arnold tongues of stability. The region of stability gets smaller as the mean baroclinic shear increases. The slivers of stability will always exist but they may occur at larger values of $h_0$.

### b. The aperiodic regime

To demonstrate that parametric instability arises in the context of baroclinic shear where the internal noise is the Kubo oscillator, Fig. 4 plots the finite-time Lyapunov exponents for the same parameters as in Fig. 3, but now we set $h_0 = 0.2$ and we consider a wide range of mean periods $T$ and phase diffusions $\sigma$. The integrations were done for 10 000 periods using the second-order Runge–Kutta method. This choice gave very accurate results to three significant digits in the periodic case. Because the growth rates in Fig. 4 range between whatever the extreme values are in the periodic case to zero, it is not possible to accurately resolve all of the growth rates. We recognize that the smaller the growth rates are the less accurate they may be; however, these calculations are sufficient to depict the stability regions of the parametric mode in an aperiodic environment.

The six different values of mean baroclinic shear are $g = 0.0, 0.2, 0.4, 0.5, 0.6$, and $0.7$. As is to be expected, in the limit as $\sigma = 0$, we recover the periodic limit as shown in Fig. 3. The plots for larger (smaller) values of $h_0$ are qualitatively similar; they simply have large (smaller) values of the growth rates.

In the four subcritical cases it is clear that the effect of stochasticity on the parametric mode is twofold. First, as in the stochastic Mathieu equation (Poulin and Flierl 2008), the growth rate of this parametric mode monotonically decreases as $\sigma$ increases. The existence of parametric instability is due to a resonance between two perturbation waves and the mean flow, as we already discussed in the previous section. When the mean flow is not exactly periodic there is a near-resonance scenario that can yield large growth; however, it is weaker than the growth that occurs in the resonant case. One interpretation of the periodic case is that the internal noise has a spectrum that is a delta function. In the aperiodic case, the spectrum possesses a finite amplitude and has a nonzero width. As long as the spectrum of the baroclinic shear has a nonzero component at the resonant frequency there is the potential for instability to occur; however, the broader the spectrum the lower the growth rates. When the spectrum becomes too broad, because of too-strong stochasticity, the growth rates of the parametric mode are negligible.

Second, as $\sigma$ increases the region of instability widens. This signifies that the stochasticity can destabilize modes that are otherwise stable in the periodic limit. This is again a result of the widening of the spectrum of the baroclinic shear with $\sigma$. Consider a particular choice of parameters that is slightly outside the Arnold instability tongue in the periodic limit. As the stochasticity increases sufficiently, the spectrum of the shear velocity broadens enough such that the power spectrum has a nonzero value at the resonant frequency. Because of this, a resonance occurs in the stochastic limit but not in the periodic one. The further the parameters are from the tongue the more difficult it is for the near resonance to contribute to yield a nonnegligible growth rate. The nature of the instability is still the same, which is why we classify all the instabilities in these regions as parametric modes.

The two supercritical cases also demonstrate how stochasticity can act as a stabilizing or destabilizing effect depending on the particular choice of parameters. We observe in the two plots that, as in the subcritical cases, the Arnold instability tongues widen and weaken as the stochasticity increases. But as well, we see that the stochasticity acts to decrease the size of the stability tongues. In these six plots, we find that as the stochasticity increases the growth rate varies less with the mean period of the baroclinic shear. There is still some variation but it is much weaker than in the periodic limit. In both cases, it is true that the growth of the stochastic states is below that of the time-averaged state. This suggests that an erratic baroclinic shear has a lower growth rate than the corresponding time-averaged state.

To better understand the dependency of the growth rate with stochasticity we did a series of calculations for $g_0 = 0.6$ and 0.7 for relatively large values of the phase diffusion: $\sigma = 2, 3, \ldots, 9, 10$. In Figs. 5a,c, we plot the growth rate of particular realizations as a function of period in solid lines and that of the corresponding steady state in dashed lines. Clearly, there is some variation between the different level of stochasticities. Figures 5b,d plot the average growth rate over all of the periods as a function of the parameter $\sigma$. The stronger supercritical case shows that for strong mean shear the average growth rate does not differ very much from the time-averaged state and that the averaged growth rate increases monotonically with $\sigma$. The case with $\sigma = 0.6$ illustrates that the weaker supercritical cases have average growth rates that have a nonmonotonic dependency on $\sigma$. For relatively small values of $\sigma$ the growth rates decrease but for larger values the averaged growth rate increases with period. These results suggest that it might be possible to parameterize the effect of random variation in baroclinic shear, based on the amount of irregularity.
6. Aperiodic baroclinic shear on an $f$ plane

The stochastic mode was discovered in Poulin and Flierl (2008) in a rather simple context, and it is of great interest to determine whether and how this mode manifests itself in a geophysical context. We take the first step in this direction by focusing our attention on aperiodic time-dependent baroclinic shear. In the Mathieu equation,
the stochastic mode appeared near a zero mean frequency. The analog in the Phillips model is to look for solutions on the $f$ plane because that is where the frequencies of the barotropic and baroclinic waves vanish. In the case of an $f$ plane, the solution to Eq. (18) is shown to follow:

$$
z = A_0 \exp \left( k \sqrt{\frac{2F - K^2}{2F + K^2}} \int_0^t U_s \frac{s}{2} \, dt \right) + B_0 \exp \left( -k \sqrt{\frac{2F - K^2}{2F + K^2}} \int_0^t U_s \frac{s}{2} \, dt \right),$$

for constants $A_0$ and $B_0$ that depend on the particular initial conditions. For the square root in the exponent to be real, it is necessary that $K^2 < 2F$. If this criterion is violated the solution oscillates and there is no possibility for growth or decay because the modulus of the solution is constant in time. Recall that this criterion is nothing more than the classical necessary condition for instability in a steady baroclinic shear (Pedlosky 1987; Pedlosky and Thomson 2003). The Lyapunov exponent of the potentially unstable case is

$$\pm k \sqrt{\frac{2F - K^2}{2F + K^2}} \lim_{t \to \infty} \frac{1}{t} \int_0^t U_s \frac{s}{2} \, dt,$$

where we must take the sign that yields a positive result.

There are three different possible types of baroclinic shears that can occur: the basic state can be steady, periodic, or aperiodic. Equation (24) signifies that for a basic state to be unstable it is necessary that the wave-number is sufficiently small. In the case of an $f$ plane, we find that the growth rate is

$$k \sqrt{\frac{2F - K^2}{2F + K^2}} \frac{|U_s|}{2}.$$  

Clearly this is what arises from Eq. (28) when the shear is constant in time. The second possibility is that the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5}
\caption{(a),(c) Plot of the growth rates as a function of the period for $\sigma = 2, 3, \ldots, 9, 10$ and $g_0 = 0.6$ and 0.7, respectively, in solid lines and the growth rate of the time-averaged state in dashed lines. (b),(d) Plot of the growth rate averaged over all the periods as a function of the stochasticity parameter $\sigma$. That there is a nonlinear relationship between the averaged growth rate and $\sigma$ is indicated in (b).}
\end{figure}
baroclinic shear is periodic. It has been shown in Flierl and Pedlosky (2007) that periodic baroclinic shear on an $f$ plane has growth rates that are equal to the growth rate of the time-averaged state. [Analogous results were obtained for barotropic shear flow and centrifugal instability in the absence of a mean shear in Poulin et al. (2003) and Rosenblat (1968), respectively.] This result also arises in the case of periodic shear flow in Eq. (28). Note that given our choice of variables the frequencies of the barotropic and baroclinic modes have the same magnitude but opposite signs. This is why parametric resonance occurs from looking at the difference between the two frequencies. The sum of the two frequencies is zero and does not yield a new resonance condition.

The final possibility is that the baroclinic shear is aperiodic. This is the one that is most realistic, but also the one that is most neglected in the literature because of its inherent complexity. Equation (28) indicates that the growth rate is proportional to the average of $U_s(t)/2$ for all time. Thus, the growth rate of any time-dependent baroclinic shear is equivalent to the growth rate of the time-averaged state. This suggests that for these types of flows it is not necessary to know the details of how the shear varies in time, it is sufficient to know the averaged behavior. This makes parameterizing aperiodic states on an $f$ plane relatively simple. However, in the presence of a $\beta$ plane, this result no longer applies and presumably these are the flows of greatest interest, which is what causes this result of limited applicability. Furthermore, the previous analysis indicates that by looking at the stability of a particular realization on an $f$ plane, the growth rates are zero and thus the stochastic mode is not present in the Phillips model for the particular scenario we have chosen.

Equation (27) presents an exact solution to the linearized equations. However, to determine this solution as a function of time it is necessary to generalize a particular realization of the baroclinic shear and then integrate this quantity over time. It is for this reason that we refer to this solution as semianalytic. We computed the numerical integration using a trapezoidal method, which is second order, and it gave qualitatively similar results to the higher-order Simpsons rule (Press et al. 2007). In Fig. 6, we plot a semilog plot of the finite-time Lyapunov exponent for the case of an aperiodic baroclinic shear that is an aperiodic Kubo oscillator. The trend is for the finite-time Lyapunov exponent to decrease but it does so very slowly. If the perturbations are large enough they will leave the linear regime and instead will be governed by the fully nonlinear dynamics, and there the system is subject to nonlinear instabilities. It is for this reason that in this case the Lyapunov exponent is not a very useful measure of the stability of the system.

7. Conclusions

We have studied stochastic baroclinic shear, in the context of the Phillips model, where the shear is the Kubo oscillator, to mimic realistic variations that occur in ocean currents such as the ACC. This extends the work of Poulin and Flierl (2008), which focused on the stochastic Mathieu equation, into a model relevant to geophysical fluid dynamics. The erratic variations can be interpreted as a result of irregular winds. We have studied aperiodic basic states by solving the linear stability problem. We have demonstrated that the parametric mode exists in aperiodic flows where its growth rates decrease with increasing stochasticity and the instability regions widen. Also, the Arnold stability tongues can be destabilized by aperiodic variations. The dual nature of stochasticity makes the problem rather complicated to parameterize.

Our results indicate that care must be taken in applying the stability criteria for periodic shears to those that are aperiodic because the periodic limit is exceptional, having the most extreme behavior. Highly erratic systems appear to have little dependency on the mean period of the baroclinic shear, but there is a strong dependency on the phase diffusion parameter and the mean shear. Future studies could try and establish a parameterization of the stability of aperiodic systems as a function of these two important parameters. Alternatively, it is clear that the power spectrum of the internal noise is
very important in determining the stability of an aperiodic system. As of yet, we have been unable to obtain explicit stability criteria directly in terms of the power spectrum of the shear; this would be desirable in future work.

In the case of an f plane, we obtained a semi-analytic solution for the perturbations with any time-dependent baroclinic shear that is uniform in the horizontal direction. The solution indicates that the growth rate of the aperiodic baroclinic shear is equal to that of the time-averaged state in the long time limit. This implies that the stochastic mode that was found in the Mathieu equation does not exist in this type of baroclinic shear; however, the Lyapunov exponent converges slowly, which is why the finite-time Lyapunov exponent is nonzero for all of the times we considered. Consequently, the solution is unbounded and thus, even though it is linearly stable in the strict sense, it is subject to nonlinear instabilities.

In the fully nonlinear simulations of periodic baroclinic shear, Flierl and Pedlosky (2007) found that a linearly stable basic state could become unstable if the amplitude of the initial perturbation is sufficiently large. This is because these periodic states are nonlinearly unstable (Holm et al. 1985; Swaters 2000). In a subsequent work (Poulin et al. 2009, manuscript submitted to J. Phys. Oceanogr.), we observe that any initial perturbation, no matter how small, grows in amplitude, and given enough time it will eventually become large enough in amplitude to enter the nonlinear regime. This is because the instances of transient growth can push the perturbation into the nonlinear regime, where nonlinear instabilities take effect. This asserts that aperiodic basic states can be more unstable than periodic ones even though the linear growth rates are smaller.

Our study here was idealized in several ways. We began with the Phillips model, focusing only on the linear dynamics, and we assumed that the shear was set by the Kubo oscillator. However, this relatively simple context illustrated clearly the qualitative effects of aperiodicity on the stability of baroclinic shear. It would be of great interest to analyze the dynamics of a more realistic aperiodic shear such as that occurring in the ACC. This would require detailed observations of how the shear changes in time. On the theoretical side, there are many possible directions in which this research can be advanced, some of which has already been mentioned. Other interesting avenues would be to consider shears with more continuous stratifications, and also shears that vary in the horizontal directions, because that would introduce the possibility of barotropic instabilities.

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