On the Estimation of the Kurtosis in Directional Sea States for Freak Wave Forecasting

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ABSTRACT

Based on Monte Carlo simulations of the nonlinear Schrödinger equation in two horizontal dimensions, the dependence of the kurtosis on the directional energy distribution of the initial conditions is examined. The parametric survey is carried out to obtain the behavior of the kurtosis as function of the Benjamin–Feir index and directional spread in directional sea states. As directional dispersion effect becomes significant, the kurtosis monotonically decreases in comparison with the unidirectional waves. A parameterization of the kurtosis estimated from directional spectra is proposed here; the error of the parameterization is at most 10%. The parameterization is verified against laboratory data, and good agreement is obtained.

1. Introduction

Freak waves are sometimes featured by a single and steep crest causing severe damage to offshore structures and ships. In the last decade, freak waves have become an important topic in engineering and science. Freak wave studies started in the late 1980s (Dean 1990), and high-order nonlinear effects on the freak waves were discussed in the early 1990s at several engineering conferences (Yasuda et al. 1992; Yasuda and Mori 1994). Evidence of freak wave generation in the real ocean was reported from field observations in the North Sea (Stansell et al. 2003; Guedes Soares et al. 2003), the Sea of Japan (Yasuda and Mori 1997; Yasuda et al. 1997), and the Gulf of Mexico (Guedes Soares et al. 2004). Owing to the many research efforts, the occurrence of freak waves, their mechanism, and detailed dynamic properties are now becoming clear (Trulsen and Dysthe 1997; Lavrenov 1998; Osborne et al. 2000; Onorato et al. 2001; Mori et al. 2002). The state of the art on freak waves was well summarized at the last three Rogue Wave Conferences, held in 2000, 2004, and 2008, and in a review by Dysthe et al. (2008) and Kharif et al. (2009).

Nowadays, it is accepted that several mechanisms are responsible for the formation of extreme waves, including freak waves. The first one is just a linear superposition of waves; in this case, the probability distribution for wave height in the limit of the narrowband approximation obeys a Rayleigh distribution (Longuet-Higgins 1952), and corrections due to finite spectral bandwidth have been obtained (Næss 1985; Boccotti 1989; Tayfun 1981). Wave crest statistics can be achieved by using the second-order theory developed by Longuet-Higgins (1963). In the narrowband approximation, the probability distribution for wave crests has been found in Tayfun (1980) (for finite bandwidth, see Fedele and Arena 2005). The second mechanism is the interaction of waves with currents: linear theory can explain the formation of extreme waves using ray theory. The statistical properties of the surface elevation as a function of the properties of the currents are, so far, unknown. The third possible mechanism is
related to crossing sea states (i.e., Onorato et al. 2006). The fourth mechanism, the one that will be mainly discussed here, concerns the generation of extreme waves as a result of modulational instabilities: that is, a four-wave quasi-resonant interaction process (e.g., Yasuda and Mori 1994; Onorato et al. 2001; Janssen 2003). This mechanism has been verified and becomes relevant for long crested waves. In the early 1990s, numerical and experimental studies have demonstrated that freaklike waves can be generated frequently without current, refraction, or diffraction (Stansberg 1990; Yasuda et al. 1992). The occurrence probability of a wave having a single steep crest generated by the third-order nonlinear interactions is considerably higher than the other mechanisms.

Freak wave generation is sometimes discussed in the context of the Benjamin–Feir (BF) instability (Yasuda et al. 1992; Onorato et al. 2001). Janssen (2003) found that the ratio between the steepness, a measure of the nonlinearity, and the spectral bandwidth, a measure of the dispersion, is an important parameter for determining the probability of finding a large wave. After Janssen, this ratio between nonlinearity and dispersion was named the Benjamin–Feir index (BFI); its relation to the kurtosis has been found in Janssen (2003) in the limit of large times and for narrowbanded spectra neglecting directional dispersion. The Benjamin–Feir index is defined as

$$\text{BFI} = \frac{\sqrt{2\epsilon}}{\delta_\omega}, \hspace{1cm} (1)$$

where $\epsilon$ and $\delta_\omega$ are the characteristic wave steepness and the typical bandwidth of frequency spectrum, respectively. Then, for narrowband, unidirectional waves the kurtosis $\mu_4$ becomes

$$\mu_4 - 3 = k_{40} = \frac{\pi}{\sqrt{3}} \text{BFI}^2, \hspace{1cm} (2)$$

where $k_{40}$ is the fourth-order cumulant of the surface elevation probability distribution, which is equal to kurtosis $-3$ ($k_{40} = \mu_4 - 3$). The final result is that the kurtosis depends on the square of the BFI. The role of the skewness in the wave height distribution is less important with respect to kurtosis. The skewness comes usually as a result of second-order corrections (bound modes) and is weakly affected by the dynamics of free waves (Onorato et al. 2005). In Mori and Janssen (2006) the formal relation between kurtosis and the maximum wave height and occurrence probability of a freak wave has been discussed. The kurtosis enters in the maximum wave height distribution function as a nonlinear correction to the Rayleigh distribution: when the kurtosis tends to 3, the expected Gaussian value, then the distribution function tends to the Rayleigh distribution. In this context, the changes in the kurtosis can be evaluated once the evolution of the wave spectrum is known (see Janssen 2003). This theoretical framework of freak wave prediction was verified by large-scale wave experiments for unidirectional waves and predicts well the spatial developments of the kurtosis, the wave height distribution, and the maximum wave height distribution in the wave tank (Mori et al. 2007; Petrova et al. 2007) and field observation (Petrova and Guedes Soares 2008). This is of course not a surprise because the kurtosis is a fourth-order moment of a probability density function (pdf); therefore, it is a measure of the importance of the tails of the distribution. Moreover, the fourth-order moment of surface elevation is related to a third-order nonlinear interactions (Longuet-Higgins 1963), providing consistency between the statistical and dynamical point of view. Thus, an accurate estimation of kurtosis is of paramount interest for freak wave prediction.

However, recent works based on numerical simulations of envelope equations and wave tank experiments have pointed out the influence of directional dispersion on the kurtosis evolution in deep water. Gramstad and Trulsen (2007) showed a decrease of kurtosis and maximum wave height for directional waves in comparison with unidirectional waves based on simulations with the modified nonlinear Schrödinger (NLS) equation. Waseda (2006) and Onorato et al. (2009a) investigated the monotonic decrease of kurtosis as directional effects become significant and the comparison of two different directional wave experiments was summarized in (Onorato et al. 2009b). These results show that the directional dispersion effects suppress the kurtosis enhancement in directional sea states. Therefore, a more general expression for the kurtosis in directional sea states is required. Although crossing sea states, consisting of a wind sea and a swell system, may be interesting examples of freak wave generation (Petrova and Guedes Soares 2009), we will restrict ourselves to single wave systems only.

The purpose of the present paper is to present an expression for the kurtosis, which is an extension of Eq. (2) and includes directional dispersion effects. This paper is organized as follows. In section 2, a summary of the theoretical background is given, which suggests the relevant parameters for a directional sea. In section 3, results of the Monte Carlo simulations using the cubic nonlinear Schrödinger equation with two-dimensional Gaussian directional spectra are described and, by introducing a natural extension of the Benjamin–Feir index to two dimensions, the kurtosis estimation formula for directional sea states is proposed. In section 4, the validity of the formula is examined by means of laboratory
data. Our results provide a practical tool to obtain useful information on extreme wave prediction in the ocean.

2. Summary of theory of kurtosis prediction in directional sea states

We briefly review the theory of kurtosis estimation following our previous work (Janssen 2003; Mori and Janssen 2006). Let us consider the potential flow of an ideal fluid of infinite depth. Coordinates are chosen in such a way that the undisturbed surface of the fluid coincides with the \( x = (x, y) \) plane. The \( z \) axis is pointed upward and the acceleration of gravity \( g \) is pointed in the negative \( z \) direction. The surface elevation \( \eta \) may be written in terms of a Fourier expansion as

\[
\eta(x,t) = \int_{-\infty}^{\infty} dk [a(k,t) + a^*(-k,t)]e^{ikx},
\]

where \( a(k,t) = \sqrt{(\omega/2g)}B(k,t) \) and \( B(k,t) \) is the normal variable; \( k \) is the wavenumber vector, \( k \) is its absolute value, and \( \omega(k) = \sqrt{gk} \) denotes the angular frequency following the dispersion relation of deep water gravity waves.

Zakharov (1968) obtained from the Hamilton equations an approximate evolution equation for the amplitude of the free surface gravity waves. To eliminate the effects of bound waves, he applied on \( B \) a canonical transformation of the type

\[
B = B(b, b^*),
\]

where \( b \) is the normal variable of the free gravity waves. The evolution equation for \( b \), called the Zakharov equation for free gravity waves, becomes

\[
\frac{\partial b}{\partial t} + i\omega_1 b = -i \int dk_2dk_3dk_4 T_{1234} b^* b_3 b_4 \delta_{1+2-3-4},
\]

where for brevity we have introduced the notation \( b_1 = b(k_1) \), etc., and the nonlinear transfer function \( T_{1234} \) is the coupling coefficient in the Zakharov equation [for its analytical form, see Krasitskii (1994)].

The second and the fourth moment of \( b(k) \) are introduced by

\[
B_{ij} = \langle b_i b^*_j \rangle = N_i \delta(k_i - k_j)
\]

and

\[
\langle b_k b_l^* b_m^* \rangle = B_{ij} B_{km} + B_{jm} B_{ik} + D_{jklm},
\]

where \( B_{ij} \) and \( D_{jklm} \) are the second and the fourth cumulant and \( N_i \) is the spectral action density. Application of the random phase approximation to the sixth cumulant and solving the evolution equation (18) of Janssen (2003) for the fourth cumulant \( D \), subject to the initial value \( D(t = 0) = 0 \), gives

\[
D_{i,j,k,l} = 2T_{i,j,k,l} \delta_{i+j-k-l} G(\Delta\omega, t) [N_i N_j (N_k + N_l)] - (N_i + N_j) N_k N_l t,
\]

where \( \Delta\omega = \omega_1 + \omega_2 - \omega_3 - \omega_4 \) and \( G(\Delta\omega, t) \) is the resonance function defined as

\[
G(\Delta\omega, t) = i \int_0^t d\tau \exp(i\Delta\omega(\tau - t)) = R_i(\Delta\omega, t) + iR_0(\Delta\omega, t),
\]

and

\[
R_i(\Delta\omega, t) = \frac{1 - \cos(\Delta\omega t)}{\Delta\omega},
\]

and

\[
R_0(\Delta\omega, t) = \frac{\sin(\Delta\omega t)}{\Delta\omega}.
\]

There are two limits of \( R_i(\Delta\omega, t) \):

\[
\lim_{t \to 0} R_i(\Delta\omega, t): \quad R_i(\Delta\omega, t) = t
\]

and

\[
\lim_{t \to \infty} R_i(\Delta\omega, t) = \pi \delta(\Delta\omega).
\]

Substitution of Eq. (8) into the evolution equation, Eq. (17) of Janssen (2003), gives the so-called kinetic equation for the action density \( N \):

\[
\frac{\partial N_i}{\partial t} = 4 \int dk_{234} \left| T_{1234} \right|^2 \delta_{i+1-2-3-4} R_i(\Delta\omega) \times [N_3 N_4 (N_1 + N_2) - N_i N_2 (N_3 + N_4)]
\]

This equation describes both resonant and nonresonant wave-wave interactions. The modulational instability is a quasi-resonant four-wave interaction process of Eq. (14) for which wavenumbers and frequencies satisfy the following conditions:

\[
k_1 + k_2 - k_3 - k_4 = 0
\]

and

\[
\omega(k_1) + \omega(k_2) - \omega(k_3) - \omega(k_4) \leq \epsilon^2.
\]

The standard kinetic equation derived by Hasselmann (1962) that describes the evolution of the wave spectrum due to exact resonances [see Eq. (13)] is formally only valid for large times, \( T_{XNL} = O(e^{-4\omega_0^2}) \). Equation (11)
also describes quasi-resonant interactions, and these occur on the much shorter time scale of the Benjamin–Feir instability, that is, on the BF time scale $T_{BF} = O(\epsilon^{-2} \omega_0^{-1})$.

Now, we turn our attention to the evolution of kurtosis for freak wave prediction. It turns out that the quasi-resonant interactions are responsible for deviations from normality. In Janssen (2003) the explicit relation between nonlinear interactions of free waves and the fourth-order moment of the surface elevation from Eq. (8) has been obtained. The result is

$$\kappa_{40} = \frac{(\eta^2)}{m_0^2} - 3 = \frac{12}{\sqrt{2} \pi m_0} \left[ d k_{1,2,3,4} T_{1,2,3,4} \sqrt{\omega_1 \omega_2 \omega_3 \omega_4} \delta_{1+2-3-4} R_c(\Delta \omega, \omega) N_1 N_2 N_3, \right]$$  \hspace{1cm} (17)

where $m_0$ is the variance of the surface elevation $\eta$, $g$ is the acceleration of gravity, and $R_c$ is the real part of the resonance function given by Eq. (10). Equation (17) is the exact form of kurtosis due to four-wave interactions. In the limit of large times $R_c \rightarrow \mathcal{P}/\Delta \omega$, where $\Delta \omega = \omega_1 + \omega_2 - \omega_3 - \omega_4$ and $\mathcal{P}$ denotes the principle value of the integral to avoid singularity in the integral. Therefore, the kurtosis is determined by both resonant and nonresonant interactions since it depends on $\mathcal{P}/\Delta \omega$.

The consequences of the general expression for the kurtosis are hard to comprehend because it involves the evaluation of a six-dimensional integral. One exception is the limit of a narrowband wave train. Mori and Janssen (2006) studied this case, but neglected directional dispersion. Assuming that the frequency spectrum $F(\omega)$ is frozen in time and has the Gaussian shape,

$$F(\omega) = \frac{m_0}{\sigma_\omega \sqrt{2\pi}} e^{-\nu^2/2},$$ \hspace{1cm} (18)

where $\nu = (\omega - \omega_0)/\sigma_\omega$ and $\sigma_\omega$ is the spectral bandwidth of Gaussian spectrum, the integral of Eq. (17) becomes for large times

$$\kappa_{40} = \frac{12 c^2}{\delta_\omega^2} \mathcal{P} \int \frac{d \nu_1 \nu_2 \nu_3 \nu_4}{(2\pi)^3} e^{-[\nu_1^2 + \nu_2^2 + \nu_3^2]} [\nu_1^2 - \nu_1^2] \left[ \nu_3^2 - \nu_3^2 \right],$$ \hspace{1cm} (19)

where $\delta_\omega = \sigma_\omega / \omega_0$ is the relative spectral bandwidth as defined in Eq. (1). The integral of Eq. (19) can be evaluated analytically and leads to Eq. (2) in the previous section. Equation (2) is the simplified prediction equation for the kurtosis of the surface elevation, assuming a narrowband frequency spectrum and unidirectional wave train. Equation (2) has been used and verified against numerical simulations and laboratory data.

Janssen and Bidlot (2009) included, in addition, directional effects and studied the kurtosis behavior of analytical short-term solutions and numerical long-term solutions, respectively. For two-dimensional propagation, $\omega_4$ in Eq. (16) becomes

$$\omega_4 = \left\{ (\omega_1^2 + \omega_2^2 - \omega_3^2)^2 + 2 \omega_1^2 \omega_2^2 [\cos(\theta_1 - \theta_2) - 1] \right. \nonumber$$

$$\left. - 2 \omega_1^2 \omega_2^2 [\cos(\theta_1 - \theta_3) - 1] \right\}^{1/4}.$$ \hspace{1cm} (19)

Denoting the directional width of the directional spectrum by $\sigma_\theta$, one may write the angular frequency and directional widths as $\delta_\omega$ and $\delta_\theta = \sigma_\theta$ in the narrowband approximation. The frequency mismatch $\Delta \omega$ in Eq. (17) becomes

$$\Delta \omega = \delta_\omega^2 \omega_0 [\nu_3 - \nu_1] (\nu_3 - \nu_2) \nonumber$$

$$- R(\phi_3 - \phi_1) (\phi_3 - \phi_2) + O(\delta_3),$$ \hspace{1cm} (20)

where $\phi$ is the normalized direction parameter $\phi = (\theta - \theta_0)/\sigma_\theta$ scaled by the directional width $\sigma_\theta$ while $R$ is a parameter that is a measure of the angular width with respect to the frequency width,

$$R = \frac{1}{2} \frac{\delta_\theta^2}{\delta_\omega^2}.$$ \hspace{1cm} (21)

In the narrowband approximation the transfer coefficient $T_{1,2,3,4}$ in Eq. (17) can be approximated by $k_{1,3,4}^2$. Janssen and Bidlot (2009) studied both short-time and long-time behavior of kurtosis by substituting Eq. (21) into Eq. (17). For short times a result is obtained that holds for general spectra. The kurtosis becomes

$$\kappa_{40} = \frac{3 \tau^2 \mathcal{B} \mathcal{F}^2}{\mathcal{L}} \int d \nu_1 \nu_2 d \phi_1 \square \{ (\nu_3 - \nu_1) (\nu_3 - \nu_2) \nonumber$$

$$\left. - R(\phi_3 - \phi_1) (\phi_3 - \phi_2) \right\} \tilde{E}_1 \tilde{E}_2 \tilde{E}_3,$$ \hspace{1cm} (22)

where $\tilde{E}$ is the frequency–direction spectrum normalized by the zeroth moment $m_0$ and $\tau = \omega_0 \delta_\omega^2 t$ is a dimensionless time. As can be seen from Eq. (23), the integrals are related to moments of the wave spectrum. Upon evaluation of the integrals, a simple relation for the kurtosis follows

$$\kappa_{40} = \frac{3 \tau^2 \mathcal{B} \mathcal{F}^2}{\mathcal{L}} (1 - R).$$ \hspace{1cm} (24)
This result shows that directional effects play an important role. We have a positive kurtosis that corresponds to nonlinear focusing when \( \delta_\theta < \sqrt{2}\delta_\omega \) while for large directional spread (\( \delta_\theta > \sqrt{2}\delta_\omega \)) we have a negative kurtosis. The condition \( R = 1 \rightarrow \delta_\theta = \sqrt{2}\delta_\omega \) corresponds to one of the boundaries of the instability diagram of the two-dimensional nonlinear Schrödinger equation (Alber and Saffman 1978).

The long-time behavior of the kurtosis can only be obtained for the special case that the two-dimensional spectrum is given by a Gaussian. Again making use of the approximate form of the frequency mismatch, Eq. (21), the large time behavior of the real part of the resonance function, and assuming that the wave spectrum has the Gaussian shape

\[
E(\omega, \theta) = \frac{m_0}{2\pi\sigma_\omega^4} e^{-\left(\frac{\omega^2 + \phi^2}{\sigma_\omega^2}\right)^2}
\]

with \( \phi = (\theta - \theta_0)/\sigma_\theta \), the kurtosis becomes for the two-dimensional case

\[
\kappa_{40} = \frac{6\text{BFI}^2}{(2\pi)^3} \int d\nu_1 d\nu_2 d\nu_3 (\nu_3 - \nu_1)(\nu_3 - \nu_2) - R(\phi_3 - \phi_1)(\phi_3 - \phi_2).
\]

The relevant six-dimensional integral has been numerically evaluated in appendix A of Janssen and Bidlot (2009). To a good approximation it is found that the kurtosis \( \kappa_{40} \) becomes

\[
\kappa_{40} = J(R)\text{BFI}^2,
\]

where, for \( R < 1 \),

\[
J(R) = \frac{3}{4\pi^2} \frac{1 - R}{R + R_0}
\]

with \( R_0 = 3\sqrt{3}/4\pi^3 \), whereas \( J(R) \) for \( R > 1 \) follows from the relation \( J(R) = -J(1/R)/R \) (long-time solution). Therefore, for large times the kurtosis depends on the square of the BFI and on the ratio of the directional width and frequency width through the parameter \( R \). Just as in the short-time limit, the kurtosis can be shown to vanish for \( R = 1 \), which is a remarkable property indeed. Based on these theoretical results, the following conjecture can be made: In reality, the spectrum will evolve in time because of the nonlinear interactions. If for large times equilibrium is achieved, then it is expected that the pdf of the surface elevation becomes close to a Gaussian: that is, kurtosis \( \kappa_{40} \) is expected to vanish. According to the present results the wave spectrum will then be reshaped in such a manner that \( R = 1 \) for large times. However, for freak wave forecasting, the interesting question is how large may kurtosis become at the intermediate stages. This question can only be answered by doing numerical simulations. Guided by these theoretical results, we will attempt to parameterize results for kurtosis obtained from a series of numerical simulations by assuming that \( \kappa_{40} \) depends on the BFI parameter and on the parameter \( R \), which measures the relative effect of directional width.

Once kurtosis is estimated accurately, a pdf of maximum wave height and occurrence probability of extreme waves can be obtained following the Mori and Janssen (2006) formulation:

\[
p_m(H_{\text{max}}) = \frac{N}{4} H_{\text{max}}^{-1/8} \left[ 1 + \kappa_{40} A_H(H_{\text{max}}) \right] \times \exp \left\{ -Ne^{-H_{\text{max}}/8} \left[ 1 + \kappa_{40} B_H(H_{\text{max}}) \right] \right\}
\]

and

\[
P_m(H_{\text{max}}) = 1 - \exp \left\{ -Ne^{-H_{\text{max}}/8} \left[ 1 + \kappa_{40} B_H(H_{\text{max}}) \right] \right\},
\]

where \( p_m \) and \( P_m \) are the pdf of maximum wave height and the exceedance probability of maximum wave height, respectively; \( H_{\text{max}} \) is the maximum wave height normalized by \( \eta_{\text{rms}} \), \( N \) is the number of waves (corresponding to duration of storm), and \( A_H \) and \( B_H \) are polynomials of \( H_{\text{max}} \). For \( \kappa_{40} = 0 \) the results are identical to the ones following from the Rayleigh distribution. If a freak wave is defined as a wave whose height \( H_{\text{freak}} \geq 2H_{\text{rms}}(=8\eta_{\text{rms}}) \), then, integrating Eq. (30), we obtain the following simple formula to predict the occurrence probability of a freak wave as functions of \( N \) and \( \kappa_{40} \):

\[
P_{\text{freak}} = 1 - \exp\left[ -\beta N (1 + 8\kappa_{40}) \right],
\]

where \( \beta = e^{-8} \) is constant. In Mori and Janssen (2006), comparisons of the theoretical distributions with field data have shown qualitative agreement. The term \( 8\kappa_{40} \) in Eq. (31) is the nonlinear correction to linear theory for the maximum wave height distribution. Thus, nonlinear corrections to maximum wave height depend on kurtosis, and therefore the kurtosis estimation from the
spectra is a key element in predicting freak waves on the ocean surface.

3. Numerical simulation of kurtosis change in directional sea states

a. Numerical method and conditions

For a weakly nonlinear narrowband wave train, the two-dimensional NLS (NLS2D) equation can be derived from the Euler equations assuming potential flow. The NLS2D has the following form:

\[
l\left( \frac{\partial A}{\partial t} + \frac{\omega_0}{2k_0} \frac{\partial A}{\partial x} \right) - \frac{\omega_0}{8k_0^2} \frac{\partial^2 A}{\partial x^2} + \frac{\omega_0}{4k_0^3} \frac{\partial^2 A}{\partial y^2} - \frac{1}{2} \omega_0 k_0^2 |A|^2 A = 0,
\]

where \(k_0\) and \(\omega_0\) are the principal wavenumber and the angular frequency, respectively, and \(A\) is the wave envelope and is related at the leading order to the free surface as follows:

\[
\eta(x, y, t) = \text{Re}\{A(x, y, t) \exp[i(k_0 x - \omega_0 t)]\}. \tag{33}
\]

The NLS2D is valid on the Benjamin–Feir time scale, \(t \ll O(e^{-2\omega_0^{-1}})\), whereas the higher-order NLS2D is valid on a longer time scale than the original NLS2D (Dysthe et al. 2003). Our aim here is to investigate the formation of extreme waves on the time scale of the Benjamin–Feir instability for the weakly nonlinear, directional sea states; therefore, to reduce computational costs, we have used the NLS2D equation.

A series of runs has been carried out to obtain statistically stable relations for kurtosis with different initial conditions. The computational domain was set to \(128L_0 \times 128L_0\) in the \(x\) and \(y\) directions, where \(L_0\) is the characteristic wavelength. Wave numbers are normalized by \(k_0\) in both the \(k_x\) and \(k_y\) directions. The initial condition for the simulations is provided by the two-dimensional Gaussian spectrum given in Eq. (25), which for completeness is written here explicitly:

\[
E(\omega, \theta) = \frac{m_0}{2\pi \sigma_\omega \sigma_\theta} \exp\left\{ -\frac{1}{2} \left[ \left( \frac{\omega - \omega_0}{\sigma_\omega} \right)^2 + \left( \frac{\theta - \theta_0}{\sigma_\theta} \right)^2 \right] \right\}, \tag{34}
\]

where \(\theta\) is direction, \(\sigma_\omega\) and \(\sigma_\theta\) are a measure of spectral width in \(\omega\) and \(\theta\), and \(\theta_0\) is the principal wave direction. Note that, in general, the directional distribution depends on both direction \(\theta\) and frequency \(\omega\) but here the simplifying assumption is made that it is independent of frequency. This assumption is valid around the peak frequency at which most of the wave variance is to be found. To build the surface elevation, the phases are taken as uniformly distributed in the interval \([0, 2\pi]\). The initial wave steepness \(\epsilon = k_p \sqrt{\overline{m_0^1}}\) was fixed at 0.07 to ensure a weakly nonlinear condition for the NLS equation and, without loss of generality, the principal wave direction was fixed at \(0 (\theta_0 = 0)\).

Note that the normalized spectral bandwidths can be related to the already introduced parameters \(\delta_\omega\) and \(\delta_\theta\). First, the frequency spectrum bandwidth is given by \(\delta_\omega = \sigma_\omega/\omega_0\). Furthermore, there is a relation between \(\sigma_\omega\) and Goda’s spectral bandwidth \(Q_p\), and there is also a relation between \(Q_p\) and the JONSWAP spectral shape parameter \(\gamma\). One finds \(Q_p = 1/(\sqrt{2\pi} \sigma)\), while numerically it is found that \(Q_p = -0.015\gamma^2 + 0.60\gamma + 1.37\). From these relations one finds that \(\gamma = 1.0\) and 3.5 of the Joint North Sea Wave Project (JONSWAP) spectrum correspond to \(\sigma_\omega = 0.29\) and 0.17, respectively. Second, using the relation of \(E(\omega, \theta) = F(\omega)D(\theta)\), where the directional distribution \(D(\theta)\) is normalized to one, the directional width parameter \(\sigma_\theta\) is found to be equal to the directional spread \(s\) of Eq. (34). Here, \(s\) is defined as

\[
s = \sqrt{\int d\theta \theta^2 D(\theta)}.\]

Therefore, \(\sigma_\theta\) of Eq. (34) is equal to the standard directional spread and is measured in radians.

Finally, for a Mitsuyasu-type \(\cos^2\theta/2\) directional distribution, the directional spread can be expressed in terms of the power \(2S\) of the cosine function as \(\sigma_\theta = \sqrt{2/(1 + S)}\).

Hence, \(\sigma_\theta\) directional spread \(s\) and \(S\) of the \(\cos^2\theta/2\) distribution can be interchanged. The observed directional spread by buoys at the peak frequency is for swell about \(10^\circ (0.17\text{ rad})\) while for windsea it is \(30^\circ (0.52\text{ rad})\) (Ewans 1998; Forristall and Ewans 1998). Banner and Young (1994) indicated that the directional spread decreases slightly as wave age increases. Thus, the directional spread \(\sigma_\theta\) ranges from 0.15 to 0.5 depending on the sea state.

Numerical simulations were performed for different values of \(\sigma_\omega\) and \(\sigma_\theta\) (BFI and \(\sigma_\theta\) ranged from 0.0 to 1.0 with a step of 0.05, respectively). The initial wave steepness \(k_p \sqrt{\overline{m_0}}\) was fixed at 0.07; therefore, BFI = 0.05 and 1.0 correspond to \(\sigma_\omega = 2.0\) and 0.1, respectively. For each initial wave spectrum 500 runs were performed while making a random draw of the initial phases. The sensitivity of the results to ensemble size \(M\) was verified by changing the size of the ensemble from 50 to 500. However, results were found to be insensitive to ensemble size for \(M\) larger than 200. With an ensemble size of 500 the total number of simulations was 200 000.
b. Parameterization of kurtosis based on MCNLS

Figure 1 shows the evolution in time of the ensemble-averaged kurtosis up to 10 dominant wave periods. The initial growth of ensemble-averaged kurtosis on the short time scale is predicted by Eq. (24). The solid lines and dashed line in the figure are the time-averaged kurtosis by the Monte Carlo simulation using the 2DNLS (MCNLS) and predicted value by Eq. (24), respectively. Note that Eq. (24) is only valid for a very short time scale on the order of a few wave periods. It is seen from the figure that, for narrow directional spectra, Eq. (24) gives good agreement with the ensemble-averaged numerical results. However, the agreement between ensemble-averaged values and Eq. (24) is less favorable for large directional width, as the numerical results never fall below \(-0.1\). However, the parameter \(R\) in Eq. (24), which depends both on frequency and directional bandwidth, seems to give the appropriate direction of kurtosis change.

Figure 2 shows examples of the temporal evolution of the spectral bandwidth of the wavenumber spectrum. We define characteristic spectral bandwidth parameters both \(x\) and \(y\) directions as mean-square slopes,

\[
\sigma_x = \frac{1}{m_0^x} \int k_x^2 E(k_x, k_0^0) \, dk_x
\]

and

\[
\sigma_y = \frac{1}{m_0^y} \int k_y^2 E(k_x^0, k_y) \, dk_y,
\]

where \(\sigma_x\) and \(\sigma_y\) are wavenumber spectral bandwidth parameters of principal and cross directions; \(k_x^0\) and \(k_y^0\) are the peak wavenumber vector \([k_x^0, k_y^0]\); and \(m_0^x\) and \(m_0^y\) are integrals of the wavenumber spectrum in the \(k_x\) and \(k_y\) direction, respectively. Basically, these spectral bandwidth parameters are equivalent to mean-square slope along two directions. Both instantaneous spectral bandwidth parameters increase monotonically as function of time. However, although the \(\sigma_x\) remains more or less constant, the instantaneous cross-directional spectra bandwidth \(\sigma_y\) increases, by comparison, significantly. For the broader directional case \((\sigma_0 = 0.10)\), both \(\sigma_x\) and \(\sigma_y\) are stable in comparison with the directionally narrowband case \((\sigma_0 = 0.05)\).

Figure 3 shows the temporal evolution of the instantaneous kurtosis with the same conditions as in Fig. 2. For small BFI (see Fig. 3a), the kurtosis changes are

\[
\begin{align*}
\text{FIG. 1. Initial time evolution of ensemble-averaged kurtosis:} & \text{ instantaneous kurtosis from wave profile (solid lines) and estimated kurtosis by Eq. (24) (dashed lines) for BFI = 1.0; } \\
& \sigma_0 = 0.10 \text{ (circles), } \sigma_0 = 0.15 \text{ (triangles), and } \sigma_0 = 0.25 \text{ (squares).}
\end{align*}
\]

\[
\begin{align*}
\text{FIG. 2. Time evolution of directional spread of an instantaneous} & \text{ spectrum: } \sigma_x \text{ of wavenumber spectrum bandwidth of the} \\
& \text{principal direction (solid symbols) and } \sigma_y \text{ of wavenumber spectrum bandwidth of the cross direction (open symbols).} \ (a) \text{ BFI = 0.5 and (b) BFI = 1.0.}
\end{align*}
\]
smaller than for the large BFI condition in Fig. 3b. This is consistent with previous results obtained for unidirectional waves. The kurtosis significantly increases on a time scale of 10–40 periods and then decreases. Similar results are observed for unidirectional wave simulations. The difference between $\sigma_0 = 0.05$ and $\sigma_0 = 0.10$ is significant for BFI = 1.0, but it is smaller for BFI = 0.5. In the same figure, we show the evolution of the kurtosis calculated from Eq. (2) directly from instantaneous spectral bandwidth in the principal direction defined by Eq. (35) as a practical point of view. It should be mentioned that a corresponding equation has been derived for the one-dimensional case as a large times limit, assuming that the spectrum does not evolve in time and the initial normality condition. However, the figure shows the similar temporal changes of kurtosis, although there is a time shift between them. This result implies that the small deviation from the normality, which corresponds to weak nonlinearity, allows us to use Eq. (2).

Also, although the instantaneous value of BFI is related to the instantaneous value of kurtosis, there is a time lag $O(\epsilon^{-2} \omega_0^{-1})$. It is negligible for wave forecasts scales which are $O(10^2$ – $10^3$ km$^2$) but is not negligible for wave tank experiments.

Figure 4 shows the maximum value of the kurtosis obtained from MCNLS as a function of BFI and $\sigma_0$ [a similar plot is shown in Gramstad and Trulsen (2007), who used a different choice of variables]. As the directional dispersion effects become significant ($\sigma_0 \rightarrow 1$), the kurtosis decreases. The reduction of kurtosis is monotonic as $\sigma_0$ increases. Similar results were obtained numerically by Gramstad and Trulsen (2007) and in the laboratory by Waseda (2006) and Onorato et al. (2009a). Theoretically, a straightforward dimensional analysis of NLS2D gives dispersion effects on kurtosis change (i.e., Alber and Saffman 1978; Waseda et al. 2009). However, the asymptotic behavior of BFI/kurtosis, $\sigma_0 \rightarrow$ large, is different from the results of the MCNLS as shown in Fig. 4. The reduction of BFI/kurtosis due to directional dispersion cannot be expected from this analysis. Now, we have explicit relations of kurtosis as functions of BFI and directional spread in comparison with previous studies.

From Fig. 4, it follows that there is a definite relation between the maximum of kurtosis and parameters such as the BFI and the direction spread, and the question is whether it is possible to estimate the expected kurtosis using an extension of the BFI parameter introduced by Janssen (2003). Based on the theoretical expression of Eq. (27) and (28), the reduction of kurtosis by directional dispersion can be expected monotonically as a function of directional spread. In a straightforward manner, a fit for maximum kurtosis as function of BFI and directional spread can be found using

FIG. 3. Time evolution of kurtosis: instantaneous kurtosis from wave profile (solid lines) and estimated kurtosis by Eq. (2) from wave spectra (dashed lines).

FIG. 4. The ensemble-averaged maximum value of kurtosis dependence on BFI and $\sigma_0$ by MCNLS for multidirectional waves.
where \( a_1 \) is the empirical coefficient obtained by adjusting Eq. (37) to the results of Fig. 4. Thus, the kurtosis is expected to be inversely proportional to \( \sigma_\theta \), as shown in Fig. 5. The rms error between the kurtosis simulated by the MCNLS and estimated by Eq. (37) is 0.0169; hence, Eq. (37) gives the expected maximum kurtosis from BFI and directional spread with reasonable accuracy. However, there is no theoretical support for the inverse dependence of kurtosis on directional spread in this simple manner. In addition, in the limit of small directional width, \( \sigma_\theta \to 0 \), Eq. (37) does not approach the unidirectional result given in Eq. (2).

Returning to the analytical formulation of the evolution of kurtosis by Eqs. (24) and (27), we know that in the narrowband approximation the kurtosis depends on two dimensionless parameters (viz., BFI and \( R \)). Based on this, the following “natural” extension of Eq. (2) is suggested:

\[
\kappa_{40} = \frac{\pi}{\sqrt{3}} BFI^2 \left( \frac{\alpha_1}{\sigma_\theta} \right)
\]

and

\[
= \frac{2\pi^2}{\sqrt{3}} e^{2} \phi^2 \left( \frac{\alpha_1}{\sigma_\theta} \right),
\]

where \( \alpha_1 \) is the empirical coefficient obtained by adjusting Eq. (37) to the results of Fig. 4. Thus, the kurtosis is expected to be inversely proportional to \( \sigma_\theta \), as shown in Fig. 5. The rms error between the kurtosis simulated by the MCNLS and estimated by Eq. (37) is 0.0169; hence, Eq. (37) gives the expected maximum kurtosis from BFI and directional spread with reasonable accuracy. However, there is no theoretical support for the inverse dependence of kurtosis on directional spread in this simple manner. In addition, in the limit of small directional width, \( \sigma_\theta \to 0 \), Eq. (37) does not approach the unidirectional result given in Eq. (2).

Fig. 5. The empirical fit and error of simulated maximum kurtosis from Eq. (37).

\[
k_{40} = \frac{\pi}{\sqrt{3}} \times BFI^2 \left( \frac{\alpha_1}{\sigma_\theta} \right)
\]

and, based on a fit to the data of Fig. 4, the constant \( \alpha_1 \) is found to be \( \alpha_1 = 7.10 \). Using the definitions of BFI and \( R \) the effective BFI may be written as \( BFI_{2D} = e^{2} \phi^2 \frac{\delta^2}{\sigma_\theta} + \alpha_2 \sigma_\theta^2/2 \). Compared to Mori and Janssen (2006), this is an extension of the BFI by including directional effects in the width parameter. Therefore, in agreement with Fig. 4, the parameterization given in Eq. (39) shows a monotonic decrease of kurtosis with increasing directional width \( \delta_\theta \). The empirical curve of Eq. (39) and the difference with the data from Fig. 4 is shown in Fig. 6. It is seen that Eq. (39) gives the numerically simulated maximum kurtosis with reasonable accuracy. As shown in Fig. 6, the maximum difference between the kurtosis simulated by the MCNLS and empirical curve is less than 0.1 and corresponds to an error of \( \sim 3\% \) on average and 10% error at most. Therefore, for practical purposes, the empirical curve represents a useful estimation of the kurtosis from directional spectra. Finally, note that, by comparing Figs. 5 and
6 with Fig. 4, it is seen that Eq. (39) provides a better fit to the numerical data than Eq. (37). This is to be expected, as Eq. (39) follows from a straightforward dimensional analysis of the NLS2D equation.

The proposed empirical formula for the kurtosis estimation in directional sea state is expected to predict maximum kurtosis in a stationary, homogeneous, and weakly nonlinear sea state such as in a wave tank experiment. However, the validation of the equations for the real ocean will be more difficult due to the nonstationary behavior of kurtosis and ocean itself. The validity of the empirical formula will be examined in the next section in more detail.

Figures 7 and 8 show temporal evolutions of the ensemble-averaged kurtosis by the MCNLS as a function of BFI and $\sigma_u$, respectively. Note that the results in Fig. 4 are obtained by taking the maximum value of the ensemble-averaged data of Figs. 7 and 8. Similar as in the case of the maximum value, the time evolution of the ensemble-averaged kurtosis depends linearly on BFI and is inversely proportional to $\sigma_u$. These relations not only can be observed at the maximum value, but they also appear through most of the evolution of the sea state. The kurtosis approaches the Gaussian value ($\kappa_{40} = 0$) in the extreme case of BFI $\rightarrow 0$, $\sigma_u \rightarrow 1$, and $t/T \rightarrow \infty$. Therefore, the period for which kurtosis is positive, corresponding to deviations from normality, is limited and the length of this excitation period is probably on the order of the BF time scale.

As indicated in the above figures, the kurtosis changes in time. To verify the kurtosis estimation formula, Eq. (39), by means of field data, it is noted that it is difficult to observe such an evolution as usually only time-averaged characteristics can be obtained from the field observations. Figure 9 shows the mean and standard deviation of the kurtosis obtained from the MCNLS as functions of BFI and $\sigma_u$. It is seen that the standard deviation of the kurtosis is similar in shape and magnitude as the mean of the kurtosis. As a consequence, the kurtosis can change significantly in the course of time. On the other hand, the mean of kurtosis is much smaller than its maximum value (cf. Figs. 6, 9). These rapid temporal features of the kurtosis make verification of the theory by field observations very difficult.

From the practical point of view, the directional width of the spectrum depends on wave age (e.g., Babanin and Soloviev 1998). The directional distribution for old sea is much narrower than that of young sea. On the other hand, the wave steepness of young sea is greater than that of old sea. Therefore, the combination of the wave steepness, frequency, and directional bandwidth are important aspects of kurtosis estimation using Eq. (39). These aspects are subject to further study.

4. Validation of the formulation

We use two different series of experimental data for validation of the maximum kurtosis estimation. One in Norway, at the Norwegian Marine Technology Research Institute (MARINTEK) ocean basin (Onorato et al. 2009a) and the other in Japan at the Institute of Industrial Science (ISI), University of Tokyo (Waseda et al. 2009). The MARINTEK basin is one of the largest wave basins in the world; it is 50 m long and 70 m wide with an adjustable depth of 10 m maximum. The experiments in
the University of Tokyo were conducted in a facility 50 m long, 10 m wide, and 5 m deep with a segmented plunger type directional wave maker equipped with 32 plungers. Both wave basins have the same length and two different series of directional wave experiments were conducted individually. The last 4000 waves have been collected for each experiment. The details of experiments are summarized in Onorato et al. (2009a) and Waseda et al. (2009), and two experiments are directly compared in Onorato et al. (2009b).

The validation of kurtosis parameterization is conducted in comparison with the experimental data. Figure 10 shows the maximum value of the kurtosis along the tank for different directional spreading from the two different wave experiments, MCNLS, and the empirical formula, respectively. The thick solid, dashed, dotted–dashed, thin solid, and dotted lines are the results of MCNLS2D, the empirical formula given by Eq. (39), the empirical plus bound mode effects, the large time solution plus bound mode effects, Eq. (27) + Eq. (41), (thin solid line); and unidirectional theory, Eq. (1), (dotted line). Solid dots and crosses are experimental data.

The equilibrium value of kurtosis, the large time solution, according to the theory of Eq. (27) in section 2, is shown. Effects of bound waves were estimated using

$$\mu_4^{\text{bound}} = 24c^2 + 3,$$  \hspace{1cm} (41)

which follows from Eq. (34) of Mori and Janssen (2006) using the expression for the Stokes wave, which is approximately valid for narrowband random waves. As shown in Fig. 10, in the case of BFI = 0.7 kurtosis increases as $\sigma_0$ further decreases, although the kurtosis remains more or less constant for $\sigma_0 \approx 0.2$ rad. As already shown above, the extreme waves are more probable for the case of narrowband directional waves; the corresponding kurtosis based on Eq. (1) is $\mu_4 = 3.88$ for the unidirectional case of BFI = 0.7. Overall, the MCNLS, parameterization [Eq. (39)], and large time solution [Eq. (27)] show qualitatively good agreement with the experimental data. However, several remarks have to be made regarding this comparison. First, both the large time solution and empirical formula with the bound mode effects in the range of $\sigma_0 > 0.2$, the large directional dispersion case, show good agreement with the experimental data. Second, the empirical formula with the bound mode effects in the range of $\sigma_0 < 0.1$, the small directional dispersion case, overestimates the...
observed kurtosis, although the large time solution, the empirical formula, and the MCNLS results show fairly good agreement with the experimental data. Third, the long-term solution increases faster than the others for the narrowband limit of $\sigma_0 \rightarrow 0$. For the large directional dispersion case, only bound mode effects contribute to the kurtosis change. The time scale of bound mode effects is on the order of the carrier wave period; therefore, the experimental data is statistically stable. On the other hand, four-wave interactions become significant for the small directional dispersion case. The time scale of four-wave interactions is the Benjamin–Feir time scale; therefore, the experimental data is statistically sensitive in this range.

There are several uncertainties regarding the parameterization and experiments. Due to the Benjamin–Feir time scale, $O(e^{-2\sigma_0^{-1}})$, the measurements have a limited validity imposed by the length of the basin and the spatial resolution of wave gages. The parameterization also has a limitation for larger wave steepness due to the restriction of the nonlinear Schrödinger equation to weak nonlinearity and narrowband spectra. Besides these discrepancies and uncertainties, the parameterization gives a good estimation of kurtosis from the directional spectra. Therefore, the long-time solution and the empirical formula of maximum kurtosis estimation, Eq. (39), work well for directional waves in stationary conditions, such as a wave tank.

Note that, as mentioned above, the observed directional spreads $\sigma_0$ by buoys at peak frequency ranges from 0.15 to 0.5 depending on the sea state. The reduction of BFI due to directional effects is significant, and kurtosis increase in these conditions becomes only 3%–12% in comparison with the case of unidirectional waves. Therefore, the directional dispersion effect is dominant rather than nonlinear focusing owing to quasi-resonant effects in Eq. (16). Therefore, for most of the ocean conditions the pdf of the surface elevation will be close to a Gaussian, and only in the special case of steep waves with a narrow frequency and directional spectrum is a significant increase in kurtosis to be expected. In other words, according to the present approach freak waves are still rare events.

5. Conclusions

For a narrowband, random wave train, the kurtosis of the surface elevation is mainly determined by four wave–wave interactions, whereas bound waves only give a small contribution. The kurtosis and related high-order cumulants can be evaluated from the frequency spectrum on the basis of Janssen (2003) and extended to the maximum wave height estimation by Mori and Janssen (2006). However, directional dispersion suppresses the kurtosis enhancement by four wave–wave interactions as indicated by several researchers.

In this study, the directional dispersion effects on the kurtosis have been investigated. Based on Monte Carlo simulations of the 2D nonlinear Schrödinger equation, a formula for the kurtosis has been presented for directional sea states as a function of a natural extension of the BFI toward two-dimensional wave propagation. Three different types of kurtosis prediction equations have been proposed as Eqs. (27), (37), and (39). The reduction of the kurtosis is monotonic as the directional bandwidth $\sigma_0$ increases. The maximum difference between the kurtosis simulated by the MCNLS and the empirical formula is 0.1 with 10% error at most. The validity of the new kurtosis formula in the directional sea states has been verified by means of results from laboratory experiments.

Note that a careful and quantitative validation of the kurtosis estimation in the field will be performed in the near future. Both free and bound modes play an important role in the magnitude of the kurtosis for directional waves. A first step toward validation in the field has been reported by Janssen and Bidlot (2009). In this reference, the theoretical pdf of maximum wave height was validated against observations of maximum wave height and fair agreement was obtained. Despite the reasonable agreement, a problem with the present theory was noted. The maximum wave height distribution is obtained under the assumption that the waves are statistically independent. In practice, ocean waves have a fairly narrow spectrum and the assumption of independence may not be true. Therefore, more work is required.

In addition, for most oceanic conditions the pdf of the surface elevation will be close to Gaussian because, in practice, directional dispersion is large, thus reducing the kurtosis. Only in the special case of steep waves with a narrow frequency and directional spectrum is a significant increase in kurtosis to be expected. In other words, according to the present approach freak waves are still rare events.

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