Thickness-Weighted Mean Theory for the Effect of Surface Gravity Waves on Mean Flows in the Upper Ocean

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ABSTRACT

The residual effect of surface gravity waves on mean flows in the upper ocean is investigated using thickness-weighted mean (TWM) theory applied in a vertically Lagrangian and horizontally Eulerian coordinate system. Depth-dependent equations for the conservation of volume, momentum, and energy are derived. These equations allow for (i) finite amplitude fluid motions, (ii) the horizontal divergence of currents, and (iii) a concise treatment of both kinematic and viscous boundary conditions at the sea surface. Under the assumptions of steady and monochromatic waves and a uniform turbulent viscosity, the TWM momentum equations are used to illustrate the pressure- and viscosity-induced momentum fluxes through the surface, which are implicit in previous studies of the wave-induced modification of the classical Ekman spiral problem. The TWM approach clarifies, in particular, the surface momentum flux associated with the so-called virtual wave stress of Longuet-Higgins. Overall, the TWM framework can be regarded as an alternative to the three-dimensional Lagrangian mean framework of Pierson. Moreover, the TWM framework can be used to include the residual effect of surface waves in large-scale circulation models. In specific models that carry the TWM velocity appropriate for advecting tracers as their velocity variable, the turbulent viscosity term should be modified so that the viscosity acts only on the Eulerian mean velocity.

1. Introduction

In the theory for surface gravity waves, the Lagrangian mean transport by the Stokes drift has been known for more than 150 years (Stokes 1847) whereas it is only in relatively recent times that the importance of Lagrangian transport by ocean mesoscale eddies has been appreciated. In both cases, the Stokes or eddy-induced velocities are corrections to an Eulerian mean (EM) velocity to account for the difference between Eulerian and Lagrangian mean motions. In the theory of oceanic mesoscale eddies, it has become common to introduce a vertically Lagrangian (VL) coordinate system, using density or neutral density for the vertical coordinate (e.g., isopycnal or isoneutral coordinates), following the example from the atmospheric literature where potential temperature is typically used (Andrews et al. 1987). In the horizontal the standard Eulerian coordinates are retained. Equations averaged in isopycnal coordinates have been widely used to develop parameterizations of the effect of mesoscale eddies in global ocean models designed for climate studies (Gent et al. 1995; Greatbatch 1998; Griffies 2004; Gent 2011). In the surface gravity wave literature, a VL coordinate system, analogous to isopycnal coordinates, has been introduced by Mellor (2003, 2008) and by Broström et al. (2008) to derive depth-dependent equations for the effect of surface gravity waves on the larger-scale flow, in an attempt to present a simpler formulation than provided by the (traditional) three-dimensional Lagrangian mean equations (e.g., Lamb 1932; Pierson 1962; Andrews and McIntyre 1978; Jenkins and Ardhuin 2004). Both Mellor and Broström et al. rely on small amplitude theory to develop their equations, using a perturbation expansion approach. As we show in the present paper, an alternative is to use Favre filtering (Hesselberg 1926; Favre 1965, 1983), following the example

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from the large-scale oceanographic and atmospheric literature and corresponding to what in this literature is known as thickness-weighted isopycnal or mass-weighted averaging, respectively. The (time-) averaged equations of motion can then be written for finite-amplitude surface waves and allow for exact conservation of volume; momentum; energy; and, if required, passive tracers.

The reason Favre filtering is common in the studies dealing with eddies in the ocean and atmosphere (e.g., Gallimore and Johnson 1981; de Szoeke and Bennett 1993; Iwasaki 2001; Greatbatch and McDougall 2003; Aiki and Richards 2008)[1] is because, when written in the VL coordinate system, the nonlinear terms in the equations of motion typically involve three independent variables and not two as when the equations are written in terms of standard Eulerian coordinates (often called height coordinates in oceanic studies). The conventional Reynolds averaging decomposition applied to the product of two variables, A and B, takes the form $\overline{AB}^c = \overline{A}^c\overline{B}^c + \overline{A'B'}^c$, where $\overline{(\cdots)}^c$ is an Eulerian low-pass temporal filter and the single prime represents the deviation from the Eulerian mean (EM). This form of averaging becomes complicated when it is applied to the product of three variables. The Favre decomposition, on the other hand, has been developed to handle terms that are the product of three variables. For example, the nonlinear terms written in flux divergence form in isopycnal coordinates appear as the divergence of terms of the form hAB, where h is the thickness (or mathematically the Jacobian determinant of the coordinate transformation between the isopycnal and height coordinates). The Favre decomposition takes the form $\overline{hAB} = \overline{h}\hat{A}\hat{B} + \overline{hA''B''}$, where $\overline{(\cdots)}$ is a low-pass temporal filter in isopycnal coordinates. The caret is the Favre filter $\hat{A} \equiv \overline{hA}/\overline{h}$ and the double prime is the Favre deviation $A'' \equiv A - \hat{A}$, the latter of which is slightly different from the conventional definition of the deviation $A''' \equiv A - \overline{A}$. An important byproduct of the Favre filtering is a concise treatment of the boundary condition at the top and bottom of the ocean (Aiki and Yamagata 2006). As we shall see, the free sea surface can itself be a coordinate surface in the VL coordinate system, and this avoids the complications inherent when using vertically Eulerian averaging. In particular, the problem of extrapolating variables to a location above surface troughs is avoided (a common feature of papers dealing with surface gravity waves).

To illustrate the power of the Favre-filtered equations in the VL coordinate system, we revisit the issue of how the classical Ekman spiral solution (which is for the EM flow) is modified in the presence of surface waves. Polton et al. (2005) point out that the Coriolis–Stokes force of Hasselmann (1970) drives an EM flow that, even though it is confined near the surface, impacts the whole depth of the Ekman layer because the additional mean flow modifies the surface boundary condition from that in the classical Ekman problem. Their solution nevertheless differs from that of some previous authors, notably Madsen (1978) and Xu and Bowen (1994), who include an additional surface stress that they associate with the so-called virtual wave stress (VWS) of Longuet-Higgins (1953, 1960). So far, the most rigorous framework for explaining the VWS is the three-dimensional Lagrangian approach of Pierson (1962).[2] We show here that the VWS is contained in the Favre-filtered momentum equations. Indeed, we are able to reproduce the equations used by all of these previous authors and are able to point out the different surface fluxes of momentum that are implicit in these papers and account for the differences between the different solutions. For this analysis, we restrict the case to a steady monochromatic wave train and a spatially uniform turbulent viscosity. More general situations, such as nonsteady waves and a spatially variable turbulent viscosity (e.g., Jenkins 1986, 1987a,b; Gnanadesikan and Weller 1995; Song and Huang 2011), will be discussed in a later paper in the context of the Favre-filtered equations in the VL coordinate system.

The manuscript is organized as follows: Section 2 presents the fundamental equations for surface waves in deep water using the VL coordinates. The basic theory derived in this section is quite general and applies to both unsteady situations and a general wave spectrum. In particular, we show that (i) the Favre decomposition allows depth-dependent equations for finite amplitude waves to be derived without resorting to perturbation or Taylor expansions and (ii) the surface boundary condition of the Favre-filtered equations is concise and straightforward in the VL coordinate system. We take advantage of the treatment of the surface boundary conditions in section 3 as part of an analytical investigation of the residual effect of linear surface waves on the momentum flux through the thin viscous boundary layer associated with the waves, where the link to the previous work on the modification of the classical Ekman spiral

[1] Greatbatch and McDougall (2003) show that the temporal residual-mean equations of McDougall and McIntosh (2001) are essentially the Favre-filtered equations in isopycnal coordinates. Thus, no expansion method, as used in the latter paper, is necessary.

[2] To our knowledge, no previous studies, except for an attempt by Ardhuin et al. (2008), have used the generalized Lagrangian-mean equations of Andrews and McIntyre (1978) to explain the VWS.

TABLE 1. List of symbols, where A is an arbitrary quantity.

$\overline{A}^c$	Time mean in Eulerian coordinates
$\hat{A} \equiv \overline{z_z^c A}$	Thickness-weighted time mean in the VL coordinates
$\overline{A}$	Unweighted time mean in the VL coordinates (mean height is $\overline{z^c} \equiv z$ and mean thickness is $\overline{z_z^c} = 1$)
$A' \equiv A - \overline{A}^c$	Deviation from the Eulerian mean, compared at fixed z^c ($\overline{A'}^c = 0$)
$A'' \equiv A - \hat{A}$	Deviation from the thickness-weighted mean, compared at fixed z ($\overline{z_z^c A''} = 0$)
$A''' \equiv A - \overline{A}$	Deviation from the unweighted mean, compared at fixed z ($\overline{A'''} = 0$)
$z''' \equiv z^c - \overline{z^c} = z^c - z$	Vertical displacement ($z_z^c = 1 + z_z'''$, $\overline{z_z^c} = 1$, $\overline{z_z'''} = 0$)
$\mathbf{\nabla} \equiv (\partial_x, \partial_y)$	Lateral gradient in the VL coordinates ($\mathbf{\nabla} z = 0$, $\mathbf{\nabla} z^c = \mathbf{\nabla} z'''$)
$\nabla^c \equiv (\partial_{x^c}, \partial_{y^c})$	Horizontal gradient in Eulerian coordinates ($\nabla^c = \mathbf{\nabla} - (\mathbf{\nabla} z^c)\partial_{z^c}$)
$\mathbf{V} \equiv (u, v)$	Horizontal component of velocity
w	Vertical component of velocity
$w^* \equiv (w - z_t^c - \mathbf{V} \cdot \mathbf{\nabla} z^c)/z_z^c$	Vertical velocity associated with volume flux through surface of fixed z
$(\hat{\mathbf{V}}, \hat{w})$	Thickness-weighted mean (TWM) velocity
$(\hat{\mathbf{V}}, \widehat{w^*})$	Total transport velocity, $\mathbf{\nabla} \cdot \hat{\mathbf{V}} + \widehat{w^*}_z = 0$
$\mathbf{V}^B \equiv \hat{\mathbf{V}} - \overline{\mathbf{V}} = \overline{z_z'''\mathbf{V}'''}$	Horizontal component of bolus velocity
$w^B \equiv \widehat{w^*} - \overline{w} = -\overline{\mathbf{V}''' \cdot \mathbf{\nabla} z'''}$	Vertical component of bolus velocity
$\mathbf{V}^{qs} \equiv \hat{\mathbf{V}} - \overline{\mathbf{V}}^c = (\overline{z'''\mathbf{V}'''})_z + \cdots$	Horizontal component of quasi-Stokes velocity
$w^{qs} \equiv \widehat{w^*} - \overline{w}^c$	Vertical component of quasi-Stokes velocity
ρ	Reference density of seawater (positive real constant)
g	Gravity acceleration (positive real constant)
η	Sea surface height
p	Sum of oceanic nonhydrostatic pressure and atmospheric sea surface pressure
$\mathcal{FS}^{\mathbf{V}}$	Divergence of the form stress $\equiv [\overline{\mathbf{\nabla} z'''(\rho g \eta''' + p''')}]_z - \mathbf{\nabla}[\overline{z_z'''(\rho g \eta''' + p''')}]$
$\mathcal{RS}^A$	Divergence of the Reynolds stress $\equiv \rho[\mathbf{\nabla} \cdot (\overline{z_z^c \mathbf{V}'' A''}) + (\overline{z_z^c w^{*\prime\prime} A'''})_z]$ for $A = u, v$, and w
F^A	Turbulent mixing term, parameterized by (28a)–(28c)
$\overline{\tau}$	Viscous stress at the sea surface by wind in the x direction (positive real constant)
ν	Turbulent viscosity coefficient (positive real constant)
f	Coriolis parameter (positive real constant)
$\epsilon = \sqrt{if/\nu}$	Vertical wavenumber/decay rate of the Ekman spiral velocity (complex constant)
k	Wavenumber in the direction of x axis (positive real constant)
σ	Wave frequency (positive real constant)
$\theta \equiv kx - \sigma t$	Wave phase (sign-indefinite real constant)
$\alpha a = \alpha/k$	Wave amplitude ($a \equiv 1/k$)
α	Nondimensional scale for surface slope (positive real constant)
$\beta \equiv \nu k^2/\sigma$	Nondimensional scale for turbulent viscosity (positive real constant)
$\gamma \equiv f/\sigma$	Nondimensional scale for the rotation of the earth (positive real constant)
$m \equiv \sqrt{-i\sigma/\nu} = \sqrt{-i/\beta}k$	Vertical wavenumber/decay rate of viscid waves (complex constant)
$n^{\pm} \equiv \sqrt{(-i\sigma + \nu k^2 \pm if)/\nu}$	Vertical wavenumber/decay rate of rotating viscid waves (complex constant)

problem is made. Section 4 presents a summary and discussion.

2. Formulation using vertically Lagrangian coordinates

We derive depth-dependent equations for finite-amplitude surface waves in incompressible deep water of constant, uniform density ρ. The equations are written in the vertically Lagrangian and horizontally Eulerian coordinate system introduced by Mellor (2003, 2008), Jacobson and Aiki (2006), and Broström et al. (2008). As a check on the formulation of the kinematic boundary condition, we take into account the presence of a background vertical flow, which might be caused by the horizontal divergence of the larger-scale flow. For convenience, Table 1 presents a list of the symbols used in the text.

a. Cartesian coordinates

Let Cartesian coordinates be labeled by the set of independent variables (x^c, y^c, z^c, t^c), where x^c and y^c are horizontal coordinates; z^c (the geopotential height) increases vertically upward; (u, v, w) are the corresponding three-dimensional components of velocity. The continuity, horizontal, and vertical momentum equations then take the form

$$\nabla^c \cdot \mathbf{V} + w_{z^c} = 0, \tag{1a}$$

$$\rho(\mathbf{V}_{t^c} + \mathbf{V} \cdot \nabla^c \mathbf{V} + w\mathbf{V}_{z^c} + f\mathbf{z} \times \mathbf{V}) = \underbrace{-\nabla^c[\rho g(\eta - z^c)]}_{-\rho g \nabla^c \eta} - \nabla^c p + F^{\mathbf{V}}, \tag{1b}$$

$$\rho(w_{t^c} + \mathbf{V} \cdot \nabla^c w + w w_{z^c}) = -p_{z^c} + F^w, \tag{1c}$$

where $\mathbf{V} \equiv (u, v)$ is the horizontal velocity; $\nabla^c = (\partial_{x^c}, \partial_{y^c})$ is the horizontal gradient operator; f is the Coriolis parameter; $\mathbf{z}$ is the unit vector in the vertical direction; and $\rho g(\eta - z^c)$ is hydrostatic pressure, which vanishes at the sea surface where $z^c = \eta$ with g being the acceleration due to gravity. Use of the hydrostatic pressure has led to no gravitational acceleration term appearing in (1c). The quantity p is the sum of oceanic nonhydrostatic pressure and atmospheric sea surface pressure. The terms $F^{\mathbf{V}}$ and F^w represent the effect of turbulent mixing on $\mathbf{V}$ and w, respectively.

The kinematic boundary condition at the sea surface, $z^c = \eta$, is

$$\eta_{t^c} + \mathbf{V} \cdot \nabla^c \eta = w. \tag{2}$$

Using (2) we take the depth integral of (1a) to give

$$\eta_{t^c} + \nabla^c \cdot \int_{-\infty}^{\eta} \mathbf{V}\, dz^c = 0, \tag{3}$$

which expresses the conservation of volume in each water column.

b. Vertically Lagrangian coordinates

The idea is to choose a coordinate system that follows the high-frequency fluid motion (i.e., waves), as in Lagrangian coordinates, but is such that the equations for the low-frequency fluid motion (i.e., currents) appear as in Eulerian coordinates. High-frequency fluid motion is distinguished from low-frequency fluid motion by using either a low-pass temporal filter with a given time scale or an ensemble average (cf. Andrews and McIntyre 1978). In what follows, we shall refer to the averaging operator as a low-pass filter. It should also be noted that the theory is quite general, applying to both finite amplitude waves and a general wave spectrum.

In this study, we use the vertically Lagrangian (VL) coordinates of Jacobson and Aiki (2006), which we label by the set of independent variables (x, y, z, t). The transformation between the Cartesian coordinates and the VL coordinates may be written as

$$x^c = x, \quad y^c = y, \quad z^c = z^c(x, y, z, t), \quad t^c = t, \tag{4}$$

with the inverse transformation given by

$$x = x^c, \quad y = y^c, \quad z = z(x^c, y^c, z^c, t^c), \quad t = t^c. \tag{5}$$

Care is required to define the value of the vertical coordinate z attached to a particular fluid particle at the horizontal location, (x^c, y^c) at time t^c. First, we let z^L be the (Lagrangian) low-pass filtered height of that same fluid particle centered around time t^c. Then we form the material surface that consists of all fluid particles with this same low-pass filtered height, z^L, centered around time t^c. We then define z to be the (Eulerian) low-pass filtered height of this material surface at the location (x^c, y^c) and again centered around the time t^c. It follows immediately that

$$z \equiv \overline{z^c}, \tag{6}$$

where the overbar indicates a temporal low-pass filter carried out in the VL coordinates. It should be noted that this particular transformation is rather special since it requires that, if one fluid particle is instantaneously situated above another fluid particle at (x^c, y^c) at time t^c, then the value of z assigned to the first fluid particle is also higher than that assigned to the second. While this property can be expected to be satisfied for surface gravity waves in the vertical plane (or indeed the heaving of isopycnals by the mesoscale eddy field), it is not likely to be satisfied, for example, by turbulent motions in the vertical plane. The expression $z^c(x, y, z, t)$ may be interpreted as a surface fluctuating in (x, y, t) space. Each surface is formed by the group of fluid particles whose (Lagrangian) low-pass filtered height z^L is a given value.[3] The members of the group are successively updated with progressing time using a sliding time window. In the analogy to isopycnal coordinates, the coordinate z corresponds to the density and the z surfaces correspond to isopycnals.[4]

Time series of surfaces of constant z at a fixed horizontal position (x^c, y^c) are illustrated in Fig. 1 by blue lines. Two cases are compared, in Fig. 1a without and in Fig. 1b with a background vertical flow (the latter may be caused by a large-scale horizontal convergence/divergence of currents). With a background vertical flow, the blue lines get left behind by the rising sea surface and are eventually below the layer of active wave motion. In this case, water clearly passes through the z surfaces (using the notation introduced below, $w^* > 0$ in this case). Without the background flow, no water passes through the z surfaces (corresponding to $w^* = 0$). In this case, the sea surface is also a z surface.

c. Mathematical development

To proceed with the mathematical development,[5] we note that spatial derivatives in the VL coordinates are given by

[3] Each surface is labeled by the value of the (Eulerian) low-pass filtered height of this material surface, written by (6).

[4] It should be noted that, in the case of mesoscale eddies, there is no guarantee that the z surfaces defined here are the same as isopycnal surfaces.

[5] Readers may find it helpful to refer to the corresponding analysis in de Szoeke and Bennett (1993).

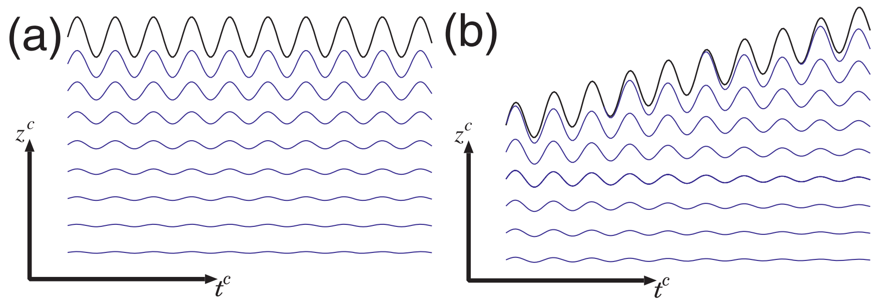

FIG. 1. Illustration of surfaces of fixed z (blue line) and the sea surface (black line) in (t^c, z^c) space (a) without a background vertical flow and (b) with a background flow.

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \\ \partial_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & z_x^c & 0 \\ 0 & 1 & z_y^c & 0 \\ 0 & 0 & z_z^c & 0 \\ 0 & 0 & z_t^c & 1 \end{pmatrix} \begin{pmatrix} \partial_{x^c} \\ \partial_{y^c} \\ \partial_{z^c} \\ \partial_{t^c} \end{pmatrix}. \qquad (7)$$

We also note that $z \equiv \overline{z^c}$ leads to

$$(\overline{z_x^c}, \overline{z_y^c}, \overline{z_z^c}, \overline{z_t^c}) = (0, 0, 1, 0), \qquad (8)$$

identities that are useful later when we average the governing equations. It should also be noted that z_z^c corresponds to the thickness (and is analogous to the thickness in isopycnal coordinates).

We now use (7) to write the governing equations (1a)–(1c) in terms of the VL coordinates,

$$(z_z^c)_t + \nabla \cdot (z_z^c \mathbf{V}) + (z_z^c w^*)_z = 0, \qquad (9a)$$

$$z_z^c w^* \equiv w - z_t^c - \mathbf{V} \cdot \nabla z^c, \qquad (9b)$$

$$\rho(\mathbf{V}_t + \mathbf{V} \cdot \nabla \mathbf{V} + w^* \mathbf{V}_z + f\mathbf{z} \times \mathbf{V}) = -\nabla(\rho g \eta + p) + p_{z^c} \nabla z^c + F^{\mathbf{V}}, \qquad (9c)$$

$$\rho(w_t + \mathbf{V} \cdot \nabla w + w^* w_z) = -p_{z^c} + F^w, \qquad (9d)$$

where $\nabla \equiv (\partial_x, \partial_y) = \nabla^c + (\nabla z^c)\partial_{z^c}$ is the lateral gradient operator in the VL coordinates. Except that (i) p is the nonhydrostatic pressure and (ii) η is a free sea surface height, (9a)–(9d) are the same as equations (9)–(14) of Jacobson and Aiki (2006).

The quantity w^* measures the flow that passes through the surfaces z = const and is caused by the horizontal divergence/convergence of the large-scale flow (Fig. 1b). In fact, using the coordinate transformation (7), it is easy to show that $w^* = (\partial_{t^c} + \mathbf{V} \cdot \nabla^c + w\partial_{z^c})z$, from which it follows that w^* is the rate of change of the coordinate z following a fluid particle: $w^* = Dz/Dt^c$, analogous to $w = Dz^c/Dt^c$ in which $D/Dt^c \equiv (\partial_{t^c} + \mathbf{V} \cdot \nabla^c + w\partial_{z^c})$. When no water passes through a z surface, as in Fig. 1a, $w^* = 0$. It follows that for the situation shown in Fig. 1a, $w^* = 0$ at the sea surface.

d. The thickness-weighted mean (TWM) governing equations

Momentum equations in a flux-divergence form can be obtained by multiplying each of (9c) and (9d) by the thickness z_z^c and then using (9a) to give

$$\rho[(z_z^c \mathbf{V})_t + \nabla \cdot (z_z^c \mathbf{V}\mathbf{V}) + (z_z^c w^* \mathbf{V})_z + f\mathbf{z} \times z_z^c \mathbf{V}] = -z_z^c \nabla(\rho g \eta + p) + p_z \nabla z^c + z_z^c F^{\mathbf{V}}, \qquad (10a)$$

$$\rho[(z_z^c w)_t + \nabla \cdot (z_z^c \mathbf{V} w) + (z_z^c w^* w)_z] = -p_z + z_z^c F^w, \qquad (10b)$$

where $z_z^c p_{z^c} = p_z$ has been used.

Hereafter, the term "thickness-weighted mean" (TWM) refers to the Favre filter associated with the thickness z_z^c. Application of a low-pass temporal filter to each of (9a), (10a), and (10b) yields TWM equations for the incompressibility condition and the horizontal and vertical components of momentum:

$$\nabla \cdot \hat{\mathbf{V}} + \widehat{w^*_z} = 0, \qquad (11a)$$

$$\rho[\hat{\mathbf{V}}_t + \nabla \cdot (\hat{\mathbf{V}}\hat{\mathbf{V}}) + (\widehat{w^*}\hat{\mathbf{V}})_z + f\mathbf{z} \times \hat{\mathbf{V}}] + \mathcal{RS}^{\mathbf{V}} = -\nabla(\rho g \overline{\eta} + \overline{p}) + \mathcal{FS}^{\mathbf{V}} + \hat{F}^{\mathbf{V}}, \qquad (11b)$$

$$\rho[\hat{w}_t + \nabla \cdot (\hat{\mathbf{V}}\hat{w}) + (\widehat{w^*}\hat{w})_z] + \mathcal{RS}^w = -\overline{p}_z + \hat{F}^w, \qquad (11c)$$

where we have used $\overline{z_z^c} \equiv 1$ [since $\overline{z^c} \equiv z$, Eq. (6)], and

$$z''' \equiv z^c - z, \qquad (12)$$

and hence $\nabla z^c = \nabla z'''$. The caret indicates the TWM operator ($\hat{A} \equiv \overline{z_z^c A}$ for an arbitrary quantity A), the double-prime indicates the deviation from the TWM

($A'' \equiv A - \hat{A}$, compared at fixed z), and the triple-prime indicates the deviation from the unweighted mean ($A''' \equiv A - \overline{A}$, compared at fixed z).

The quantity $\mathcal{RS}^A$ in (11b) and (11c) with $A = u, v$, and w is the divergence of the Reynolds stress (or, more correctly here, the Favre stress),

$$\mathcal{RS}^A \equiv \rho[\nabla \cdot (\overline{z_z^c \mathbf{V}'' A''}) + (\overline{z_z^c w^{*''} A''})_z]. \tag{13}$$

The Reynolds stress represents the effect of wave motions, while the turbulent mixing is represented by F^A in (11b) and (11c). Equation (13) shows that the vertical component of the Reynolds stress is based on $w^{*''}$ (not w'') and thus is nearly zero. Indeed, for the situation shown in Fig. 1a, $w^* = 0$ everywhere at all times, showing that the second term on the rhs of (13) is zero in this case—an issue we return to in section 3 [it means, in particular, that, in the VL coordinate system, the Coriolis–Stokes force of Hasselmann (1970) does not arise from a Reynolds stress, unlike the situation in Cartesian coordinates]. This is attributed to the way the VL coordinates have been designed so that w^* represents fluid motions associated with low-frequency fluid motions and not with the waves themselves. The quantity $\mathcal{FS}^{\mathbf{V}}$ in (11b) is the divergence of the layer-thickness form stress:

$$\begin{aligned}\mathcal{FS}^{\mathbf{V}} &\equiv -\overline{z_z''' \nabla(\rho g \eta''' + p''')} + \overline{p_z''' \nabla z'''} \\ &= [\overline{\nabla z'''(\rho g \eta''' + p''')}]_z - \nabla[\overline{z_z'''(\rho g \eta''' + p''')}].\end{aligned} \tag{14}$$

The TWM momentum equations (11b) and (11c) contain two types of three-dimensional velocity, the TWM velocity $(\hat{\mathbf{V}}, \hat{w})$ and the total transport velocity $(\hat{\mathbf{V}}, \widehat{w^*})$. Here $\hat{w}$ and $\widehat{w^*}$ are not the same mathematically, but the difference is negligible as far as the present study is concerned. The total transport velocity is three-dimensionally nondivergent, as shown by (11a), and can be written as the sum of the unweighted mean velocity $(\overline{\mathbf{V}}, \overline{w})$ (averaged in VL coordinates) and a velocity $(\mathbf{V}^B, w^B)$, analogous to the bolus velocity (Rhines 1982) in the mesoscale eddy literature. In particular,

$$\hat{\mathbf{V}} \equiv \overline{\underbrace{(1 + z_z''')}_{z_z^c}(\overline{\mathbf{V}} + \mathbf{V}''')} = \overline{\mathbf{V}} + \underbrace{\overline{z_z'''\mathbf{V}'''}}_{\mathbf{V}^B}, \tag{15a}$$

$$\widehat{w^*} \equiv \overline{z_z^c w^*} = \overline{w} \underbrace{- \overline{z_t^c} - \overline{\mathbf{V}} \cdot \nabla \overline{z^c}}_{0} \underbrace{- \overline{\mathbf{V}''' \cdot \nabla z'''}}_{w^B}, \tag{15b}$$

where (9b) and (6) have been used. The explicit form of the vertical component of the bolus velocity as in (15b) has been little mentioned in previous studies because the bolus velocity was originally defined for large-scale horizontal flows in layered, hydrostatic ocean models (Rhines 1982). The bolus velocity can be compared with the quasi-Stokes velocity.[6] The bolus velocity is referenced to the unweighted mean velocity in the VL coordinates, whereas the quasi-Stokes velocity is referenced to the EM velocity, $(\overline{\mathbf{V}}^c, \overline{w}^c)$, averaged in Cartesian coordinates. In particular,

$$\begin{aligned}(\mathbf{V}^{qs}, w^{qs}) &\equiv (\hat{\mathbf{V}} - \overline{\mathbf{V}}^c, \widehat{w^*} - \overline{w}^c) \\ &= (\hat{\mathbf{V}} - \overline{\mathbf{V}}, \widehat{w^*} - \overline{w}) + (\overline{\mathbf{V}} - \overline{\mathbf{V}}^c, \overline{w} - \overline{w}^c) \\ &= (\mathbf{V}^B, w^B) + (\overline{z'''\mathbf{V}_z'''} + \cdots, \overline{z'''w_z'''} + \cdots),\end{aligned} \tag{16}$$

where the last term represents a Taylor expansion in the vertical direction (cf. McDougall and McIntosh 2001). Combining with (15a) gives the following expression for the horizontal component of the quasi-Stokes velocity, an expression that will prove useful later (Smith 2006; Mellor 2008):

$$\mathbf{V}^{qs} = (\overline{z'''\mathbf{V}'''})_z + \cdots. \tag{17}$$

It should be noted that, although the definitions of the quasi-Stokes velocity and traditional Stokes drift are different, they are closely related. The conventional definition of the Stokes-drift velocity based on a Taylor expansion in Cartesian coordinates is

$$\mathbf{V}^{\mathrm{Stokes}} \equiv \overline{\left(\int^{t^c} \mathbf{V}\, dt^c\right) \cdot \nabla^c \mathbf{V}}^c + \overline{\left(\int^{t^c} w\, dt^c\right) \mathbf{V}_{z^c}}^c,$$

which can be transformed to

$$\nabla^c \cdot \left[\overline{\left(\int^{t^c} \mathbf{V}\, dt^c\right)\mathbf{V}}^c\right] + \left[\overline{\left(\int^{t^c} w\, dt^c\right)\mathbf{V}}^c\right]_{z^c},$$

[6] The concept of the quasi-Stokes velocity was introduced in the studies of mesoscale eddies to develop an eddy-induced velocity, which satisfies both an incompressible condition and a no-normal-flow boundary condition at the top and bottom of the ocean, as does the EM velocity (McDougall and McIntosh 2001; Aiki and Yamagata 2006). The sea surface is assumed to be rigid in the theoretical studies of mesoscale eddies. For surface gravity waves, the sea surface is not rigid, and this means that the EM velocity, $(\overline{\mathbf{V}}^c, \overline{w}^c)$, is not strictly defined at depths that spend part of the averaging time above the sea surface. However, in such cases, it is sometimes possible to use the Taylor expansion on the rhs of (16) to define the quasi-Stokes velocity at such depths. The bolus velocity, by contrast, is always defined, even at finite amplitude.

in which the first term vanishes for horizontally homogeneous waves and the second term is analogous to the first term on the rhs of (17) within an approximation $\int^{t^c} w\, dt^c \simeq z'''$.

e. The free surface

A nice feature of the VL coordinates used here is the handling of the free surface. Indeed, as can be seen from Fig. 1a, since the free surface is itself a surface of constant z in that case, it follows that, when averaging in the VL coordinate system, there is no need to deal with regions beyond the sea surface, that is, above troughs when the surface is below its mean height, as happens when averaging in Eulerian coordinates. Mathematically, the ease with which averaging can be carried out in the VL coordinates arises because the kinematic boundary condition is not only preserved in the VL coordinates but also avoids products of quantities varying at high frequency, making averaging straightforward.

We begin by noting that $\eta_t = \eta_{t^c}$ and $\nabla\eta = \nabla^c\eta$. We then note that the sea surface is given by $z^c(x, y, z, t) = \eta(x, y, t)$ and, since at the sea surface[7] $z = \overline{\eta}$, this means that

$$z^c(x, y, \overline{\eta}, t) = \eta(x, y, t). \tag{18}$$

It then follows that

$$z^c_t = \eta_t - z^c_z\overline{\eta}_t, \tag{19a}$$

$$\nabla z^c = \nabla\eta - z^c_z\nabla\overline{\eta}. \tag{19b}$$

One immediate consequence is that $(z^c_t)|_{z^c=\eta} \neq \eta_t$ and $(\nabla z^c)|_{z^c=\eta} \neq \nabla\eta$ in general. We now substitute the set of (19a) and (19b) to (9b) and obtain

$$\begin{aligned} z^c_z w^* &= w - \underbrace{(\eta_t - z^c_z\overline{\eta}_t)}_{z^c_t|_{z=\overline{\eta}}} - \mathbf{V}\cdot\underbrace{(\nabla\eta - z^c_z\nabla\overline{\eta})}_{\nabla z^c|_{z=\overline{\eta}}} \\ &= z^c_z(\overline{\eta}_t + \mathbf{V}\cdot\nabla\overline{\eta}), \end{aligned} \tag{20}$$

where the kinematic boundary condition (2) at the sea surface has been used. Equivalently, dividing by the thickness z^c_z,

$$w^* = \overline{\eta}_t + \mathbf{V}\cdot\nabla\overline{\eta}, \tag{21}$$

showing that the form of the kinematic boundary condition is preserved in the VL coordinates and that, further, only one high frequency variable, $\mathbf{V} = \hat{\mathbf{V}} + \mathbf{V}''$, appears in the expression for the kinematic boundary condition in our VL coordinates. The advantage of the form taken by the kinematic boundary condition in the VL coordinates can be seen when applying a low-pass temporal filter to (20) to yield

$$\widehat{w^*} \equiv \overline{z^c_z w^*} = \overline{\eta}_t + \hat{\mathbf{V}}\cdot\nabla\overline{\eta}, \tag{22a}$$

$$w^{*\prime\prime} \equiv w^* - \widehat{w^*} = \mathbf{V}''\cdot\nabla\overline{\eta}, \tag{22b}$$

at the sea surface (where it is assumed that $\overline{\eta}_t$ and $\nabla\overline{\eta}$ are effectively constant during the filtering). Equations (19a)–(22b) have not, to our knowledge, been shown before the present study. These equations are cornerstones for (i) treating the slow variations of the sea surface in both time and horizontal space and (ii) taking the depth integral of various quantities.

To illustrate (ii), we note that an equation for volume conservation can be derived by taking the depth integral of (9a):

$$\begin{aligned} 0 &= (z^c_t + z^c_z w^*)|_{z=\overline{\eta}} + \int_{-\infty}^{\overline{\eta}} \nabla\cdot(z^c_z\mathbf{V})\, dz \\ &= (z^c_t + z^c_z w^* - z^c_z\mathbf{V}\cdot\nabla\overline{\eta})|_{z=\overline{\eta}} + \nabla\cdot\int_{-\infty}^{\overline{\eta}} z^c_z\mathbf{V}\, dz \\ &= (z^c_t + z^c_z\overline{\eta}_t + z^c_z\mathbf{V}\cdot\nabla\overline{\eta} - z^c_z\mathbf{V}\cdot\nabla\overline{\eta})|_{z=\overline{\eta}} \\ &\quad + \nabla\cdot\int_{-\infty}^{\overline{\eta}} z^c_z\mathbf{V}\, dz \\ &= \eta_t + \nabla\cdot\int_{-\infty}^{\overline{\eta}} z^c_z\mathbf{V}\, dz, \end{aligned} \tag{23}$$

where (20) has been used to derive the third line and (19a) has been used to derive the last line. Equation (23) is consistent with (3), resulting in validating (19a) and (19b). Equation (22a) allows the depth integral of (11a) to be written as

$$\begin{aligned} 0 &= \widehat{w^*}|_{z=\overline{\eta}} + \int_{-\infty}^{\overline{\eta}} \nabla\cdot\hat{\mathbf{V}}\, dz \\ &= (\widehat{w^*} - \hat{\mathbf{V}}\cdot\nabla\overline{\eta})|_{z=\overline{\eta}} + \nabla\cdot\int_{-\infty}^{\overline{\eta}} \hat{\mathbf{V}}\, dz \\ &= \overline{\eta}_t + \nabla\cdot\int_{-\infty}^{\overline{\eta}} \hat{\mathbf{V}}\, dz, \end{aligned} \tag{24}$$

which is consistent with both (3) and (23). Exact equations for depth-integrated momentum can also be derived easily (not shown).

The kinematic boundary conditions (19a)–(22b) also allow exact energy equations for finite amplitude waves

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[7] Strictly, for the situation shown in Fig. 1b, it is not correct to say that $z = \overline{\eta}$. This is because, in that situation, the sea surface is no longer coincident with the material surface used to define z [see how z is defined just before (6)]. The analysis, nevertheless, remains the same, and for convenience we continue to label the value of z at the sea surface by $\overline{\eta}$.

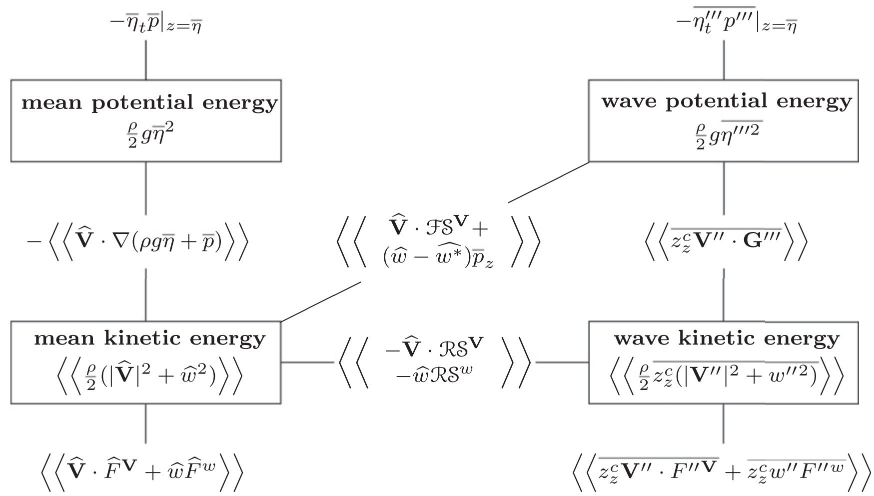

FIG. 2. Energy diagram based on (A8a)–(A9b) where $\langle\langle\cdots\rangle\rangle \equiv \int_{-\infty}^{\overline{\eta}}\cdots dz$. The sign of energy conversion terms is referenced to the budget of (mean) kinetic energy. The symbol $\mathbf{G} \equiv -\nabla^c(\rho g\eta + p)$ is the negative of the horizontal gradient of the combined hydrostatic and nonhydrostatic pressure.

and currents to be derived (appendix A). An associated four-box energy diagram is illustrated in Fig. 2. The boxes of mean kinetic energy and wave potential energy are connected by a conversion path by the form stress $\mathcal{FS}^{\mathbf{V}}$. The mean kinetic energy is defined by the TWM velocity

$$\frac{\rho}{2}(|\hat{\mathbf{V}}|^2 + \hat{w}^2),$$

which includes the horizontal component of the quasi-Stokes velocity (i.e., $\hat{\mathbf{V}} = \overline{\mathbf{V}}^c + \mathbf{V}^{qs}$). These characteristics are the same as that of an energy diagram based on hydrostatic equations (Bleck 1985; Røed 1997; Iwasaki 2001; Aiki and Yamagata 2006; Aiki and Richards 2008). Because the present study includes a free surface, the work of air pressure disturbances $-\overline{\eta_t''' p'''}|_{z=\overline{\eta}}$ is included in Fig. 2.

3. Linear waves with a viscous boundary layer

We now consider viscid surface waves in the presence of wind forcing and show how waves can modify the classical Ekman spiral velocity near the sea surface, a problem that has been investigated previously using a number of different approaches. As noted in the introduction, the solution of Polton et al. (2005) (and also Huang 1979) differs from that of Madsen (1978) and Xu and Bowen (1994) in that the former do not include an additional surface stress at the surface associated with the so-called VWS of Longuet-Higgins (1953, 1960). So far, the most rigorous framework for explaining the VWS is the three-dimensional Lagrangian approach of Pierson (1962), Piedra-Cueva (1995), and Ng (2004). The use of the VL coordinate system and the TWM approach allows for a careful reexamination of the surface boundary conditions used in these studies as well as the budget of momentum in each vertical column. As in the previous studies, we use a perturbation expansion approach appropriate for small-amplitude waves.[8]

a. Perturbation expansion

We work with the situation as in Fig. 1a in which there is no horizontal convergence/divergence of the large-scale flow and $w^* = 0$ everywhere (including at the sea surface). In addition we assume for simplicity that wave statistics are equilibrated in both time and horizontal space so that $\partial_t\overline{A} = 0$ and $\nabla\overline{A} = 0$ for an arbitrary quantity A. An immediate consequence is that $\overline{\eta} = 0$ (strictly $\overline{\eta} = \text{const}$, but we can put the constant to zero). Below, we use $z = \overline{\eta} = 0$ as the label for the sea surface in the VL coordinates.

We work with small amplitude waves, with smallness measured by the parameter α. In particular, we let the slope of the sea surface be scaled by $\alpha \ll 1$ and use it to make a perturbation expansion

[8] We restrict here to the case of a spatially uniform viscosity with steady waves and steady wind forcing in order to illustrate the power of the TWM approach. Readers are referred to Jenkins (1986, 1987a,b) for discussion of nonsteady waves and spatially varying viscosity using a Lagrangian coordinate system.

$$z^c = z + \alpha z_1''' + \alpha^2 z_2''' + O(\alpha^3), \tag{25a}$$

$$\eta = \alpha\eta_1''' + \alpha^2\eta_2''' + O(\alpha^3), \tag{25b}$$

$$p = \alpha p_1''' + \alpha^2 p_2 + O(\alpha^3), \tag{25c}$$

$$\mathbf{V} = \alpha\mathbf{V}_1''' + \alpha^2\mathbf{V}_2 + O(\alpha^3), \tag{25d}$$

$$w = \alpha w_1''' + \alpha^2 w_2 + O(\alpha^3), \tag{25e}$$

$$w^* = 0, \tag{25f}$$

where $p_2 = \overline{p}_2 + p_2'''$, $\mathbf{V}_2 = \overline{\mathbf{V}}_2 + \mathbf{V}_2'''$, and $w_2 = \overline{w}_2 + w_2'''$. For simplicity, we have assumed no mean flow at $O(\alpha)$. The thickness-weighted governing Eqs. (9a), (9b), (10a), and (10b) become

$$\underbrace{z_{1zt}'''}_{w_{1z}'''} + \nabla\cdot\mathbf{V}_1''' = 0, \tag{26a}$$

$$\rho(\mathbf{V}_{1t}''' + f\mathbf{z}\times\mathbf{V}_1''') = -\rho g\nabla\eta_1''' - \nabla p_1''' + (z_z^c F^{\mathbf{V}})_1, \tag{26b}$$

$$\rho w_{1t}''' = -p_{1z}''' + (z_z^c F^w)_1, \tag{26c}$$

at $O(\alpha)$ and

$$z_{2zt}''' + \nabla\cdot(\mathbf{V}_2 + z_{1z}'''\mathbf{V}_1''') = 0, \tag{27a}$$

$$w_2 = z_{2t}''' + \mathbf{V}_1'''\cdot\nabla z_1''', \tag{27b}$$

$$\begin{aligned}\rho[(\mathbf{V}_2 + z_{1z}'''\mathbf{V}_1''')_t + \nabla\cdot(\mathbf{V}_1'''\mathbf{V}_1''') + f\mathbf{z}\times(\mathbf{V}_2 + z_{1z}'''\mathbf{V}_1''')] \\ = -\nabla(\rho g\eta_2''' + p_2) - z_{1z}'''\nabla(\rho g\eta_1''' + p_1''') \\ + p_{1z}'''\nabla z_1''' + (z_z^c F^{\mathbf{V}})_2,\end{aligned} \tag{27c}$$

$$\rho[(w_2 + z_{1z}'''w_1''')_t + \nabla\cdot(\mathbf{V}_1'''w_1''')] = -p_{2z} + (z_z^c F^w)_2, \tag{27d}$$

at $O(\alpha^2)$.

b. *Turbulent mixing term*

In the following, ν is a real, uniform constant representing turbulent viscosity and the momentum mixing is represented using a conventional symmetric tensor in Cartesian coordinates. The turbulent mixing term may, therefore, be expressed in the VL coordinates as

$$\begin{aligned}z_z^c F^u &\equiv \rho\nu z_z^c[(2u_{x^c})_{x^c} + (u_{y^c} + v_{x^c})_{y^c} + (u_{z^c} + w_{x^c})_{z^c}] \\ &= \rho\nu z_z^c[(2u_{x^c})_x + (u_{y^c} + v_{x^c})_y + (u_{z^c} + w_{x^c})_{z^c} - z_x^c(2u_{x^c})_{z^c} - z_y^c(u_{y^c} + v_{x^c})_{z^c}] \\ &= \rho\nu[z_z^c(2u_{x^c})_x + z_z^c(u_{y^c} + v_{x^c})_y + (u_{z^c} + w_{x^c})_z - z_x^c(2u_{x^c})_z - z_y^c(u_{y^c} + v_{x^c})_z] \\ &= \rho\nu[(z_z^c 2u_{x^c})_x + (z_z^c(u_{y^c} + v_{x^c}))_y + (u_{z^c} + w_{x^c} - z_x^c 2u_{x^c} - z_y^c(u_{y^c} + v_{x^c}))_z],\end{aligned} \tag{28a}$$

$$\begin{aligned}z_z^c F^v &= \rho\nu z_z^c[(v_{x^c} + u_{y^c})_{x^c} + (2v_{y^c})_{y^c} + (v_{z^c} + w_{y^c})_{z^c}] \\ &= \rho\nu[(z_z^c(v_{x^c} + u_{y^c}))_x + (z_z^c 2v_{y^c})_y + (v_{z^c} + w_{y^c} - z_x^c(v_{x^c} + u_{y^c}) - z_y^c 2v_{y^c})_z],\end{aligned} \tag{28b}$$

$$\begin{aligned}z_z^c F^w &\equiv \rho\nu z_z^c[(w_{x^c} + u_{z^c})_{x^c} + (w_{y^c} + v_{z^c})_{y^c} + (2w_{z^c})_{z^c}] \\ &= \rho\nu[(z_z^c(w_{x^c} + u_{z^c}))_x + (z_z^c(w_{y^c} + v_{z^c}))_y + (2w_{z^c} - z_x^c(w_{x^c} + u_{z^c}) - z_y^c(w_{y^c} + v_{z^c}))_z],\end{aligned} \tag{28c}$$

where $z_z^c\partial_{z^c} = \partial_z$ has been used. Perturbation expansion of (28a)–(28c) yields

$$\begin{aligned}(z_z^c F^u)_1 &= \rho\nu[(2u_{1x}''')_x + (u_{1y}''' + v_{1x}''')_y + (u_{1z}''' + w_{1x}''')_z] \\ &= \rho\nu(\nabla^2 + \partial_z^2)u_1''',\end{aligned} \tag{29a}$$

$$\begin{aligned}(z_z^c F^v)_1 &= \rho\nu[(v_{1x}''' + u_{1y}''')_x + (2v_{1y}''')_y + (v_{1z}''' + w_{1y}''')_z] \\ &= \rho\nu(\nabla^2 + \partial_z^2)v_1''',\end{aligned} \tag{29b}$$

$$\begin{aligned}(z_z^c F^w)_1 &= \rho\nu[(w_{1x}''' + u_{1z}''')_x + (w_{1y}''' + v_{1z}''')_y + (2w_{1z}''')_z] \\ &= \rho\nu(\nabla^2 + \partial_z^2)w_1''',\end{aligned} \tag{29c}$$

at $O(\alpha)$ where (26a) has been used and

$$\begin{aligned}(z_z^c F^u)_2 = \rho\nu[(\cdots)_x + (\cdots)_y + (u_{2z} + w_{2x} - z_{1z}'''u_{1z}''' \\ - z_{1x}'''w_{1z}''' - z_{1x}'''2u_{1x}''' - z_{1y}'''(u_{1y}''' + v_{1x}'''))_z],\end{aligned} \tag{30a}$$

$$\begin{aligned}(z_z^c F^v)_2 = \rho\nu[(\cdots)_x + (\cdots)_y + (v_{2z} + w_{2y} - z_{1z}'''v_{1z}''' \\ - z_{1y}'''w_{1z}''' - z_{1x}'''(v_{1x}''' + u_{1y}''') - z_{1y}'''2v_{1y}''')_z],\end{aligned} \tag{30b}$$

$$(z^c_z F^w)_2 = \rho\nu[(\cdots)_x + (\cdots)_y + (2w_{2z} - 2z'''_{1z}w'''_{1z} - z'''_{1x}(w'''_{1x} + u'''_{1z}) - z'''_{1y}(w'''_{1y} + v'''_{1z}))_z], \quad (30c)$$

at $O(\alpha^2)$ where $\partial_{z^c} = (1/z^c_z)\partial_z = [1/(1 + z'''_z)]\partial_z \simeq (1 - z'''_z)\partial_z$ has been used.

Substitution of (30a)–(30c) to (27c) and time averaging yields the TWM momentum balance at $O(\alpha^2)$,

$$-\rho f\underbrace{(\overline{v}_2 + \overline{z'''_{1z}v'''_1})}_{\hat{v}_2} = [\overline{z'''_{1x}(\rho g\eta'''_1 + p'''_1)}]_z + \rho\nu[\overline{u}_{2z} - (\overline{z'''_{1z}u'''_{1z} + z'''_{1x}(u'''_{1x} - v'''_{1y}) + z'''_{1y}(u'''_{1y} + v'''_{1x})})]_z, \quad (31a)$$

$$\rho f\underbrace{(\overline{u}_2 + \overline{z'''_{1z}u'''_1})}_{\hat{u}_2} = [\overline{z'''_{1y}(\rho g\eta'''_1 + p'''_1)}]_z + \rho\nu[\overline{v}_{2z} - (\overline{z'''_{1z}v'''_{1z} + z'''_{1x}(v'''_{1x} + u'''_{1y}) + z'''_{1y}(v'''_{1y} - u'''_{1x})})]_z, \quad (31b)$$

where (26a) has been used. The above equations can be rewritten using the TWM velocity at $O(\alpha^2)$,

$$-\rho f\hat{v}_2 = [\overline{z'''_{1x}(\rho g\eta'''_1 + p'''_1)}]_z + \rho\nu[\hat{u}_{2z} - (\overline{z'''_{1zz}u'''_1 + 2z'''_{1z}u'''_{1z} + z'''_{1x}(u'''_{1x} - v'''_{1y}) + z'''_{1y}(u'''_{1y} + v'''_{1x})})]_z, \quad (32a)$$

$$\rho f\hat{u}_2 = [\overline{z'''_{1y}(\rho g\eta'''_1 + p'''_1)}]_z + \rho\nu[\hat{v}_{2z} - (\overline{z'''_{1zz}v'''_1 + 2z'''_{1z}v'''_{1z} + z'''_{1x}(v'''_{1x} + u'''_{1y}) + z'''_{1y}(v'''_{1y} - u'''_{1x})})]_z, \quad (32b)$$

where (15a) has been used.

c. Monochromatic wave

We consider a monochromatic wave propagating in the x direction: $\eta'''_1 = ae^{i\theta}$ in which $\theta = kx - \sigma t$ is wave phase (real constant), k is wavenumber (positive real constant), and σ is wave frequency (positive real constant). Because η'''_1 is $O(\alpha)$, wave amplitude becomes αa so that $a \equiv 1/k$. See Table 2 for the value of physical parameters assumed in this section. The governing equations (26a)–(26c) can be rewritten in wave space:

$$\underbrace{-i\sigma z'''_{1z}}_{w'''_{1z}} + iku'''_1 = 0, \quad (33a)$$

$$-fv'''_1 = -ik(gae^{i\theta} + p'''_1/\rho) + (i\sigma - \nu k^2 + \nu\partial^2_z)u'''_1, \quad (33b)$$

$$fu'''_1 = (i\sigma - \nu k^2 + \nu\partial^2_z)v'''_1, \quad (33c)$$

$$0 = -p'''_{1z}/\rho + (i\sigma - \nu k^2 + \nu\partial^2_z)w'''_1, \quad (33d)$$

where (29a)–(29c) have been used.

We consider a Poisson equation for p'''_1, derived from the three-dimensional divergence of (26b) and (26c):

$$-\rho f\nabla \times \mathbf{V}'''_1 = -\nabla^2(g\rho\eta'''_1 + p'''_1) - p'''_{1zz} + \rho\nu(\nabla^2 + \partial^2_z)(\underbrace{\nabla \cdot \mathbf{V}'''_1 + w'''_{1z}}_{0}), \quad (34)$$

where (29a)–(29c) have been used. The viscosity term vanishes because $O(\alpha)$ velocity satisfies an incompressible condition (26a). A wave-space expression of (34) is

$$ik\rho fv'''_1 = -k^2(g\rho ae^{i\theta} + p'''_1) + p'''_{1zz}. \quad (35)$$

Substitution of (33a) and (35) to the lhs and rhs of (33c), respectively, yields

$$-\rho f^2 w'''_{1z} = (i\sigma - \nu k^2 + \nu\partial^2_z)[-k^2(g\rho ae^{i\theta} + p'''_1) + p'''_{1zz}]. \quad (36)$$

Substitution of (33d) to the vertical derivative of (36) yields a characteristic equation for the vertical profile of w'''_1,

$$-f^2 w'''_{1zz} = (i\sigma - \nu k^2 + \nu\partial^2_z)^2(-k^2 w'''_1 + w'''_{1zz}), \quad (37)$$

which can be approximated by two separate equations:

$$-f^2 w'''_{1zz} \simeq -\sigma^2(-k^2 w'''_1 + w'''_{1zz}), \quad (38a)$$

$$-f^2 w'''_{1zz} \simeq (i\sigma - \nu k^2 + \nu\partial^2_z)^2 w'''_{1zz}. \quad (38b)$$

The first equation can be reduced further to $0 \simeq (-k^2 w'''_1 + w'''_{1zz})$ because $f/\sigma \equiv \gamma \ll 1$ (nondimensional positive real constant, Table 2). Thus, w'''_1 is written by the composite of e^{kz} and $e^{n^{\pm}z}$, where

$$n^{\pm} \equiv \sqrt{\frac{-i\sigma + \nu k^2 \pm if}{\nu}} = m\sqrt{1 + i\beta \mp \gamma}, \quad (39)$$

TABLE 2. The value of physical parameters assumed in section 3.

		Main text	Figure 3
α		10^{-1}	0.1 or 0.2
$\beta \equiv \nu k^2/\sigma$		between 10^{-6} and 10^{-4}	10^{-5}
$\gamma \equiv f/\sigma$		10^{-4}	10^{-4}
ρ	kg m^{-3}	10^3	10^3
σ	s^{-1}	1	1
f	s^{-1}	10^{-4}	10^{-4}
k	m^{-1}	10^{-1}	10^{-1}
$\alpha a = \alpha/k$	m	1	1.0 or 2.0
ν	m^2 s^{-1}	between 10^{-4} and 10^{-2}	10^{-3}
$\sqrt{\nu/\sigma} = \sqrt{\beta}/k$	m	between 10^{-2} and 10^{-1}	3.2×10^{-2}
$\sqrt{\nu/f}$	m	between 1 and 10	3.2
$2\rho\nu\sigma k^2(\alpha a)^2$	N m^{-2}	between 10^{-3} and 10^{-1}	0.02 or 0.08
$\alpha^2\overline{\tau_2}$	N m^{-2}	arbitrary	0.1
$\sigma k(\alpha a)^2$	m s^{-1}	10^{-1}	0.1 or 0.4
$\alpha^2\overline{\tau_2}/(\rho\sqrt{2\nu f})$	m s^{-1}	arbitrary	0.22

in which $\beta \equiv \nu k^2/\sigma \ll 1$ (nondimensional positive real constant, Table 2) and $m \equiv \sqrt{-i\sigma/\nu} = \sqrt{-i/\beta}k$ (complex constant). Using both $(i\sigma - \nu k^2 + \nu\partial_z^2)e^{kz+i\theta} = i\sigma e^{kz+i\theta}$ and $(i\sigma - \nu k^2 + \nu\partial_z^2)e^{n^{\pm}z+i\theta} = \pm ife^{n^{\pm}z+i\theta}$, we solve (33a)–(33d) and obtain both a general solution

$$p_1''' = \mathrm{Re}\left\{e^{i\theta}\left[e^{kz}(a + b^+ + b^-) + \left(\frac{b^+}{n^+}e^{n^+z} - \frac{b^-}{n^-}e^{n^-z}\right)\gamma k - a\right]\right\}\rho g, \tag{40a}$$

$$z_1''' = \mathrm{Re}\{e^{i\theta}[e^{kz}(a + b^+ + b^-) - b^+e^{n^+z} - b^-e^{n^-z}]\}, \tag{40b}$$

$$u_1''' = \mathrm{Re}\left\{e^{i\theta}\left[e^{kz}(a + b^+ + b^-) - (b^+n^+e^{n^+z} + b^-n^-e^{n^-z})\frac{1}{k}\right]\right\}\sigma, \tag{40c}$$

$$v_1''' = \mathrm{Im}\left\{e^{i\theta}\left[e^{kz}(a + b^+ + b^-) - (b^+n^+e^{n^+z} - b^-n^-e^{n^-z})\frac{1}{\gamma k}\right]\right\}f, \tag{40d}$$

$$w_1''' = \mathrm{Im}\{e^{i\theta}[e^{kz}(a + b^+ + b^-) - b^+e^{n^+z} - b^-e^{n^-z}]\}\sigma, \tag{40e}$$

and a dispersion relation $\sigma^2 = gk$. Each of b^+ and b^- is a complex constant to be determined in the next subsection. The above solution is given in the VL coordinates so that there is no need to extrapolate the solution using the Taylor expansion to include regions above the free surface, for example, where there are surface troughs, as in previous studies.

d. Case of no air pressure disturbance

One way to determine b^+ and b^- is to assume at the sea surface that (i) there is no air pressure disturbance, $p_1'''|_{z=0} = 0$, and (ii) there is no stress in the direction of wave crests, $v_{1z}'''|_{z=0} = 0$. With a straightforward manipulation (appendix B), the general solution (40a)–(40e) is reduced to

$$p_1''' = \mathrm{Re}\{e^{i\theta}(e^{kz} - 1)\}a\rho g, \tag{41a}$$

$$z_1''' = \mathrm{Re}\{e^{i\theta+kz}\}a, \tag{41b}$$

$$u_1''' = \mathrm{Re}\{e^{i\theta+kz}\}a\sigma, \tag{41c}$$

$$v_1''' = \mathrm{Im}\left\{e^{i\theta}\left[e^{kz} - e^{mz}\left(1 - i\frac{\beta}{2} + i\frac{\beta}{2}mz\right)\sqrt{i\beta}\right]\right\}af, \tag{41d}$$

$$w_1''' = \mathrm{Im}\{e^{i\theta+kz}\}a\sigma. \tag{41e}$$

Substitution of (41a)–(41d) to (32a) and (32b) yields the TWM momentum balance at $O(\alpha^2)$,

$$-\rho f\hat{v}_2 = \rho\nu(\hat{u}_{2z} - \underbrace{2\sigma k^2a^2e^{2kz}}_{z_{1zz}'''u_1'''+2z_{1z}'''u_{1z}'''+z_{1x}'''u_{1x}'''})_z, \tag{42a}$$

$$\rho f\hat{u}_2 = \rho\nu(\hat{v}_{2z} - \underbrace{fk^2a^2\mathrm{Re}\{e^{(k+m)z}\}}_{z_{1zz}'''v_1'''+2z_{1z}'''v_{1z}'''+z_{1x}'''v_{1x}'''})_z, \tag{42b}$$

where the pressure term has vanished owing to the phase relationship of $O(\alpha)$ waves. Substitution of (41b)–(41d) to (16) yields the expression of the quasi-Stokes velocity,

$$u_2^{qs} \equiv (\overline{z_1'''u_1'''})_z = \sigma ka^2e^{2kz}, \tag{43a}$$

$$v_2^{qs} \equiv (\overline{z_1'''v_1'''})_z = \frac{fka^2}{2}\mathrm{Re}\{ie^{(k+m)z}\}, \tag{43b}$$

where u_2^{qs}, at this order in α, is identical to the Stokes-drift velocity in the inviscid theory (hereafter, the inviscid Stokes velocity refers to σka^2e^{2kz}).[9] The viscous stress in (42a) can be rewritten as $\rho\nu(\hat{u}_{2z} - 2\sigma k^2a^2e^{2kz}) = \rho\nu(\hat{u}_{2z} - u_{2z}^{qs}) = \rho\nu\overline{u}_{2z}^c$. The stress acts on the EM component of velocity and not the TWM velocity. This result holds for irrotational waves in general, as shown using $u_{1z}''' - w_{1x}''' = 0$, $u_{1x}''' + w_{1z}''' = 0$ and (43a) to obtain

[9] It should be noted that the expression of (u_2^{qs}, v_2^{qs}) is not guaranteed to be the same as the Stokes drift for inviscid waves, a case in point being the solution in the next subsection.

$$\begin{aligned}
\hat{u}_{2z} &- (\overline{z'''_{1zz}u'''_1 + 2z'''_{1z}u'''_{1z} + z'''_{1x}u'''_{1x}}) \\
&= \overline{u}^c_{2z} + \overline{z'''_1u'''_{1zz} - z'''_{1x}u'''_{1x}} \\
&= \overline{u}^c_{2z} + \overline{z'''_1w'''_{1xz} - z'''_{1x}u'''_{1x}} \\
&= \overline{u}^c_{2z} - \overline{z'''_{1x}w'''_{1z}} - \overline{z'''_{1x}u'''_{1x}} \\
&= \overline{u}^c_{2z} + \overline{z'''_{1x}u'''_{1x} - z'''_{1x}u'''_{1x}} \\
&= \overline{u}^c_{2z}. \qquad (44)
\end{aligned}$$

Indeed, (41c) and (41e) satisfy both the irrotational and incompressibility conditions.

The boundary condition for $(\hat{u}_{2z}, \hat{\upsilon}_{2z})$ is set by the rate of momentum input at the sea surface,

$$\rho\nu(\hat{u}_{2z}|_{z=0} - 2\sigma k^2a^2) = \overline{\tau_2}, \qquad (45a)$$

$$\rho\nu(\hat{\upsilon}_{2z}|_{z=0} - fk^2a^2) = 0, \qquad (45b)$$

which represents the wind blowing in the direction of wave propagation, with $\overline{\tau_2}$ being the wind stress (here introduced at order α^2). The fk^2a^2 term in (45b) may be omitted because of $f/\sigma = \gamma \ll 1$. Equations (43b) and (42b) contain terms proportional to $e^{mz} = e^{\sqrt{-i/\beta}kz}$ that are effective only within the thin viscous boundary layer associated with the waves (hereafter referred to simply as the viscous boundary layer, not to be confused with the Ekman layer of depth $\sqrt{\nu/f}$). At depths below this layer, (42a) and (42b) can be rendered into

$$-f\hat{\upsilon}_2 = \nu(\hat{u}_{2z} - 2\sigma k^2a^2e^{2kz})_z, \qquad (46a)$$

$$f\hat{u}_2 = \nu\hat{\upsilon}_{2zz}, \qquad (46b)$$

which may be solved by using the boundary condition $(\hat{u}_{2z}, \hat{\upsilon}_{2z})|_{z=0} = (\overline{\tau_2}/(\rho\nu) + 2\sigma k^2a^2, 0)$ to yield

$$\hat{u}_2 + i\hat{\upsilon}_2 = \frac{(2k/\epsilon)e^{\epsilon z} + (4ik^2\nu/f)e^{2kz}}{1 + 4ik^2\nu/f}\sigma ka^2 + \frac{\overline{\tau_2}}{\rho\nu\epsilon}e^{\epsilon z}, \qquad (47)$$

where the second term is the classical Ekman spiral velocity with $\epsilon = \sqrt{if/\nu}$ being a complex constant. By subtracting $u^{qs}_2 + i\upsilon^{qs}_2 \simeq \sigma ka^2e^{2kz}$ from (47), we obtain the EM velocity

$$\overline{u}^c_2 + i\overline{\upsilon}^c_2 = \frac{(2k/\epsilon)e^{\epsilon z} - e^{2kz}}{1 + 4ik^2\nu/f}\sigma ka^2 + \frac{\overline{\tau_2}}{\rho\nu\epsilon}e^{\epsilon z}, \qquad (48)$$

which corresponds to the solution of Huang (1979) and Polton et al. (2005). The latter point out that the classical Ekman spiral solution is modified by the presence of surface waves because the waves drive flow near the surface through the Coriolis–Stokes force. Even though the flow that is directly driven by the Coriolis–Stokes force is surface confined, its effect is felt throughout the whole depth of the surface Ekman layer, as shown by the first term of (48) and illustrated in Fig. 3. This is because the presence of the surface-confined flow modifies the surface boundary condition from that in the classical Ekman problem.

Setting the turbulent viscosity $\nu = 0$ in (46a) and (46b), it follows immediately that in the inviscid case (no turbulent mixing and no surface wind stress) $\hat{\mathbf{V}}_2 = 0$. Since $\overline{\mathbf{V}}_1 = \hat{\mathbf{V}}_1 = 0$, as follows from (25a) and (25d), it follows that there is no net horizontal transport by the waves up to $O(\alpha^2)$, corresponding to the result of Ursell (1950), Pollard (1970), and Hasselmann (1970) that surface waves propagating without change of form in a rotating system have no net mass transport associated with them.

To understand the budget of mean kinetic energy, $(\rho/2)(\hat{u}^2_2 + \hat{\upsilon}^2_2)$, in each vertical column, we take the depth integral of the inner product of $(\hat{u}_2, \hat{\upsilon}_2)$ and (46a) and (46b),

$$\begin{aligned}
0 &= \hat{u}_2\rho\nu(\hat{u}_{2z} - 2\sigma k^2a^2)|_{z=0} - \rho\nu\int_{-\infty}^{0}[\hat{u}_{2z}(\hat{u}_{2z} - 2\sigma k^2a^2e^{2kz}) + \hat{\upsilon}_{2z}(\hat{\upsilon}_{2z} - fk^2a^2\mathbf{Re}\{e^{(k+m)z}\})]\,dz \\
&\simeq \hat{u}_2|_{z=0}\overline{\tau_2} - \rho\nu\int_{-\infty}^{-\delta}[\hat{u}_{2z}(\hat{u}_{2z} - 2\sigma k^2a^2e^{2kz}) + \hat{\upsilon}_{2z}\hat{\upsilon}_{2z}]\,dz \\
&= \underbrace{\hat{u}_2|_{z=0}\overline{\tau_2}}_{\text{surface viscous stress}} - \underbrace{\rho\nu\int_{-\infty}^{-\delta}(\overline{u}^c_{2z}\overline{u}^c_{2z} + \overline{\upsilon}^c_{2z}\overline{\upsilon}^c_{2z})\,dz}_{\text{viscous stress on EM velocity}} - \underbrace{\rho\nu\int_{-\infty}^{-\delta}2\sigma k^2a^2e^{2kz}\overline{u}^c_{2z}\,dz}_{\text{viscous stress on Stokes velocity}}, \qquad (49)
\end{aligned}$$

where integration by parts has been used and we have assumed that the depth integral is not sensitive to complicated terms in the viscous boundary layer of thin thickness $\delta \sim \sqrt{\beta}/k$. The first term in the last line of (49) represents the work of wind stress on the TWM velocity at surface. The second term represents the dissipation of mean kinetic energy (i.e., production of turbulent kinetic energy) based on the vertical shear of the EM

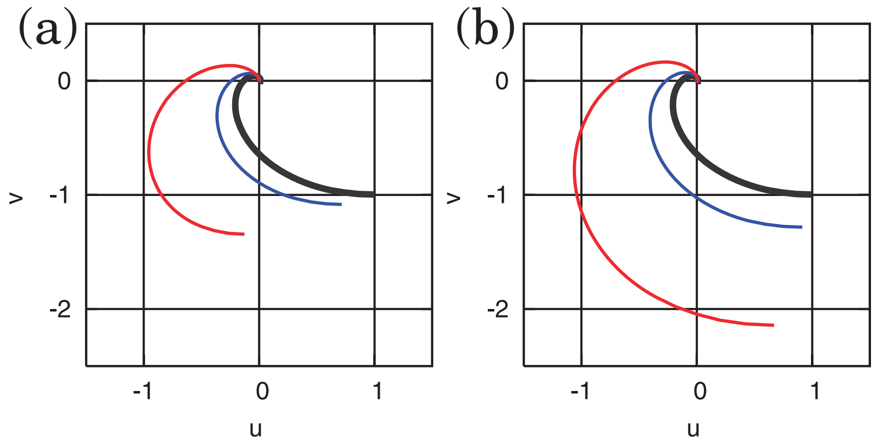

FIG. 3. Hodograph of the Eulerian mean velocity for the case of (a) no surface pressure disturbances and (b) no variations in tangential stress. The blue line is the case with dimensional wave amplitude $\alpha a = 1.0$ m and for the red line $\alpha a = 2.0$ m. The black line is the classical Ekman spiral velocity: $[\alpha^2\overline{\tau}_2/(\rho\nu\epsilon)]e^{\epsilon z} = [\alpha^2\overline{\tau}_2/(\rho\sqrt{i\nu f})]e^{\epsilon z}$. The wind stress is $\alpha^2\overline{\tau}_2 = 0.1$ N m^{-2}. The values of the other parameters are listed in Table 2. Both axes are scaled by the magnitude of the x component of the classical Ekman spiral velocity at the sea surface, $\alpha^2\overline{\tau}_2/(\rho\sqrt{2\nu f}) = 0.22$ m s^{-1}. The Ekman layer depth is $\sqrt{\nu/f} = 3.2$ m. Note that the velocity at the surface is different in each case, with the velocity spiraling to zero at depth.

velocity. The third term is given by the vertical shear of the inviscid Stokes velocity $2\sigma k^2a^2e^{2kz} = (\sigma ka^2e^{2kz})_z$ and might be related to the Stokes production of turbulent kinetic energy that has been considered in Teixeira and Belcher (2002) and Kantha and Clayson (2004). However, when wave amplitude is large, the x component of the EM velocity $\overline{u}_2^c$ tends to be against the direction of the wave propagation and the wind (Fig. 3a), with the result that the Stokes shear term of (49) is actually sign indefinite (and therefore not necessarily a production term). It should be noted that, although the EM velocity is against the wind stress, for the red curve in Fig. 3a, the wind stress nevertheless inputs energy through the work that is done by the wind stress on the quasi-Stokes (i.e., Stokes drift) component of the TWM velocity.

e. *Case of no variation in tangential stress arising from the presence of the waves*

Another way to determine b^+ and b^- in (40a)–(40e) is to assume that (i) there is no variation in the tangential component of surface stress arising from the presence of the waves $(u_{1z}''' + w_{1x}''')|_{z=0} = 0$ (Longuet-Higgins 1953, 1960) and (ii) there is no surface stress in the direction of wave crests $\upsilon_z'''|_{z=0} = 0$ (this is as in the previous subsection). With a straightforward manipulation (appendix B), the general solution (40a)–(40e) is reduced to

$$p_1''' = \mathrm{Re}\{e^{i\theta}[e^{kz}(1 + 2\beta i) - 1]\}a\rho g, \tag{50a}$$

$$z_1''' = \mathrm{Re}\{e^{i\theta}[e^{kz}(1 + 2\beta i) + e^{mz}(-2i + \beta mz)\beta]\}a, \tag{50b}$$

$$u_1''' = \mathrm{Re}\{e^{i\theta}[e^{kz}(1 + 2\beta i) + e^{mz}(-2i + \beta + \beta mz)\sqrt{-i\beta}]\}a\sigma, \tag{50c}$$

$$\upsilon_1''' = \mathrm{Im}\left\{e^{i\theta}\left[e^{kz}(1 + 2\beta i) + e^{mz}\left(-2i + \frac{1}{2}\beta + imz + \frac{3}{2}\beta mz\right)\sqrt{-i\beta}\right]\right\}af, \tag{50d}$$

$$w_1''' = \mathrm{Im}\{e^{i\theta}[e^{kz}(1 + 2\beta i) + e^{mz}(-2i + \beta mz)\beta]\}a\sigma. \tag{50e}$$

Substitution of (50a)–(50e) to (32a) and (32b) yields the TWM momentum balance at $O(\alpha^2)$,

$$-\rho f\hat{\upsilon}_2 = [\underbrace{\rho\nu\sigma k^2a^2\mathrm{Re}\{e^{(k+m)z}\}}_{\overline{z_{1x}'''(\rho g\eta_1''' + p_1''')}}]_z + \rho\nu[\hat{u}_{2z} - \underbrace{\sigma k^2a^2(2e^{2kz} - 3\mathrm{Re}\{e^{(k+m)z}\})}_{\overline{z_{1zz}'''u_1''' + 2z_{1z}'''u_{1z}''' + z_{1x}'''u_{1x}'''}}]_z, \tag{51a}$$

$$\rho f\hat{u}_2 = \rho\nu\left[\hat{\upsilon}_{2z} - \underbrace{fk^2a^2\mathrm{Re}\left\{\frac{imz - 3i}{2}e^{(k+m)z}\right\}}_{\overline{z_{1zz}'''\upsilon_1''' + 2z_{1z}'''\upsilon_{1z}''' + z_{1x}'''\upsilon_{1x}'''}}\right]_z. \tag{51b}$$

Equation (51a) is the cornerstone for explaining the so-called VWS of Longuet-Higgins (1953) concerning the vertical transfer of the x component of momentum. Our explanation follows the following steps (as explained in the text that follows):

(i) $\hat{u}_{2z} = \overline{\tau_2}/(\rho\nu)$ at the sea surface (where the wind stress $\overline{\tau_2}$ is introduced at second order in α, as before);
(ii) the momentum flux through the sea surface is $\overline{\tau_2} + 2\rho\nu\sigma k^2a^2$, of which the wave-induced flux $2\rho\nu\sigma k^2a^2$ is attributed in equal measure to form stress and viscous stress;
(iii) the momentum flux through the base of the thin viscous boundary layer is $\overline{\tau_2} + 2\rho\nu\sigma k^2a^2$, all of which is maintained by viscous stress; and
(iv) $\hat{u}_{2z} = \overline{\tau_2}/(\rho\nu) + 4\sigma k^2a^2$ at the base of the thin viscous boundary layer.

The condition of no variation in the tangential stress gives a constraint for the vertical gradient of the TWM velocity. This constraint is written by (B10) to which we substitute an identity $(\sigma/k)z'''_{1zz} = u'''_{1z} = -w'''_{1x} = -(\sigma k)\eta'''_1$, which has been derived from (33a) and $(u'''_{1z} + w'''_{1x})|_{z=0} = 0$. It follows that

$$
\begin{aligned}
\hat{u}_{2z}|_{z=0} &= \overline{\tau_2}/(\rho\nu) + \overline{z'''_{1zz}u'''_1} + 2\overline{z'''_{1z}u'''_{1z}} + 3\overline{\eta'''_{1x}u'''_{1x}} \\
&= \overline{\tau_2}/(\rho\nu) - k^2\overline{\eta'''_1u'''_1} - 2k^2\overline{u'''_1\eta'''_1} + 3k^2\overline{\eta'''_1u'''_1} \\
&= \overline{\tau_2}/(\rho\nu).
\end{aligned} \tag{52}
$$

This is (i). The rhs of (51a) has been written as the vertical divergence of a pressure-induced momentum flux (i.e., form stress) and a viscosity-induced momentum flux (i.e., viscous stress). At the sea surface where $z = 0$, the form stress becomes $\overline{\eta'''_{1x}p'''_1} = \rho\nu\sigma k^2a^2$ and the viscous stress becomes $\overline{\tau_2} - \rho\nu(\overline{z'''_{1zz}u'''_1 + 2z'''_{1z}u'''_{1z} + z'''_{1x}u'''_{1x}}) = \overline{\tau_2} + \rho\nu\sigma k^2a^2$, yielding a total momentum flux of $\overline{\tau_2}+2\rho\nu\sigma k^2a^2$, where the last term represents the effect of waves. This is (ii). The viscous boundary layer is so thin (thickness is scaled by $\sqrt{\nu/\sigma} = \sqrt{\beta}/k$) that the Coriolis term on the lhs of (51a) would have little contribution to the momentum balance within the layer. Thus, the vertical profile of the total momentum flux is nearly constant, $\overline{\tau_2} + 2\rho\nu\sigma k^2a^2$, within the boundary layer. This is (iii). The $2\rho\nu\sigma k^2a^2$ part is what has been called the VWS in previous studies. Below the viscous boundary layer, terms proportional to $e^{mz} = e^{\sqrt{-i/\beta}kz}$ in (51a) vanish so that the vertical transfer of momentum is done by only the viscous stress, $\rho\nu(\hat{u}_{2z} - 2\sigma k^2a^2)$. This stress should match $\overline{\tau_2} + 2\rho\nu\sigma k^2a^2$, which comes from (iii). The result is that, at the base of the viscous boundary layer, $\hat{u}_{2z} = \overline{\tau_2}/(\rho\nu) + 4\sigma k^2a^2$, whose last term is twice the vertical gradient of the inviscid Stokes velocity (Longuet-Higgins 1953). This is (iv). It should be noted that in the above analysis, the TWM momentum equation has been written in a flux-divergence form, which is suitable for identifying the route of the momentum transfer.

To understand the budget of mean kinetic energy, $(\rho/2)(\hat{u}_2^2 + \hat{v}_2^2)$, in each vertical column, we take the depth integral of the inner product of $(\hat{u}_2, \hat{v}_2)$ and (51a) and (51b),

$$
\begin{aligned}
0 &= \hat{u}_2|_{z=0}(\overline{\tau_2} + 2\rho\nu\sigma k^2a^2) - \rho\nu\int_{-\infty}^{0}\hat{u}_{2z}[\hat{u}_{2z} - \sigma k^2a^2(2e^{2kz} - 3\mathbf{Re}\{e^{(k+m)z}\})]\,dz \\
&\quad - \rho\nu\int_{-\infty}^{0}\hat{v}_{2z}\left(\hat{v}_{2z} - fk^2a^2\mathbf{Re}\left\{\frac{imz - 3i}{2}e^{(k+m)z}\right\}\right)dz \\
&\simeq \hat{u}_2|_{z=0}(\overline{\tau_2} + 2\rho\nu\sigma k^2a^2) - \rho\nu\int_{-\infty}^{-\delta}[\hat{u}_{2z}(\hat{u}_{2z} - 2\sigma k^2a^2e^{2kz}) + \hat{v}_{2z}\hat{v}_{2z}]\,dz \\
&= \underbrace{\hat{u}_2|_{z=0}(\overline{\tau_2} + \rho\nu\sigma k^2a^2)}_{\text{surface viscous stress}} + \underbrace{\hat{u}_2|_{z=0}\rho\nu\sigma k^2a^2}_{\text{surface form stress}} - \underbrace{\rho\nu\int_{-\infty}^{-\delta}(\overline{u}^c_{2z}\overline{u}^c_{2z} + \overline{v}^c_{2z}\overline{v}^c_{2z})\,dz}_{\text{viscous stress on EM velocity}} - \underbrace{\rho\nu\int_{-\infty}^{-\delta}2\sigma k^2a^2e^{2kz}\overline{u}^c_{2z}\,dz}_{\text{viscous stress on Stokes velocity}},
\end{aligned} \tag{53}
$$

where integration by parts has been used. In addition to wind stress $\overline{\tau_2}$, wave viscous stress $\rho\nu\sigma k^2a$ and form stress $\rho\nu\sigma k^2a$ at the surface feed the mean kinetic energy, as illustrated in Fig. 4.

Below the viscous boundary layer, (51a) and (51b) are reduced to (46a) and (46b), which can be solved using an adjusted boundary condition $\rho\nu(\hat{u}_{2z}, \hat{v}_{2z})|_{z=-\delta} = (\overline{\tau_2}/(\rho\nu) + 4\sigma k^2a^2, 0)$ to yield

$$
\begin{aligned}
\hat{u}_2 + i\hat{v}_2 &= \frac{(2k/\epsilon)e^{\epsilon(z+\delta)} + (4ik^2\nu/f)e^{2k(z+\delta)}}{1 + 4ik^2\nu/f}\sigma ka^2 \\
&\quad + \frac{\overline{\tau_2} + 2\rho\nu k^2a^2}{\rho\nu\epsilon}e^{\epsilon(z+\delta)},
\end{aligned} \tag{54}
$$

which corresponds to Eq. (16) of Madsen (1978), who considered the same problem using the three-dimensional Lagrangian mean equations of Pierson (1962). Indeed,

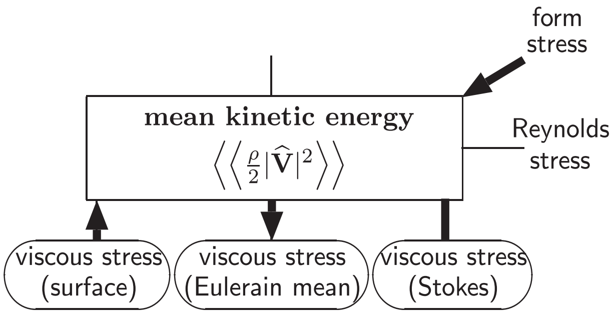

FIG. 4. Illustration of the budget of the mean kinetic energy in the case of no variation in the tangential component of surface stress (section 3e).

(46a) and (46b) are identical to (5) and (6) of Madsen (1978) with our TWM velocity corresponding to their Lagrangian mean velocity. Substitution of (50b)–(50d) to (16) yields the expression of the quasi-Stokes velocity

$$u_2^{qs} \equiv (\overline{z_1'''u_1'''})_z = \sigma k a^2 (e^{2kz} + \mathrm{Re}\{e^{(k+m)z}\}), \tag{55a}$$

$$v_2^{qs} \equiv (\overline{z_1'''v_1'''})_z = \frac{fka^2}{2}\mathrm{Re}\{(i - imz)e^{(k+m)z}\}, \tag{55b}$$

which is slightly different from (43a) and (43b), but both reduce to $(u_2^{qs}, v_2^{qs}) = (\sigma k a^2 e^{2kz}, 0)$ —the inviscid Stokes velocity—below the thin viscous boundary layer. By subtracting $u_2^{qs} + i v_2^{qs} \simeq \sigma k a^2 e^{2kz}$ from (54), we obtain the EM velocity

$$\overline{u}_2^c + i\overline{v}_2^c = \frac{(2k/\epsilon)e^{\epsilon z} - e^{2kz}}{1 + 4ik^2\nu/f}\sigma k a^2 + \frac{\overline{\tau_2} + 2\rho\nu k^2 a^2}{\rho\nu\epsilon}e^{\epsilon z}, \tag{56}$$

which corresponds to the solution of Xu and Bowen [1994, their Eq. (87)]. The first term of (56) is almost identical to that of (48). The characteristics of this term have already been explained in the previous subsection concerning Fig. 3. The second term of (56) can be regarded as the classical Ekman velocity caused by the combined wind stress and VWS, $\overline{\tau_2} + 2\rho\nu\sigma k^2 a^2$. We estimate the strength of the VWS, $2\rho\nu\sigma k^2(\alpha a)^2$, based on the values of physical parameters used to plot in Fig. 3. When wave amplitude is small ($\alpha a = 1.0$ m), the VWS becomes 0.02 N m^{-2}. When wave amplitude is large ($\alpha a = 2.0$ m), the VWS becomes 0.08 N m^{-2}, which is close to the strength of wind stress $\alpha^2\overline{\tau_2} = 0.1$ N m^{-2}. See, for example, Weber et al. (2006) for detailed comparisons of the strengths of wind stress and VWS from a model simulation.

f. Discussion of non-Lagrangian approaches

Below the thin surface viscous layer associated with the waves, the TWM velocity satisfies the equation system (46a) and (46b) in both cases considered above (i.e., surface boundary conditions of both no pressure disturbance and no variation of the tangential stress). Transforming to the EM system, the resulting equation system is identical to the EM momentum equations that have been used by Huang (1979), Xu and Bowen (1994), and Polton et al. (2005):

$$-f\overline{v}_2^c = \nu\overline{u}_{2zz}^c, \tag{57a}$$

$$f\overline{u}_2^c = \underbrace{-f\sigma k a^2 e^{2kz}}_{-(\overline{w'v'}^c)_z} + \nu\overline{v}_{2zz}^c. \tag{57b}$$

The term looking like the Coriolis force induced by the inviscid Stokes velocity is the Coriolis–Stokes force and can be derived by substituting an inviscid wave solution for w' and v' to the Reynolds stress term (Hasselmann 1970), where A' indicates deviation from the Eulerian time mean $\overline{A}^c$ for an arbitrary quantity A. The boundary condition of Huang (1979) and Polton et al. (2005) is $(\overline{u}_{2z}^c, \overline{v}_{2z}^c)|_{z=0} = (\overline{\tau_2}/(\rho\nu), 0)$, which corresponds to our boundary condition for the TWM velocity, $(\hat{u}_{2z}, \hat{v}_{2z})|_{z=0} = (\overline{\tau_2}/(\rho\nu) + 2\sigma k^2 a^2, 0)$, in the case of no air pressure disturbance. The boundary condition of Xu and Bowen (1994) is $(\overline{u}_{2z}^c, \overline{v}_{2z}^c)|_{z=-\delta} = (\overline{\tau_2}/(\rho\nu) + 2\sigma k^2 a^2, 0)$, which corresponds to our boundary condition for the TWM velocity, $(\hat{u}_{2z}, \hat{v}_{2z})|_{z=-\delta} = (\overline{\tau_2}/(\rho\nu) + 4\sigma k^2 a^2, 0)$, in the case of no variation in the tangential stress. Interestingly, although the surface boundary condition used by Xu and Bowen (1994) corresponds to our case of no variation of the tangential stress, corresponding to step (iv) in our derivation (see the previous subsection), these authors appear to have arrived at (iv) without apparently using steps (i)–(iii) (they do not take explicit account of the thin viscous boundary layer). The large difference between the different solutions is apparent from Fig. 3, where the left panel shows the solution of Huang (1979) and Polton et al. (2005) for two different wave amplitudes and the right panel shows the corresponding solution of Xu and Bowen (1994) for the same two wave amplitudes. Clearly, the different surface boundary conditions applied to the waves can have a big effect on the resulting EM velocity, not only at the surface but also throughout the water column.

To summarize the difference between the different solutions, it is of interest to understand the budget of momentum in each vertical column, starting with the horizontal momentum equations in Cartesian coordinates. For the problem being considered here, the instantaneous momentum equation in the x direction is

$$\rho[u_{t^c} + (uu)_{x^c} + (wu)_{z^c} - fv] = -(\rho g\eta + p)_{x^c} + \rho\nu(u_{x^c x^c} + u_{z^c z^c}). \quad (58)$$

Vertically integrating over the depth of the ocean, we obtain equations for the volume transport given by

$$\rho\left(\frac{\partial}{\partial t^c}\int_{-\infty}^{\eta} u\,dz^c + \frac{\partial}{\partial x^c}\int_{-\infty}^{\eta} uu\,dz^c - f\int_{-\infty}^{\eta} v\,dz^c\right) = \frac{\partial}{\partial x^c}\int_{-\infty}^{\eta}(-\rho g\eta - p + \nu u_{x^c})\,dz^c + [(\rho g\eta + p)\eta_{x^c} + \rho\nu(-u_{x^c}\eta_{x^c} + u_{z^c})]|_{z^c=\eta}. \quad (59)$$

Time averaging the above equation, as the problem is horizontally homogeneous, then gives

$$-\rho f\overline{\int_{-\infty}^{\eta} v\,dz^c} = [\overline{\eta_{x^c}p} + \rho\nu(\overline{u_{z^c}} - \overline{\eta_{x^c}u_{x^c}})]|_{z^c=\eta}, \quad (60)$$

where there is no Reynolds stress term, consistent with the absence of Reynolds stress terms in (31a) and (32b). These terms drop out because there is no net convergence/divergence of momentum into the water column by the waves. The pressure term of (60) corresponds to the form stress at the surface and vanishes in the case of no pressure perturbations at the sea surface but is nonetheless nonzero in the case of no variations of the tangential stress, as we saw in the previous subsection. Next we note that the viscosity term of (60) can be written as

$$\begin{aligned}\rho\nu(\overline{u_{z^c}} - \overline{\eta_{x^c}u_{x^c}})|_{z^c=\eta} &\simeq \rho\nu(\overline{u}_z - \overline{z_z'''u_z} - \overline{\eta_{x^c}u_{x^c}})|_{z^c=\eta}\\ &= \rho\nu(\hat{u}_z - (\overline{z_z'''u'''})_z - \overline{z_z'''u_z'''} - \overline{\eta_{x^c}'''u_{x^c}'''})|_{z^c=\eta}\\ &= \rho\nu(\hat{u}_z - \overline{z_{zz}'''u'''} - 2\overline{z_z'''u_z'''} - \overline{\eta_{x^c}'''u_{x^c}'''})|_{z^c=\eta},\end{aligned} \quad (61)$$

where $u_{z^c} = u_z/z_z^c = u_z/(1 + z_z''') \simeq u_z(1 - z_z''')$ has been used. In the case of no pressure disturbance at the surface, the total momentum input at the surface is given by the rhs of (61) and was set equal to the surface wind stress [Eq. (45a)]. In the case of no variation of the tangential stress, $\hat{u}_z$ is set by the surface wind stress and the remaining terms on the rhs correspond to the viscous wave stress input noted in step (ii) of the previous subsection.

4. Summary and discussion

A theory is presented to investigate the effect of surface gravity waves on ocean currents in the presence of a uniform turbulent viscosity. Depth-dependent equations for the conservation of volume, momentum, and energy are derived using a thickness-weighted mean (TWM) approach in a vertically Lagrangian (VL) and horizontally Eulerian coordinate system, analogous to the TWM approach in isopycnal coordinates in theories describing the impact of mesoscale eddies on the large-scale ocean circulation. Some advantages of the TWM approach are (i) the theory allows for both finite amplitude fluid motions and the background vertical flows associated with the horizontal divergence/convergence of currents, without resorting to Taylor or perturbation expansions, and (ii) a concise treatment of the surface kinematic condition as well as the boundary condition for the viscosity term, avoiding complexity in the boundary conditions of Eulerian-mean (EM) approaches.

To illustrate the advantage of the TWM approach, we have revisited the classical Ekman spiral problem, including surface wave effects, using an analytical treatment. The TWM approach can reproduce both the Lagrangian-mean equation system of Madsen (1978) and the EM equations of Xu and Bowen (1994) and Polton et al. (2005). We have also explored the different surface boundary conditions implicit in these studies. The case studied by Polton et al. and also Huang (1979) corresponds to applying a boundary condition of no pressure disturbance at the free surface to the waves (implying no form stress), whereas the solutions of Madsen, and Xu and Bowen, correspond to applying a condition of no variations in the tangential component of surface stress to the waves. In this second case, both the form stress and the viscous stress provide a net momentum flux through the surface to the vertically integrated momentum budget that, in turn, leads to a momentum input, corresponding to the virtual wave stress of Longuet-Higgins (1953, 1960), at the base of the thin viscous boundary layer associated with the waves. By writing the TWM momentum equation in a flux-divergence form, we were able to easily identify the route of momentum transfer, an advantage over using the three-dimensional Lagrangian equations of Pierson (1962).

There are many examples of attempts to couple large-scale circulation models with surface wave models, such as the Wave Ocean Model (WAM) (e.g., Komen et al. 1994; Jenkins 1989), WAVEWATCH (e.g., Tolman 1991; Moon 2005; Tang et al. 2007; Tamura et al. 2010; Waseda et al. 2011), and Simulating Waves Nearshore (SWAN) (e.g., Booij et al. 1999) models. Nevertheless, a motivation for using the VL coordinate system and the TWM approach is the ease with which this framework allows surface wave effects to be incorporated into large-scale circulation models, with a concise treatment of surface boundary conditions as well as a clear view of energy interactions. Indeed, there is a direct analogy between the VL coordinate system and the isopycnal coordinate system that has been advocated for use when incorporating the effects of mesoscale eddies in ocean circulation models (e.g., Gent et al. 1995; Greatbatch 1998; Greatbatch and McDougall 2003; Griffies 2004). However, because the viscosity acts only on the EM velocity (at least below the thin viscous boundary layer associated with the waves and which, in any case, will not be resolved by a large-scale circulation model), this means that for models that step forward the TWM velocity, such as Mellor et al. (2008), an additional term should be included to offset the effect of viscosity on the Stokes-drift velocity.

Concerning the forcing of the momentum equations in large-scale models, we speculate that a realistic model for the waves might be a linear combination of the two solutions that we have presented (the case of no air pressure disturbance leads to no VWS, whereas the case of no variations in the tangential stress leads to a significant VWS). The ratio of the linear combination is highly relevant to the maintenance mechanism of waves and is an important issue for the parameterization of wave forcing for use in large-scale models [cf. Weber et al. (2006) and also the papers by Jenkins (1986, 1989)], an issue to be addressed using the TWM framework in future work.

Finally, we note that the VL-mean equations of Mellor (2003, 2008) have sometimes been compared to the quasi-EM equations, including the generalized Lagrangian-mean equations that are expressed in terms of the quasi-EM velocity (cf. McWilliams et al. 2004; Ardhuin et al. 2008). The Craik and Leibovich (1976) vortex force, which enables simulations of Langmuir circulations, has been derived only for the quasi-EM equations, a topic we shall discuss in the context of the TWM equations in a later paper.

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APPENDIX A

Derivation of TWM Energy Equations

Using (9a)–(9d), one can derive pressure and kinetic energy (KE) equations,

$$[z_t^c(\rho g\eta + p)]_z + \nabla\cdot[z_z^c\mathbf{V}(\rho g\eta + p)] + [z_z^c w^*(\rho g\eta + p)]_z = \mathbf{V}\cdot[z_z^c\nabla(\rho g\eta + p)] + \underbrace{(z_t^c + z_z^c w^*)}_{w-\mathbf{V}\cdot\nabla z^c} p_z, \tag{A1a}$$

$$\left[z_z^c\frac{\rho}{2}(|\mathbf{V}|^2 + w^2)\right]_t + \nabla\cdot\left[z_z^c\mathbf{V}\frac{\rho}{2}(|\mathbf{V}|^2 + w^2)\right] + \left[z_z^c w^*\frac{\rho}{2}(|\mathbf{V}|^2 + w^2)\right]_z = \mathbf{V}\cdot[-z_z^c\nabla(\rho g\eta + p) + p_z\nabla z^c] - wp_z + z_z^c(\mathbf{V}\cdot F^{\mathbf{V}} + wF^w), \tag{A1b}$$

where the last term of (A1a) can be rewritten as $(w - \mathbf{V}\cdot\nabla z^c)p_z$ using (9b).

a. Depth-dependent equations

Equations for pressure and KE in the total (mean plus wave) field can be derived by low-pass temporal filtering (A1a) and (A1b),

$$[\overline{z_t'''(\rho g\eta''' + p''')}]_z + \nabla\cdot[\overline{z_z^c\mathbf{V}(\rho g\eta + p)}] + [\overline{z_z^c w^*(\rho g\eta + p)}]_z = -\overline{z_z^c\mathbf{V}\cdot\mathbf{G}} + \overline{wp_z}, \tag{A2a}$$

$$\left[\overline{z_z^c\frac{\rho}{2}(|\mathbf{V}|^2 + w^2)}\right]_t + \nabla\cdot\left[\overline{z_z^c\mathbf{V}\frac{\rho}{2}(|\mathbf{V}|^2 + w^2)}\right] + \left[\overline{z_z^c w^*\frac{\rho}{2}(|\mathbf{V}|^2 + w^2)}\right]_z = \overline{z_z^c\mathbf{V}\cdot\mathbf{G}} - \overline{wp_z} + \overline{z_z^c(\mathbf{V}\cdot F^{\mathbf{V}} + wF^w)}, \tag{A2b}$$

where $z_t^c = z_t + z_t''' = z_t'''$ has been used for the first term on the lhs of (A2a) and $\mathbf{G} \equiv -\nabla^c(\rho g\eta + p) = -\nabla(\rho g\eta + p) + p_{z^c}\nabla z^c$ is the negative of the horizontal gradient of the combined hydrostatic and nonhydrostatic pressure for simplicity. Total KE is a third moment quantity and can be decomposed into mean KE and wave KE using the Favre filtering,

$$\frac{\rho}{2}\overline{z_z^c(|\mathbf{V}|^2 + w^2)} = \underbrace{\frac{\rho}{2}(|\hat{\mathbf{V}}|^2 + \hat{w}^2)}_{\text{mean KE}} + \underbrace{\frac{\rho}{2}\overline{z_z^c(|\mathbf{V}''|^2 + w''^2)}}_{\text{wave KE}}, \tag{A3}$$

where each of mean KE and wave KE is clearly a positive-definite quantity. On the other hand, terms on the rhs side of (A2a) and (A2b) can be expanded as

$$\overline{z_z^c \mathbf{V} \cdot \mathbf{G}} = \hat{\mathbf{V}} \cdot \hat{\mathbf{G}} + \overline{z_z^c \mathbf{V}'' \cdot \mathbf{G}''}$$
$$= \hat{\mathbf{V}} \cdot [-\nabla(\rho g \overline{\eta} + \overline{p}) + \mathcal{FS}^{\mathbf{V}}] + \overline{z_z^c \mathbf{V}'' \cdot \mathbf{G}''}, \tag{A4a}$$

$$\overline{w p_z} = \overline{(\hat{w} + w'') p_z} = \hat{w} \overline{p}_z + \overline{w'' p_z}, \tag{A4b}$$

$$\overline{z_z^c (\mathbf{V} \cdot F^{\mathbf{V}} + w F^w)} = \hat{\mathbf{V}} \cdot \hat{F}^{\mathbf{V}} + \hat{w} \hat{F}^w + \overline{z_z^c (\mathbf{V}'' \cdot F''^{\mathbf{V}} + w'' F''^w)}, \tag{A4c}$$

where $\mathcal{FS}^{\mathbf{V}}$ is defined at (14).

Equations for pressure and KE in the mean field (i.e., currents) can be derived from (11a)–(11c),

$$\nabla \cdot [\hat{\mathbf{V}}(\rho g \overline{\eta} + \overline{p})] + [\widehat{w*}(\rho g \overline{\eta} + \overline{p})]_z = \hat{\mathbf{V}} \cdot \nabla(\rho g \overline{\eta} + \overline{p}) + \widehat{w*} \overline{p}_z, \tag{A5a}$$

$$\left[\frac{\rho}{2}(|\hat{\mathbf{V}}|^2 + \hat{w}^2)\right]_t + \nabla \cdot \left[\hat{\mathbf{V}} \frac{\rho}{2}(|\hat{\mathbf{V}}|^2 + \hat{w}^2)\right] + \left[\widehat{w*} \frac{\rho}{2}(|\hat{\mathbf{V}}|^2 + \hat{w}^2)\right]_z = \hat{\mathbf{V}} \cdot [-\nabla(\rho g \overline{\eta} + \overline{p}) + \mathcal{FS}^{\mathbf{V}} - \mathcal{RS}^{\mathbf{V}} + \hat{F}^{\mathbf{V}}] + \hat{w}(-\overline{p}_z - \mathcal{RS}^w + \hat{F}^w). \tag{A5b}$$

Equations for pressure and KE in the wave field can be derived from the difference of (A5a) − (A5b) and (A2a) − (A2b),

$$[\overline{z_t'''(\rho g \eta''' + p''')}]_z + N^{(\rho g \eta''' + p''')} = -\hat{\mathbf{V}} \cdot \mathcal{FS}^{\mathbf{V}} - \overline{z_z^c \mathbf{V}'' \cdot \mathbf{G}''} + \overline{w'' p_z} + (\hat{w} - \widehat{w*}) \overline{p}_z, \tag{A6a}$$

$$\left[\overline{z_z^c \frac{\rho}{2}(|\mathbf{V}''|^2 + w''^2)}\right]_t + N^{(\rho/2)(|\mathbf{V}''|^2 + w''^2)} + \rho \nabla \cdot (\overline{z_z^c \mathbf{V}'' \mathbf{V}''} \cdot \hat{\mathbf{V}} + \overline{z_z^c \mathbf{V}'' w''} \hat{w}) + \rho (\overline{z_z^c w*'' \mathbf{V}''} \cdot \hat{\mathbf{V}} + \overline{z_z^c w*'' w''} \hat{w})_z$$
$$= \overline{z_z^c \mathbf{V}'' \cdot \mathbf{G}''} - \overline{w'' p_z} + (\hat{\mathbf{V}} \cdot \mathcal{RS}^{\mathbf{V}} + \hat{w} \mathcal{RS}^w) + (\overline{z_z^c \mathbf{V}'' \cdot F''^{\mathbf{V}}} + \overline{z_z^c w'' F''^w}), \tag{A6b}$$

where (A4a)–(A4c) have been used and $N^A \equiv \nabla \cdot (\overline{z_z^c \mathbf{V} A}) + (\overline{z_z^c w* A})_z$ is the divergence of the total advective flux of an arbitrary quantity A.

b. Depth-integrated equations

The depth integral of (A2a) and (A2b) yields explicit equations for total PE[A1] and total KE,[A2] respectively,

$$\left[\frac{\rho}{2} g(\overline{\eta}^2 + \overline{\eta'''^2})\right]_t + \nabla \cdot \langle\langle \overline{z_z^c \mathbf{V}(\rho g \eta + p)} \rangle\rangle = \langle\langle -\overline{z_z^c \mathbf{V} \cdot \mathbf{G}} + \overline{w p_z} \rangle\rangle - \overline{\eta}_t \overline{p}|_{z=\overline{\eta}} - \overline{\eta_t''' p'''}|_{z=\overline{\eta}}, \tag{A7a}$$

$$\left\langle\left\langle \overline{z_z^c \frac{\rho}{2}(|\mathbf{V}|^2 + w^2)} \right\rangle\right\rangle_t + \nabla \cdot \left\langle\left\langle \overline{z_z^c \mathbf{V} \frac{\rho}{2}(|\mathbf{V}|^2 + w^2)} \right\rangle\right\rangle = \langle\langle \overline{z_z^c \mathbf{V} \cdot \mathbf{G}} - \overline{w p_z} \rangle\rangle + \langle\langle \overline{z_z^c \mathbf{V} \cdot F^{\mathbf{V}}} + \overline{z_z^c w F^w} \rangle\rangle, \tag{A7b}$$

where $\langle\langle \cdots \rangle\rangle \equiv \int_{-\infty}^{\overline{\eta}} \cdots dz$ and $p|_{z=\overline{\eta}}$ is air pressure at the sea surface. The depth integral of (A5a) and (A5b) yields explicit equations for mean PE[A3] and mean KE,[A4] respectively,

[A1] Equation (A7a) is derived using (19a), (19b), and (21). First, $\int_{-\infty}^{\overline{\eta}} [\overline{z_t''' A'''} + \overline{z_z^c w* A}]_z \, dz = [\overline{z_t''' A'''} + \overline{z_z^c w* A}]_{z=\overline{\eta}} = [\overline{(\eta_t''' - z_z^c \overline{\eta}_t) A'''} + \overline{z_z^c (\overline{\eta}_t + \mathbf{V} \cdot \nabla \overline{\eta}) A}]_{z=\overline{\eta}} = [\overline{\eta_t''' A'''} + \overline{z_z^c \overline{\eta}_t A} + \overline{z_z^c \mathbf{V} A} \cdot \nabla \overline{\eta}]_{z=\overline{\eta}}$, where $A = \rho g \eta + p$. Second, $\int_{-\infty}^{\overline{\eta}} \nabla \cdot (\overline{z_z^c \mathbf{V} A}) \, dz = \nabla \cdot \int_{-\infty}^{\overline{\eta}} \overline{z_z^c \mathbf{V} A} dz - (\overline{z_z^c \mathbf{V} A} \cdot \nabla \overline{\eta})|_{z=\overline{\eta}}$, based on the Leibniz rule.

[A2] Equation (A7b) is derived using (21). First, $\int_{-\infty}^{\overline{\eta}} [\overline{w* A}]_z \, dz = [\overline{w* A}]_{z=\overline{\eta}} = [\overline{(\overline{\eta}_t + \mathbf{V} \cdot \nabla \overline{\eta}) A}]_{z=\overline{\eta}}$, where $A = z_z^c (\rho/2)(|\mathbf{V}|^2 + w^2)$. Second, $\int_{-\infty}^{\overline{\eta}} \overline{A}_t + \nabla \cdot (\overline{\mathbf{V} A}) \, dz = (\int_{-\infty}^{\overline{\eta}} \overline{A} \, dz)_t + \nabla \cdot \int_{-\infty}^{\overline{\eta}} \overline{\mathbf{V} A} \, dz - [\overline{(\overline{\eta}_t + \mathbf{V} \cdot \nabla \overline{\eta}) A}]_{z=\overline{\eta}}$, based on the Leibniz rule.

[A3] Equation (A8a) is derived using (22a). First, $\int_{-\infty}^{\overline{\eta}} [\widehat{w*} A]_z \, dz = [\widehat{w*} A]_{z=\overline{\eta}} = [(\overline{\eta}_t + \hat{\mathbf{V}} \cdot \nabla \overline{\eta}) A]_{z=\overline{\eta}}$, where $A = \rho g \overline{\eta} + \overline{p}$. Second, $\int_{-\infty}^{\overline{\eta}} \nabla \cdot (\hat{\mathbf{V}} A) \, dz = \nabla \cdot \int_{-\infty}^{\overline{\eta}} \hat{\mathbf{V}} A \, dz - (\hat{\mathbf{V}} A \cdot \nabla \overline{\eta})|_{z=\overline{\eta}}$, based on the Leibniz rule.

[A4] Equation (A8b) is derived using (22a). First, $\int_{-\infty}^{\overline{\eta}} [\widehat{w*} A]_z \, dz = [\widehat{w*} A]_{z=\overline{\eta}} = [(\overline{\eta}_t + \hat{\mathbf{V}} \cdot \nabla \overline{\eta}) A]_{z=\overline{\eta}}$, where $A = (\rho/2)(|\hat{\mathbf{V}}|^2 + \hat{w}^2)$. Second, $\int_{-\infty}^{\overline{\eta}} A_t + \nabla \cdot (\hat{\mathbf{V}} A) \, dz = (\int_{-\infty}^{\overline{\eta}} A \, dz)_t + \nabla \cdot \int_{-\infty}^{\overline{\eta}} \hat{\mathbf{V}} A \, dz - [(\overline{\eta}_t + \hat{\mathbf{V}} \cdot \nabla \overline{\eta}) A]_{z=\overline{\eta}}$, based on the Leibniz rule.

$$\left(\frac{\rho}{2}g\overline{\eta}^2\right)_t + \nabla\cdot\langle\langle\hat{\mathbf{V}}(\rho g\overline{\eta} + \overline{p})\rangle\rangle = \langle\langle\hat{\mathbf{V}}\cdot\nabla(\rho g\overline{\eta} + \overline{p}) + \widehat{w^*\overline{p}_z}\rangle\rangle - \overline{\eta}_t\overline{p}|_{z=\overline{\eta}}, \tag{A8a}$$

$$\left\langle\left\langle\frac{\rho}{2}(|\hat{\mathbf{V}}|^2 + \hat{w}^2)\right\rangle\right\rangle_t + \nabla\cdot\left\langle\left\langle\hat{\mathbf{V}}\frac{\rho}{2}(|\hat{\mathbf{V}}|^2 + \hat{w}^2)\right\rangle\right\rangle = -\langle\langle\hat{\mathbf{V}}\cdot\nabla(\rho g\overline{\eta} + \overline{p}) + \widehat{w^*\overline{p}_z}\rangle\rangle + \langle\langle\hat{\mathbf{V}}\cdot\mathcal{FS}^{\mathbf{V}} + (\widehat{w^*} - \hat{w})\overline{p}_z\rangle\rangle - \langle\langle\hat{\mathbf{V}}\cdot\mathcal{RS}^{\mathbf{V}} + \hat{w}\mathcal{RS}^{w}\rangle\rangle + \langle\langle\hat{\mathbf{V}}\cdot\hat{F}^{\mathbf{V}} + \hat{w}\hat{F}^{w}\rangle\rangle. \tag{A8b}$$

Explicit equations for wave PE and wave KE can be derived from the difference of (A7a) − (A7b) and (A8a) − (A8b),

$$\left(\frac{\rho}{2}g\overline{\eta'''^2}\right)_t + \nabla\cdot\langle\langle\overline{z_z^c\mathbf{V}(\rho g\eta''' + p''')}\rangle\rangle = -\langle\langle\hat{\mathbf{V}}\cdot\mathcal{FS}^{\mathbf{V}} + (\widehat{w^*} - \hat{w})\overline{p}_z\rangle\rangle + \langle\langle-\overline{z_z^c\mathbf{V}''\cdot\mathbf{G}''} + \overline{w''p_z}\rangle\rangle - \overline{\eta'''_t p'''}|_{z=\overline{\eta}}, \tag{A9a}$$

$$\left\langle\left\langle\overline{z_z^c\frac{\rho}{2}(|\mathbf{V}''|^2 + w''^2)}\right\rangle\right\rangle_t + \nabla\cdot\left\langle\left\langle\overline{z_z^c\mathbf{V}\frac{\rho}{2}(|\mathbf{V}''|^2 + w''^2)} + \rho\overline{z_z^c\mathbf{V}''\mathbf{V}''}\cdot\hat{\mathbf{V}} + \rho\overline{z_z^c\mathbf{V}''w''}\hat{w}\right\rangle\right\rangle = \langle\langle\overline{z_z^c\mathbf{V}''\cdot\mathbf{G}''} - \overline{w''p_z}\rangle\rangle + \langle\langle\hat{\mathbf{V}}\cdot\mathcal{RS}^{\mathbf{V}} + \hat{w}\mathcal{RS}^{w}\rangle\rangle + \langle\langle\overline{z_z^c\mathbf{V}''\cdot F''^{\mathbf{V}}} + \overline{z_z^c w''F''^{w}}\rangle\rangle. \tag{A9b}$$

The set of (A8a)–(A9b) yields an energy diagram as in Fig. 2.

APPENDIX B

First-Order Waves

The general solution (40a)–(40e) has a component whose vertical profile is characterized by $e^{n^{\pm}z}$. This function can be approximated as follows:

$$e^{n^{\pm}z} = e^{mz\sqrt{1+i\beta\mp\gamma}} \simeq e^{mz\left(1+i\frac{\beta}{2}\mp\frac{\gamma}{2}\right)} = e^{mz}e^{mz\left(i\frac{\beta}{2}\mp\frac{\gamma}{2}\right)} \simeq e^{mz}\left[1 + \left(i\frac{\beta}{2}\mp\frac{\gamma}{2}\right)mz\right], \tag{B1}$$

where $\beta \ll 1$ and $\gamma \ll 1$ have been used. Using (B1), we derive utility equations,

$$n^{\pm}e^{n^{\pm}z} = m\sqrt{1 + i\beta \mp \gamma}\left[1 + \left(i\frac{\beta}{2}\mp\frac{\gamma}{2}\right)mz\right]e^{mz} \simeq m\left[1 + \left(i\frac{\beta}{2}\mp\frac{\gamma}{2}\right)(1 + mz)\right]e^{mz}, \tag{B2a}$$

$$\frac{e^{n^{\pm}z}}{n^{\pm}} \simeq \frac{1}{m}\sqrt{1 - (i\beta \mp \gamma)}\left[1 + \left(i\frac{\beta}{2}\mp\frac{\gamma}{2}\right)mz\right]e^{mz} \simeq \frac{1}{m}\left[1 + \left(i\frac{\beta}{2}\mp\frac{\gamma}{2}\right)(-1 + mz)\right]e^{mz}. \tag{B2b}$$

Substitution of (B1)–(B2b) to (40a)–(40e) yields

$$p_1''' = \mathrm{Re}\left\{e^{i\theta}\left[e^{kz}(a + b^+ + b^-) + e^{mz}\left[(b^+ - b^-)\left(1 + i\frac{\beta}{2}(-1 + mz)\right) - (b^+ + b^-)\frac{\gamma}{2}(-1 + mz)\right]\gamma\frac{k}{m} - a\right]\right\}\rho g, \tag{B3a}$$

$$z_1''' = \mathrm{Re}\left\{e^{i\theta}\left[e^{kz}(a + b^+ + b^-) - e^{mz}\left[(b^+ + b^-)\left(1 + i\frac{\beta}{2}mz\right) - (b^+ - b^-)\frac{\gamma}{2}mz\right]\right]\right\}, \tag{B3b}$$

$$u_1''' = \mathrm{Re}\left\{e^{i\theta}\left[e^{kz}(a + b^+ + b^-) - e^{mz}\left[(b^+ + b^-)\left(1 + i\frac{\beta}{2}(1 + mz)\right) - (b^+ - b^-)\frac{\gamma}{2}(1 + mz)\right]\frac{m}{k}\right]\right\}\sigma, \tag{B3c}$$

$$\upsilon_1''' = \mathrm{Im}\left\{e^{i\theta}\left[e^{kz}(a + b^+ + b^-) - e^{mz}\left[(b^+ - b^-)\left(1 + i\frac{\beta}{2}(1 + mz)\right) - (b^+ + b^-)\frac{\gamma}{2}(1 + mz)\right]\frac{m}{\gamma k}\right]\right\}f, \quad \text{and} \tag{B3d}$$

$$w_1''' = \mathrm{Im}\left\{e^{i\theta}\left[e^{kz}(a + b^+ + b^-) - e^{mz}\left[(b^+ + b^-)\left(1 + i\frac{\beta}{2}mz\right) - (b^+ - b^-)\frac{\gamma}{2}mz\right]\right]\right\}\sigma. \tag{B3e}$$

The approximated solution (B3a)–(B3e) still has four free parameters: the real and imaginary parts of each of b^+ and b^-. These parameters can be determined by assuming either (i) no air pressure disturbance or (ii) no variation in the tangential stress at the sea surface. In both (i) and (ii), we also assume no surface stress in the direction of wave crests, which may be written as

$$\begin{aligned}0 &= \rho\nu v'''_{1z}|_{z=0}\\ &= \mathrm{Im}\left\{e^{i\theta}\left[(a + b^+ + b^-)\gamma k - [(b^+ - b^-)(1 + i\beta) - (b^+ + b^-)\gamma]\frac{m^2}{k}\right]\right\}\rho\nu\sigma\\ &= \mathrm{Im}\{e^{i\theta}[(a + b^+ + b^-)\gamma\beta + [(b^+ - b^-)(1 + i\beta) - (b^+ + b^-)\gamma]i]\}\rho\nu\sigma k/\beta\\ &= \mathrm{Im}\{e^{i\theta}[a\gamma\beta + (b^+ + b^-)\gamma(\beta - i) + (b^+ - b^-)(i - \beta)]\}\rho\nu\sigma k/\beta.\end{aligned} \tag{B4}$$

The result is $(b^+ - b^-) = [b^+ + b^- + a\beta(i + \beta)]\gamma = [b + a\beta(i + \beta)]\gamma$, where $b \equiv b^+ + b^-$. Substitution of this to (B3a)–(B3e) yields

$$p'''_1 = \mathrm{Re}\{e^{i\theta}[e^{kz}(a + b) - a]\}\rho g, \tag{B5a}$$

$$z'''_1 = \mathrm{Re}\left\{e^{i\theta}\left[e^{kz}(a + b) - e^{mz}b\left(1 + i\frac{\beta}{2}mz\right)\right]\right\}, \tag{B5b}$$

$$u'''_1 = \mathrm{Re}\left\{e^{i\theta}\left[e^{kz}(a + b) - e^{mz}b\left[1 + i\frac{\beta}{2}(1 + mz)\right]\frac{m}{k}\right]\right\}\sigma, \tag{B5c}$$

$$v'''_1 = \mathrm{Im}\left\{e^{i\theta}\left[e^{kz}(a + b) - e^{mz}\left[(b + a\beta i + a\beta^2)\left(1 + i\frac{\beta}{2}(1 + mz)\right) - b\frac{1}{2}(1 + mz)\right]\frac{m}{k}\right]\right\}f, \quad \text{and} \tag{B5d}$$

$$w'''_1 = \mathrm{Im}\left\{e^{i\theta}\left[e^{kz}(a + b) - e^{mz}b\left(1 + i\frac{\beta}{2}mz\right)\right]\right\}\sigma, \tag{B5e}$$

where terms proportional to γ^2 have been omitted. The number of free parameters has reduced to two: the real and imaginary parts of b.

a. Case of no air pressure disturbance

One way to determine b is to assume no air pressure disturbance at the sea surface. Equation (B5a) yields $0 = p'''_1|_{z=0} = \mathrm{Re}\{e^{i\theta}b\}\rho g$ so that $b = 0$. Substitution of this to (B5a)–(B5e) yields (41a)–(41e).

b. Case with no variation in tangential stress

Another way to determine b is to assume no tangential component of surface stress except for a constant wind stress $\alpha^2\overline{\tau_2}$. Namely, there is no variability in the tangential component of surface stress (Longuet-Higgins 1953, 1960). Let (s, n) denote the tangential and outward normal directions at a horizontally fixed point on the free surface, $z^c = \eta$ (Fig. B1). Following Longuet-Higgins (1969),[B1] we transform the stress tensor in Cartesian coordinates to P^{ss} and P^{sn} acting on

[B1] In previous literature concerning the VWS, the stress tensor has been transformed into the tangential and normal components using two different formulas, one based on Longuet-Higgins (1969, hereafter LH69) and one based on Chang (1969, hereafter C69). The formula of LH69 has been adopted in Xu and Bowen (1994), Piedra-Cueva (1995), Ng (2004), and the present study [see our Eq. (B6)]. The formula of C69 has been adopted in Ünlüata and Mei (1970), Weber (1983), and Jenkins (1986). Obviously, Eq. (5.4a) of Piedra-Cueva (1995) is different from Eq. (99b) of C69, despite the fact that both equations are presented as expressions for the tangential stress at second order using the framework of Pierson (1962). The two equations become identical if the normal stress at the sea surface is zero, an assumption that allowed C69 and Ünlüata and Mei (1970) to derive the VWS (this is for waves in a water tank). Weber (1983) made one of the first attempts to relax the condition of no normal stress while retaining the condition of no variations in the tangential stress, in order to consider steady and horizontally homogeneous waves in an open ocean. However, Weber (1983) used Eq. (99b) of C69, which is why he could not obtain the VWS.

the s component of velocity and P^{ns} and P^{nn} acting on the n component of velocity,

$$\begin{pmatrix} P^{ss} & P^{sn} \\ P^{ns} & P^{nn} \end{pmatrix} = \begin{pmatrix} 1 & \eta_x \\ -\eta_x & 1 \end{pmatrix} \begin{pmatrix} -p + 2\rho\nu u_{x^c} & \rho\nu(u_{z^c} + w_{x^c}) \\ \rho\nu(u_{z^c} + w_{x^c}) & -p + 2\rho\nu w_{z^c} \end{pmatrix} \begin{pmatrix} 1 & -\eta_x \\ \eta_x & 1 \end{pmatrix}, \tag{B6}$$

where $P^{sn} = P^{ns}$ is the instantaneous tangential stress and $|\eta_x| \ll 1$ is assumed. The condition of $P^{sn} = \alpha^2\overline{\tau_2}$ becomes

$$\begin{aligned} \alpha^2\overline{\tau_2} &= \rho\nu(u_{z^c} + w_{x^c}) + 2\rho\nu\eta_x(w_{z^c} - u_{x^c}) - \rho\nu(\eta_x)^2(u_{z^c} + w_{x^c}) \\ &= \rho\nu[(u_{z^c} + w_x - \eta_x w_{z^c}) + 2\eta_x(w_{z^c} - u_x + \eta_x u_{z^c}) - (\eta_x)^2(u_{z^c} + w_x - \eta_x w_{z^c})]. \end{aligned} \tag{B7}$$

Multiplying the above equation with $z^c_z/(\rho\nu) = (1 + z'''_z)/(\rho\nu)$ and application of the perturbation expansion yields

$$\begin{aligned} \alpha^2\overline{\tau_2}(1 + z'''_z)/(\rho\nu) &= (u_z + w_x + z'''_z w_x - \eta_x w_z) + 2\eta_x(w_z - u_x - z'''_z u_x + \eta_x u_z) - (\eta_x)^2(\cdots\cdots) \\ &= \alpha(u'''_{1z} + w'''_{1x}) + \alpha^2[u_{2z} + w_{2x} + z'''_{1z}w'''_{1x} - \eta'''_{1x}w'''_{1z} + 2\eta'''_{1x}(w'''_{1z} - u'''_{1x})] + O(\alpha^3) \\ &= \alpha(u'''_{1z} + w'''_{1x}) + \alpha^2(u_{2z} + w_{2x} + z'''_{1z}w'''_{1x} - 3\eta'''_{1x}u'''_{1x}) + O(\alpha^3), \end{aligned} \tag{B8}$$

where $u'''_{1x} + w'''_{1z} = 0$ has been used. We substitute the approximated solution of the first-order waves, (B5c) and (B5e), to the $O(\alpha)$ component of (B8),

$$\begin{aligned} 0 &= (u'''_{1z} + w'''_{1x})|_{z=0} \\ &= \mathbf{Re}\left\{e^{i\theta}\left[(a + b) - b(1 + i\beta)\frac{-i}{\beta} + (a + b) - b\right]\right\}\sigma k \\ &= \mathbf{Re}\{e^{i\theta}[2a + ib/\beta]\}\sigma k. \end{aligned} \tag{B9}$$

The result is $b = 2a\beta i$. Substitution of this to (B5a)–(B5e) yields (50a)–(50e).

Time average of (B8) yields a boundary condition for the unweighted mean velocity $\overline{u}_{2z} = \overline{\tau_2}/(\rho\nu) - \overline{z'''_{1z}w'''_{1x}} + 3\overline{\eta'''_{1x}u'''_{1x}}$, which can be rewritten for the TWM velocity,

$$\begin{aligned} \hat{u}_{2z} &= \overline{\tau_2}/(\rho\nu) + (\overline{z'''_{1z}u'''_1})_z - \overline{z'''_{1z}w'''_{1x}} + 3\overline{\eta'''_{1x}u'''_{1x}} \\ &= \overline{\tau_2}/(\rho\nu) + \overline{z'''_{1zz}u'''_1} + 2\overline{z'''_{1z}u'''_{1z}} + 3\overline{\eta'''_{1x}u'''_{1x}}, \end{aligned} \tag{B10}$$

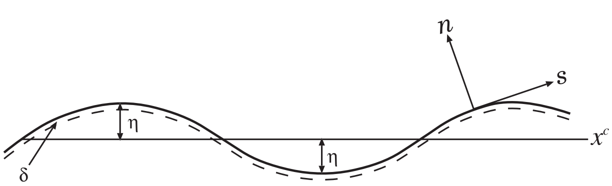

FIG. B1. Illustration of tangential vector s and normal vector n at the free surface η. The dashed line shows the base of the viscous boundary layer of thickness δ.

where $\hat{u}_2 = \overline{u}_2 + \overline{z'''_{1z}u'''_1}$ and $u'''_{1z} + w'''_{1x} = 0$ have been used. Equation (B10) corresponds to Eq. (36) of Ünlüata and Mei (1970) for the three-dimensionally Lagrangian-mean velocity. We substitute (B10) to the combined form and viscous stress on the rhs of the momentum equation (51a) in the x direction,

$$\begin{aligned} &\overline{\eta'''_{1x}p'''_1} + \rho\nu[\hat{u}_{2z} - (\overline{z'''_{1zz}u'''_1 + 2z'''_{1z}u'''_{1z} + \eta'''_{1x}u'''_{1x}})] \\ &\quad = \overline{\tau_2} + \overline{\eta'''_{1x}(p'''_1 + 2\rho\nu u'''_{1x})} \\ &\quad = \overline{\tau_2} - \overline{\eta'''_{1x}(-p'''_1 + 2\rho\nu w'''_{1z})}, \end{aligned} \tag{B11}$$

where $u'''_{1x} + w'''_{1z} = 0$ has been used. The factor of $-\eta'''_{1x}$ on the last line looks like the projection of a vector in the n direction to the x direction. Indeed, $-p'''_1 + 2\rho\nu w'''_{1z}$ is identical to the $O(\alpha)$ component of P^{nn} in (B6). This interpretation of the wave stress—the projection in the direction of wave propagation of the normal component of the stress—has been developed by Weber et al. (2006) based on a different approach.[B2]

In the main text, we have estimated the rate of momentum input at the sea surface to be $\overline{\tau_2} + 2\rho\nu\sigma k^2a^2$, by substituting the analytical solution of viscid waves to the lhs of (B11). Interestingly the wave-induced part, $2\rho\nu\sigma k^2a^2$, is available more easily by considering the budget of wave energy (Phillips 1977; Xu and Bowen 1994; Weber et al. 2006). Equations for the depth-integrated

[B2] The interpretation can be traced back to Weber (2003) and Phillips (1977).

budget of wave energy have been given by (A9a)–(A9b). We specialize to the energy budget of horizontally homogeneous waves, and for simplicity $v_1''' = 0$ and $\partial_y = 0$. To leading order of α, the sum of (A9a) and (A9b) becomes

$$\begin{aligned} 0 &= \left(\frac{\rho}{2} g \overline{\eta_1'''^2}\right)_t + \left\langle\left\langle \frac{\rho}{2} \overline{(|u_1''|^2 + |w_1''|^2)} \right\rangle\right\rangle_t \\ &= -\overline{\eta_{1t}''' p_1'''}|_{z=0} + \langle\langle \overline{u_1''(F''^u)_1} + \overline{w_1''(F''^w)_1} \rangle\rangle \\ &= -\overline{\eta_{1t}''' p_1'''}|_{z=0} + \rho\nu \overline{u_1'''(u_{1z}''' + w_{1x}''') + 2w_1''' w_{1z}'''}|_{z=0} \\ &\quad - \rho\nu \langle\langle \overline{2(u_{1x}''')^2 + 2(w_{1z}''')^2 + (w_{1x}''' + u_{1z}''')^2} \rangle\rangle \\ &= \overline{\eta_{1t}'''(-p_1''' + 2\rho\nu w_{1z}''')}|_{z=0} \\ &\quad - \rho\nu \langle\langle \overline{2(u_{1x}''')^2 + 2(w_{1z}''')^2 + (w_{1x}''' + u_{1z}''')^2} \rangle\rangle, \end{aligned} \tag{B12}$$

where $u_1'' = u_1'''$ and $w_1'' = w_1'''$ are understood and $(u_{1z}''' + w_{1x}''')|_{z=0} = 0$ and $\eta_{1t}''' = w_1'''|_{z=0}$ have been used. According to Phillips (1977), the dissipation rate can be estimated using the analytical solution of inviscid waves, because the viscous boundary layer associated with the waves is so thin that the detailed profile of viscid waves in the boundary layer does not affect the depth-integrated rate of dissipation. Substitution of $u_1''' = a\sigma e^{i\theta + kz}$ and $w_1''' = -ia\sigma e^{i\theta + kz}$ to the dissipation term of (B12) yields $2\rho\nu\sigma^2 ka^2$. Thus, the rate of wave energy input through the sea surface is $\overline{\eta_{1t}'''(-p_1''' + 2\rho\nu w_{1z}''')}|_{z=0} = 2\rho\nu\sigma^2 ka^2$. Use of $\eta_{1t}''' = -(\sigma/k)\eta_{1x}'''$ yields $-\overline{\eta_{1x}'''(-p_1''' + 2\rho\nu w_{1z}''')}|_{z=0} = 2\rho\nu\sigma k^2 a^2$ for the rate of wave-induced momentum input on the rhs of (B11).

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