Large-Scale Dynamics of Circulations with Open-Ocean Convection

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ABSTRACT

Formation of the densest water masses in the North Atlantic and its marginal seas involves open-ocean convection. The main goal of this study is to contribute to the general understanding of how such convective regions connect to the large-scale ocean circulation. Specifically, analytic and numerical versions of a variable density layer model are used to explore the processes underlying the circulation in an idealized ocean basin. The models are forced by a surface buoyancy flux, which generates a density maximum in the ocean interior. In response to the forcing, a region forms that is characterized by the closed Rossby wave characteristics and where the eddy–mean transport converges toward the convective site. Outside of that region, characteristics extend from the eastern boundary and a distorted $\beta$-plume circulation develops, linking the convection site with the western boundary. The overturning strength in the model can be related to several environment variables and forcings and is constrained by the surface density field, stratification, eddy mixing strength and by Rossby wave dynamics. Solutions forced by an interior ocean density minimum are also considered. Although no convection occurs, the dynamics underlying the circulation are closely related to the case with cooling.

1. Introduction

a. Background

The North Atlantic meridional overturning circulation (MOC) is a key climate process that significantly contributes to the transport of heat and tracers in the ocean. On the Atlantic basin scale, it is dynamically linked with a with large-scale meridional surface density gradient, which drives a northeastward convergence of the MOC surface branch (Marotzke 1997; Schloesser et al. 2012). In the Labrador Sea and Greenland–Iceland–Norwegian (GIN) Seas, where the North Atlantic Deep Water (NADW) is formed, the basic surface density pattern is different; because boundary currents transport relatively warm Atlantic waters, the densest waters and deep convection occur in the interior ocean (Mauritzen 1996). The processes linking the interior marginal seas, their boundary currents, and formation of NADW have been subject to numerous investigations, which reveal a complex interplay of processes on a wide range of scales from convective plumes (~1 km; Schott and Leaman 1991; Jones and Marshall 1993) to basin-scale currents and circulation (Spall 2010).

Convection in the marginal seas predominantly occurs in winter, when intense buoyancy fluxes result in unstable stratification. High-resolution, idealized modeling studies show that in response to the surface cooling available potential energy is released, and the fluid column overturns in multiple turbulent plumes (Jones and Marshall 1993; Marshall and Schott 1999). Together, these plumes form larger, deep patches of dense water in the cooling regions, which are encircled by cyclonic rim currents driven by the surface density gradient. When these rim currents go baroclinically unstable, mesoscale eddies develop. If the cooling is sustained sufficiently long, density advection associated with the eddies can eventually balance the surface buoyancy flux, and the convection site is in a quasi-steady state (Visbeck et al. 1997). Although convective sites get restratified during summer, they generally remain characterized by relatively weak stratification throughout the year (Marshall and Schott 1999).

Because of the requirement to balance vortex stretching in the potential vorticity (PV) budget, flow convergence in the ocean interior is strongly constrained (Send and Marshall 1995), and it has been argued that
the linkage between the convective sites and the large-scale circulation must be controlled by Rossby wave dynamics (Talley 1979; Davey and Killworth 1989; Spall and Pickart 2001; Pedlosky and Spall 2005). Gill et al. (1979) and Talley (1979) consider 2-layer systems, where a vortex is driven by a point source of mass, which is balanced by a diffusive upwelling of the form \( \omega = \frac{\epsilon h}{L^2} \), where \( h \) is the anomaly of the layer thickness, and \( \epsilon \) is an inverse damping time scale. Talley (1979) notes that this diffusion is essential for allowing for a steady-state solution. This is, of course, correct in the sense that the mass transfer across the layer interface has to be balanced to allow for an equilibrium state. In case of the large-scale overturning circulation, on the other hand, this balance is only expected to hold globally and not locally in a region of deep-water formation. Thus, the most relevant case is the limit when diffusion \( \epsilon \) becomes small. On the \( \beta \) plane, then, Rossby wave propagation leads to a westward stretching of the vortex, with increasingly zonal currents and remote upwelling.

Davey and Killworth (1989) consider a reduced-gravity model without diffusion, which is forced by a mass transfer \( W \), uniformly distributed over a circular region with radius \( L \). When the forcing is weak, the circulation is governed by linear Sverdrup dynamics (i.e., vortex stretching is balanced by changes in planetary vorticity), and \( W \) drives a meridional flow \( V \):

\[
V = \frac{f}{\beta L} W, \tag{1}
\]

where \( f \) is the Coriolis parameter, and \( \beta \) is the background PV gradient. Outside the forcing region, no meridional flow is allowed, and hence the circulation is closed by zonal currents extending toward the western boundary, forming a \( \beta \) plume (Stommel 1982; Spall 2000). Because the horizontal scale \( L \) in (1) is typically small compared to the radius of the earth, the flow convergence is weak when \( \beta \) represents the planetary vorticity gradient, \( V \gg W \) (the difference between \( W \) and \( V \) can be reduced in \( \beta \) plumes over topographic slopes owing to the larger, topographic \( \beta \); Kida et al. 2009). When the forcing strength is increased, the circulation transitions to a turbulent Sverdrup regime (Davey and Killworth 1989; Kida 2003). Relative vorticity then allows for transient, closed circulations (eddies) to form in the forcing region, which propagate westward with the Rossby wave speed.

Such dynamical considerations, as well as observational evidence (Pickart and Spall 2007), have led to the prevailing perception that the NADW is primarily formed near the boundaries of the marginal seas. That is, relatively warm and salty North Atlantic waters lose heat as they circulate around the marginal seas in boundary currents trapped to the continental slope (Mauritzen 1996; Walin et al. 2004; Spall 2004, 2010, 2011; Straneo 2006; Eldevik et al. 2009; Våge et al. 2013). The primary mechanism by which the boundary currents are cooled is mixing with colder waters in the ocean interior, with the energy for the vigorous mixing being provided by baroclinic instability of the boundary currents. In this view, the interior of the marginal seas is the dynamical equivalent to the dense-water patches in the convection models. Furthermore, the marginal sea circulation resembles the \( \beta \) plumes in that interior convergence is weak relative to the horizontal circulation along its boundaries.

**b. Present research**

In the present study, we seek to contribute to the dynamical understanding of the large-scale circulations driven by interior ocean cooling and convection. In particular, we explore the dynamical linkage of the surface density field and \( \beta \) plumes: How do the surface density gradient and the outcropping of a layer affect the dynamics of a \( \beta \) plume? What is the role of Rossby waves, the surface density field, and the (parameterized) eddy mixing in setting the scales of the horizontal circulation? How is the surface density contrast between the interocean and the boundary linked to the strength of the flow convergence, and how does it differ from circulations driven by an alongshore density difference?

To address these questions, we develop and analyze analytic and numerical solutions to variable density 1.5- and 2.5-layer models, where the surface density is strongly relaxed toward a prescribed “atmospheric” value, and the effects of eddies are parameterized. Topography and wind forcing are not considered, and solutions are obtained in an idealized, rectangular, flat bottom ocean basin. As such, the models lack several processes that are likely of fundamental importance in the North Atlantic marginal seas. On the other hand, these simplifications allow for analytic progress and to focus on the dynamical implications of the surface density field.

The study extends previous ones exploring \( \beta \)-plume circulations (Talley 1979; Davey and Killworth 1989; Kida 2003) in that it allows for the dynamical impact of surface density gradients to be considered as well as for an internal adjustment of the overturning strength in response to the forcing and other external parameters. It is also an extension of Schloessner et al. (2012, 2014), who use similar idealized models and configurations. The experimental setup differs from these studies, however, in that the surface density reaches a maximum in the
interior of the ocean basin and not along a boundary (we also report solutions with a surface density minimum).

The manuscript is organized as follows: The ocean model and experimental design are introduced in section 2. Solutions to the 1.5- and 2.5-layer models are presented in sections 3 and 4, respectively. Results are summarized and discussed in section 5. The manuscript also includes three appendixes, which motivate the set of equations used in the main text and comment on solutions with different eddy parameterizations and a surface density minimum instead of a maximum in the ocean interior.

2. Model

a. Configurations

The models used in this study are analytic and numerical versions of reduced-gravity, variable density, 1.5- and 2.5-layer models, in which the upper-layer density \( \rho_1(x,y) \) varies horizontally, whereas density in the deeper layers, \( \rho_2 \) and \( \rho_3 \) (in case of the 2.5-layer model), remains constant (Schloesser et al. 2012, 2014). Solutions are obtained in an idealized ocean basin, extending from \( x = x_c \) in the east to \( x = x_w = -x_c \) in the west and meridionally from \( y = y_n \) in the north to \( y = y_s = -y_n \) in the south. In numerical solutions, \( x_c = y_n = 1000 \text{ km} \) is used. If not stated otherwise, \( x_c \) and \( y_n \) are assumed to be sufficiently large for boundary effects to be negligible in analytical solutions. We assume a mid-latitude, \( \beta \) plane, such that the Coriolis parameter has the form \( f = f_0 + \beta y \). The standard values used are \( f_0 = 1.2 \times 10^{-4} \text{s}^{-1} \) and \( \beta = 1.3 \times 10^{-11} \text{s}^{-1} \text{m}^{-1} \), which roughly corresponds to \( y = 0 \) being located at \( \text{55°N} \).

No wind forcing is considered in the present study, and solutions are forced by prescribing the surface density field:

\[
\rho_1(x,y) = \rho_2 - \Delta \rho - \Delta \rho \begin{cases} 
0, & r \leq R_1, \\
\frac{r - R_1}{\Delta R}, & R_1 < r \leq R_2, \\
1, & r > R_2,
\end{cases} \tag{2}
\]

where \( r = \sqrt{x^2 + y^2} \) is the distance from the center of the domain, and \( \Delta R = R_2 - R_1 \) is the length over which the surface density gradient occurs (Fig. 1). This forcing can be thought of as a Haney (1971)-type buoyancy flux in the limit that the relaxation time to the atmospheric temperature is very (indefinitely) small. Assuming a balance of the surface buoyancy flux and advection, \( Q = \nabla \cdot \mathbf{V} \rho_1 \), \( Q \) can be inferred after a solution is found (Schloesser 2014).

b. Equations of motion

The basic model configuration for solutions presented here is with \( \Delta \rho = 0 \), and standard values used are \( \rho_2 = 1030 \text{ kg m}^{-3} \) and \( \Delta \rho = 1.5 \text{ kg m}^{-3} \) \( \rho_2 = 1030.45 \text{ kg m}^{-3} \) in the 2.5-layer model). Then according to (2), \( \rho_1 = \rho_2 - \Delta \rho \) outside a circle of radius of \( R_2 = 500 \text{ km} \) in the center of the domain, and it decreases monotonically until \( r = R_1 = 100 \text{ km} \); closer to the center \( \rho_1 = \rho_2 \) such that the layer interface outcrops at \( r = R_1 \), and layer 1 only exists in the region \( r \geq R_1 \) (upper cross section in Fig. 1; Schloesser et al. 2012, 2014). Solutions without outcropping of the second layer (\( \Delta \rho > 0 \); lower cross section in Fig. 1) are discussed in section 4c, and solutions with interior ocean heating (\( \Delta \rho < 0 \)) are discussed in appendix C.
assumes that the eddy–mean flow is proportional to the gradient of mean available potential energy, similar to the Gent and McWilliams (1990) parameterization (see appendix A). With these assumptions, the set of equations considered for the (residual mean) horizontal transports \( \mathbf{V}_i \) and layer thickness \( h_i \) in each layer is

\[
\begin{align*}
\nabla \cdot \mathbf{V}_i &= 0, \\
\nabla \times \mathbf{V}_i &= -\nabla p_i - \nu \mathbf{V}_i, \quad \text{and}
\end{align*}
\]

with \( i \) being the layer index and \( \nu \) being the Rayleigh friction. Temporal derivatives are enclosed in parentheses to indicate that only the steady-state versions are considered for the analytic solutions. The depth-integrated pressure terms \( \langle \nabla p_i \rangle \) are derived by vertical integration of the horizontal pressure gradients over each layer, with pressure being derived by vertical integration of the hydrostatic equation (Schloesser et al. 2012). For the 1.5- and 2.5-layer models considered here, they can be written as

\[
\begin{align*}
\langle \nabla p_i \rangle &= \nabla \rho_i \quad (1.5 \text{-layer model}), \quad \text{and} \\
\langle \nabla p_{1,2} \rangle &= \frac{h_1}{h} \nabla \rho_1 \pm \frac{h_2}{h} \nabla \rho_1 \quad (2.5 \text{-layer model}),
\end{align*}
\]

where the total depth \( h = h_1 + h_2; \rho_i = g_2 h_i^2/2; \rho_1 = g_2 h_1^2/2 + \rho_1^1 \); the reduced-gravity \( g_2 = (\rho_i - \rho) \rho_i \); the subscript 1,2 means 1 and 2, and \( \pm \) means + and −, in the case of layers 1 and 2, respectively.

Note that the horizontal density gradient in the surface layer drives a thermal wind shear in addition to the depth-averaged velocities. While the shear velocities do not contribute to the layer transports \( \mathbf{V}_i \), they can conceptually be added to the depth-averaged velocities to infer a three-dimensional flow field (Schloesser et al. 2014). In this manuscript, however, we focus on the layer depth-integrated flow. Note also that because the right-hand side of (3b) is zero, water cannot be transferred vertically between layers (except in the sponge layer discussed next). Thus, detrainment from layer 1 to layer 2 can only occur horizontally across \( r = R_1 \) when \( \delta \rho = 0 \), such that the second layer outcrops at \( r \leq R_1 \) (cf. Fig. 1).

c. Boundary conditions

A condition of no normal flow is imposed at all boundaries. Furthermore, the stratification is prescribed in a sponge layer along the southern boundary:

\[
h_i(x, y_s) = H_i,
\]

with standard values \( H_1 = 500 \text{ m} \) and \( H_2 = 1000 \text{ m} \) being used if not stated otherwise. To maintain this stratification, upwelling (or downwelling) occurs in the sponge layer, which closes the MOC. In reality, the MOC is closed outside the North Atlantic, for example, by diapycnal mixing (Stommel and Arons 1960) and wind-driven upwelling in the Southern Ocean (Wyrtki 1961; Toggweiler and Samuels 1995), and it is thought that the global stratification adjusts to equilibrate these diapycnal MOC transports (Gnanadesikan 1999; Schloesser et al. 2014). In that sense, the sponge layer provides a basic parameterization of such processes.

d. Numerical implementation

To obtain numerical model solutions, the equations in (3) are discretized and, starting from an initial condition with \( h_i = H_i \), integrated forward in time using a third-order Adams–Bashford scheme until the model reaches an equilibrium state. The horizontal discretization is of second-order precision on a C grid with a resolution of either 10 or 25 km. With Rayleigh damping, the typical width of boundary currents is that of a Stommel layer (Stommel 1948) with a decay scale

\[
L_s \sim \frac{\nu}{\beta},
\]

which is resolved in all experiments. The sponge layer along the southern boundary is 100 km wide, and layer thickness is restored toward the prescribed \( H_i \) with a time scale of 1 day.

3. 1.5-layer model solutions

In this section, we present analytic and numerical solutions to the 1.5-layer model. Because the pressure term \([4a]\) is a perfect differential, dynamical processes resolved in the 1.5-layer model, that is, Rossby waves, are rather insensitive to the specific form of the surface density field \([2]\). As a consequence, solutions can be derived entirely in terms of \( \rho_1 \), and \( \mathbf{V} \) enters the problem only in terms of a boundary condition, that is, \( \rho_1 = 0 \) in the outcropping region \( r \leq R_1 \).

Although a general analytic solution can be found (section 3b), we start by assuming that the eddy mixing is weak \((\nu \to 0)\) to emphasize the role of wave adjustment processes (section 3a). The solution provides a basis for the discussion of the 2.5-layer model response, where the surface density field does have more profound dynamical implications. Finally, this section provides a brief comparison between solutions where the surface density maximum occurs in the interior ocean and at a boundary (section 3c).

a. Limit \( \nu \to 0 \)

We begin this discussion by considering a model spin up in the limit \( \nu \to 0 \). According to (5), the layer thickness in the sponge layer is set to \( h_1(x, y_s) = H_1 \) and
$P_{1}(x, y_{s}) = \hat{P}_{1}$, where $\hat{g}_{12} = g(\Delta \rho + \delta \rho)/\rho_{2}$. Boundary (Kelvin) waves then propagate northward along the eastern boundary and westward along the northern boundary, eliminating the alongshore pressure gradient to satisfy the no normal flow boundary condition. After completion of that adjustment, the alongshore pressure is set to the prescribed value in the sponge layer: $P_{1}(x_{e}, y_{n}) = \hat{P}_{1}$.

Subsequently, Rossby waves propagate the eastern boundary structure westward across the ocean basin, setting $\mathcal{P}_{1} = \hat{\mathcal{P}}_{1}$ in the interior; however, two exceptions occur. First, because the ocean is unstratified, no baroclinic Rossby waves exist in the outcropping region ($r \leq R_{1}$), and $g_{21} = 0$ forces $\mathcal{P}_{1} = 0$ at $r = R_{1}$. Second, because Rossby waves propagate due west, a shadow region exists to the west of the outcrop ($|y| < R_{1}$ and $x < 0$). In that shadow region, westward propagation of Rossby waves requires that the zonal pressure gradient vanishes, and hence the pressure must adjust to the value at the western boundary of the outcropping region: $\mathcal{P}_{1} = 0$ (upper-left panel of Fig. 2).

In the region where $\mathcal{P}_{1} = \hat{\mathcal{P}}_{1}$, it follows by definition that the layer thickness

$$h_{1} = H_{1}\sqrt{\frac{\Delta \rho}{\rho_{2} - \rho_{1}}},$$

that is, $h_{1}$ increases as the density difference between layers 1 and 2 decreases (Schloesser et al. 2012). Furthermore, $h_{1} \rightarrow \infty$ as $\rho_{1} \rightarrow \rho_{2}$, a consequence of

FIG. 2. Solutions to the 1.5-layer model for varying $\nu$. (top left) The limit $\nu \rightarrow 0$ is illustrated (the thin gray line indicates the northern boundary in the case with cooling along the boundary). Analytic solutions with (top right) $\nu = 0.25 \times 10^{-6}$ s$^{-1}$ and (bottom left) $\nu = 10^{-6}$ s$^{-1}$, and (bottom right) a numerical solution with $\nu = 10^{-6}$ s$^{-1}$. 

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L_S \to 0 as \nu \to 0 \text{ [see the discussion of (10) below]. In contrast, } \mathcal{P}_1 = 0 \text{ implies } h_1 = 0 \text{ in the shadow region. Arguably, then, stratification is eliminated and baroclinic waves no longer exist. This situation can be avoided by including additional processes in the model, which maintain a minimum (mixed) layer thickness (Schloesser et al. 2012). For brevity, no such processes are included here.}

The difference between \mathcal{P}_1 = 0 \text{ in the outcropping and shadow regions and } \mathcal{P}_1 = \mathcal{P}_1 \text{ elsewhere drives a current, which cyclonically encircles the eastern part of the outcropping region and extends zonally to the western boundary layer. Finally, western boundary waves propagate southward, establishing a western boundary current and upwelling in the sponge layer to close the circulation (upper-left panel of Fig. 2). Since } \mathcal{P}_1 \text{ is constant, the depth-integrated flow vanishes in the rest of the basin. The horizontal density gradient does, however, drive a thermal wind shear in layer 1 in the region } R_2 \approx r \approx R_1 \text{ and with } \mathcal{P}_1 = \mathcal{P}_1. \text{ The shear velocity is cyclonic at the surface and turns anticyclonic at depth; (7) ensures that the layer depth integral of the shear velocity vanishes.}

Assuming that the depth-integrated flow in the main direction of the currents is geostrophically balanced, integration of (3a) with (4a) across the two zonal currents gives their transports

\[ h_1^+ = \frac{1}{f_0} \int_1 \mathcal{P}_1 = \frac{1}{f_0} \mathcal{P}_1 + \frac{\beta}{f_0} \mathcal{P}_1, \]

where \( h_1^+ \) corresponds to the northern current, and \( h_1^- \) corresponds to the southern current (cf. Fig. 2), respectively, and \( \mathcal{P}_1 \) is the meridional distance from the center of \( h_1^+ \) to \( y = 0 \) to the west of the outcropping region, that is, \( \mathcal{P}_1 = R_1 \) in the limit \( \nu \to 0 \) considered here.

According to (8) then, \( |h_1^+| > |h_1^-| \) because of the poleward increase in the Coriolis parameter. Furthermore, because no “vertical” mass transfer between the layers is allowed in the model, the difference between the two transports must detrain from layer 1 horizontally across \( r = R_1 \) as the current flows along the outcropping region. Consequently, the overturning strength is

\[ \mathcal{M} = h_1^+ + h_1^- = \frac{2\beta}{f_0} \mathcal{P}_1, \]

which is equivalent to the \( \beta \)-plume convergence [(1); see also Spall 2000]. Since \( \mathcal{P}_1 = 0 \) along the outcrop, it follows that the Eulerian-mean geostrophic flow does not contribute to detrainment across \( r = R_1 \), and the entire detrainment must be due to the eddy–mean flow. With (3a), (4a), and \( g_{21} = 0 \), the radial flow across \( R_1 \) is

\[ V_1^r = \frac{-\nu h_1^2}{2L^2} g_{21}, \quad r = R_1, \]

which illustrates that the overturning strength (9) can only remain finite as \( \nu \to 0 \) because the pressure gradient at \( R_1 \) becomes indefinitely large (\( h_1 \to \infty \)). Since (10) holds regardless of the strength of \( \nu \), it also follows that when \( \nu \to 0, h_1(R_1) \) must remain finite (\( h_1 \to \infty \)) for \( V_1^r(R_1) \) to be finite.

b. General solution

When finite strength eddy mixing is considered, the same basic processes are active as in the limit \( \nu \to 0 \); the mixing allows for a continuous \( \mathcal{P}_1 \), however, and currents and boundary layers attain a finite width. To derive a general solution, we cross differentiate (3a) and substitute into (3b) to obtain an equation for \( \mathcal{P}_1 \). Assuming that meridional scales associated with the solution are small compared to the radius of Earth and \( \nu f \ll 1 \),

\[ \beta \mathcal{P}_1 + \nu \nabla^2 \mathcal{P}_1 = 0. \]

Similar to Talley (1979), we proceed with a Longuet-Higgins transformation and seek solutions of the form

\[ \mathcal{P}_1 = \mathcal{P}_1 - \mathcal{P}_1 \phi(x, y) e^{-\kappa x}, \]

with \( \kappa = \beta/(2\nu) = 1/(2L_S) \). Substitution gives

\[ \nabla^2 \phi - \kappa^2 \phi = 0, \]

which is the time-independent Klein–Gordon equation. After a transformation to nondimensional polar coordinates, \( r = \kappa \sqrt{x^2 + y^2} \) and \( \theta, (13) \) becomes

\[ \phi_{rr} + r^{-1} \phi_r + r^{-2} \phi_{\theta\theta} - \phi = 0. \]

Separation of variables, \( \phi = \mathcal{R}(r) \Theta(\theta) \), gives equations

\[ \Theta_{\theta\theta} + \lambda^2 \Theta = 0, \quad \text{and} \]

\[ r^2 \mathcal{R}_{rr} + r \mathcal{R}_r - (r^2 + \lambda^2) \mathcal{R} = 0, \]

with the general solutions

\[ \Theta_n = A_n \cos(\lambda_n \theta) + B_n \sin(\lambda_n \theta), \quad \text{and} \]

\[ \mathcal{R}_n = a_n K_n + b_n I_n, \]

where \( \lambda_n = n = 0, 1, \ldots \) follows from the requirement that \( \Theta \) is cyclic, and \( K_n \) and \( I_n \) are the modified Bessel functions of first and second kind of order \( n \). Assuming
that the ocean basin is sufficiently large, we keep only decaying solutions and set \( b_n = 0 \) and, without loss of generality, \( a_n = 1 \). To find \( A_n \) and \( B_n \), we use the boundary condition that \( \hat{\mathcal{P}}_1 = 0 \) at \( r = \kappa R_1 \), and hence \( \phi(\kappa R_1, \theta) = e^{\kappa R_1 \cos \theta} \). It follows that \( A_n \) and \( B_n \) are the Fourier transformations of the boundary condition \( \phi(\kappa R_1, \theta) \), scaled by the corresponding modified Bessel function

\[
A_n = \frac{1}{\max(2-n, 1) \pi} \int_{-\pi}^{\pi} e^{\kappa R_1 \cos \theta} \cos(n\theta) K_n(\kappa R_1) \, d\theta, \tag{17a}
\]

\[
B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{\kappa R_1 \cos \theta} \sin(n\theta) K_n(\kappa R_1) \, d\theta. \tag{17b}
\]

and since the cosine is symmetric and the sine asymmetric across 0, \( B_n = 0 \). In summary, the full solution, now written in terms of the dimensional radius \( r = \sqrt{x^2 + y^2} \), is

\[
\hat{\mathcal{P}}_1 = \hat{\mathcal{P}}_0 - \hat{\mathcal{P}}_1 e^{-\kappa R_1 \cos \theta} \sum_{n=0}^{\infty} A_n \cos(n\theta) K_n(\kappa r). \tag{18}
\]

For (18) to be applicable requires that the coefficients (17a) converge sufficiently fast to 0 with increasing \( n \); the convergence rate depends on the value of \( \kappa R_1 = R_1/(2L_S) = R_1\beta/(2\nu) \). In the limit that \( \kappa R_1 \to 0 \), that is, the outcropping region is small compared to the width of the Stommel layer, \( e^{\kappa R_1 \cos \theta} \to 1 \) in (17a), and \( A_n \to 0 \) for \( n > 0 \). In that limit, the solution essentially reduces to that of a point source considered by Talley (1979). Higher-order terms have to be computed in (18) as \( \kappa R_1 \) increases, and convergence becomes indefinitely slow in the limit \( \kappa R_1 \to \infty \) (the limit \( \nu \to 0 \) discussed in section 3a).

Figure 2 shows two solutions computed from (18), one with \( \nu = 0.25 \times 10^{-6} \text{s}^{-1} \) (upper-right panel), and one with \( \nu = 10^{-8} \text{s}^{-1} \) (lower-left panel). Both solutions converge reasonably fast, with \( A_S/A_0 < 10^{-4} \) and \( 10^{-8} \), respectively. The lower-right panel of Fig. 2 shows a solution to the numerical model with \( \nu = 10^{-6} \text{s}^{-1} \). It compares well with the analytic one, except for the effects of the boundary, which show up in the numerical solution but are not included in the analytic one.

Sufficiently far away from the outcrop region, the modified Bessel functions can be approximated by \( K_n \approx \sqrt{\pi n} (\nu/(\beta r))^{n/2} e^{-\nu \beta r/(2\nu)} \) (Abramowitz and Stegun 1965). Consequently, \( \mathcal{P}_1 \) behaves roughly like a Stommel layer on the eastern side of the outcropping region, currents broaden toward the west, and the pressure anomaly decreases like \( \sqrt{nL_S/x} \) on the western side of the outcropping region, where the exponential terms cancel.

Unfortunately, (18) does not allow for simple integration to obtain a general and exact expression for the overturning strength, and thus a semiempirical approach is taken. Specifically, it is assumed that \( M \) has the form of a \( \beta \) plume (9). With \( \nu \) being finite, the distance of the currents \( \mathcal{M} \) from the origin now also depends on the finite width of the currents \( \mathcal{L}_1 \), where \( L_C \) is the current width at \( x = 0 \). The width \( L_C \) can be estimated by assuming the current broadens like a zonal Stommel layer, that is, \( L_C = \sqrt{L_S \Delta x} \), where \( \Delta x \) is set to the arc length of a quarter circle starting from \( x = R_1 \) and \( y = 0 \). With these assumptions, the half-width of the \( \beta \) plume in (9) is given by

\[
\mathcal{L}_1 = R_1 + \sqrt{\frac{\pi}{2} \pi \left( R_1 + \sqrt{\frac{1}{2} L_S} \right)}. \tag{19}
\]

According to (19), two different regimes exist in which \( \mathcal{L}_1 \) and hence the overturning \( M \) scale differently with \( \nu \) (which is proportional to \( L_S \)); when \( R_1 \gg L_S \), that is, the convection region is much wider than the Stommel layer, the term in the brackets is dominated by \( R_1 \), and hence \( M \) increases like the square root of \( \nu \). When \( L_S \gg R_1 \), on the other hand, \( M \) increases linearly with \( \nu \). The overturning strength (9) with (19) compares well with the numerical solutions, and both regimes can be identified in Fig. 3.
c. Boundary cooling

It is instructive to compare the solutions discussed above with a similar one where the northern basin boundary is shifted to the center \( y_n = 0 \) and hence cuts right through the cooling region. Since the largest surface densities then occur along the northern boundary, this corresponds to the situation discussed in Schloesser et al. (2012).

Similar as in the case with interior cooling, boundary (Kelvin) waves propagate northward along the eastern boundary and then westward along the northward boundary to eliminate the alongshore pressure gradient, and planetary Rossby waves propagate westward, canceling the interior pressure gradient. Northern boundary waves are blocked at \( x = R_1 \), where baroclinicity is lost, and a pressure jump occurs, which channels a flow into the convection region. Away from this jump, a cusp forms, and the flow broadens like a Stommel layer southward around the outcrop and toward the western boundary (Schloesser et al. 2012). To the west of the outcrop \( (x < -R_1, \ y = y_n) \), boundary waves adjust \( \mathcal{P}_1 = 0 \).

Then in the limit that \( \nu \to 0 \) (where boundary layers remain indefinitely thin), the \( \mathcal{P}_1 \) field remains unchanged compared to the solution with interior cooling (cf. section 3a and the upper-left panel of Fig. 2), that is, \( \mathcal{P}_1 = \mathcal{P}_1^* \) to the east and south of the outcropping region, and \( \mathcal{P}_1 = 0 \) in and to the west of the outcropping regions. In contrast to the situation with interior cooling, however, the flow cannot be closed around the cooling region, and current \( \mathcal{M}^- \) detrains entirely at \( x = R_1, \ y = 0 \). Consequently, the overturning strength, determined by the meridional integration across the zonal geostrophic flow east of the outcrop, is now

\[
\mathcal{M} = \int_{-y_n}^{0} U \, dy = \mathcal{M}^- \approx \frac{1}{f_0} \mathcal{P}_1 + \frac{\beta}{f_0^2} \mathcal{P}_1, \quad x < -R_1, \tag{20}
\]

where (8) has been used. The overturning strength given by (20) is dominated by the first term on the right-hand side as long as \( \mathcal{P}_1 \) is small compared to the radius of Earth. That implies that \( \mathcal{M} \) is an order of magnitude larger than in the case of interior cooling [cf. (9)], where the first-order term is canceled. It also implies that (20) is relatively insensitive to changes in \( \nu \) (or \( \mathcal{P}_1 \)), which only affects the second-order term. The differences between relations (9) and (20) illustrate the importance of alongshore density gradients for generating a strong MOC. Because the alongshore density gradient drives a thermal wind circulation into the boundary, geostrophy must ultimately break down, and strong sinking can occur in narrow boundary layers (Spall and Pickart 2001; Pedlosky and Spall 2005; Schloesser et al. 2012). In contrast, the thermal wind circulation is closed, without requiring boundary layers and sinking, in case of an interior ocean density maximum in the surface layer.

4. 2.5-layer model solutions

We now proceed by discussing solutions to the 2.5-layer model. Although the forcing and boundary conditions are identical as in the previous section, the model response is somewhat different. As will be shown, the main dynamical reason for this difference is that the circulation is now constrained by PV conservation in layer 2. As in the previous section, we start with the limit \( \nu \to 0 \) (section 4a), which allows for solutions to be found by integration along characteristics of undamped Rossby waves. Next, we discuss the impact of finite strength mixing (section 4b). We conclude this section by considering related solutions without layer outcropping (section 4c).

a. Limit \( \nu \to 0 \)

The same basic processes are active in the 2.5-layer models as in the 1.5-layer model discussed above. That is, the stratification is externally prescribed in the sponge layer along the southern boundary, boundary waves adjust the stratification along the eastern and northern boundaries, and the Rossby waves propagate the eastern boundary structure across the interior ocean. Because the model has two active layers, however, it also has two vertical modes.

1) Rossby waves

The first mode is associated with the combined transports in layers 1 and 2: \( \mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 \). Summation over the steady-state model equations in (3) for the two layers gives

\[
fk \times \mathbf{V} = -\nabla \mathcal{P} - \nu \mathbf{V}, \quad \nabla \cdot \mathbf{V} = 0. \tag{21}
\]

Because the pressure gradient term in (21) is a perfect differential, corresponding Rossby wave characteristics are directed zonally. Furthermore, because the deep layer does not outcrop, and there is no transfer between layers 2 and 3, it follows that characteristics cover the entire domain, and the model adjusts to a state with constant \( \mathcal{P} \) and \( \mathbf{V} = 0 \). With the southern boundary condition [(5)], it then follows that

\[
\mathcal{P} = \mathcal{P}_1 = g_0 H^2/2 + \mathcal{P}_1, \quad \text{with} \quad H = H_1 + H_2.
\]

Substitution of \( \nabla \mathcal{P} = 0 \) into (4b), and then substitution in and cross differentiation of the inviscid form of (3) for layer 1, yields the characteristic equation for
the second mode, which is associated with the shear between layers 1 and 2:
\[
e_R \cdot \nabla h_1 = \left(\frac{\beta g_{21}h_1}{f^2} \frac{h - h_1}{h} + \frac{g_{21}h_1^2}{2fh}\right)h_{1x} + \left(-\frac{g_{21}h_1^2}{2fh}\right)h_{1y} = \frac{\beta}{f} \frac{h - h_1}{h} \frac{g_{21}h_1^2}{2}, \tag{22}
\]
where \(e_R\) is the Rossby wave speed of mode 2. An important difference to the 1.5-layer model discussed in the previous section is that the Rossby wave speed [the terms in the parentheses in (22)] now includes terms depending on the horizontal density gradient in layer 1 (i.e., the terms \(\approx \nabla g_{21}\)). Because the density gradient drives a thermal wind shear within the surface layer (Schloesser et al. 2014), these terms are closely related to the vertical shear terms in the Rossby wave speed derived from standard quasigeostrophic theory (Pedlosky 1987). Note also that (22) is identical to the characteristic equation for the baroclinic mode in a 2-layer model (Schloesser et al. 2012, 2014). Hence, the solutions discussed in this section can be analogously derived for such a model as well.

2) REGION A

As in the 1.5-layer model, boundary waves adjust \(\mathcal{P}_1 = \mathcal{P}_1\) along the eastern and northern boundaries. A consequence of the effect of the density gradient on the Rossby wave speed is, however, that Rossby wave characteristics are not directed zonally. Specifically, characteristics are determined by two constraints: PV in layer 2, or
\[
q = \frac{h_2}{f}, \tag{23}
\]
and \(\mathcal{P}_1\) are constant along characteristics (i.e., \(e_R \cdot \nabla q = 0\) and \(e_R \cdot \nabla \mathcal{P}_1 = 0\); Schloesser et al. 2014).

A practical method for obtaining a solution is to first assume that the entire domain is covered by the eastern boundary Rossby wave characteristics, and hence \(\mathcal{P}_1 = \mathcal{P}_1\). In a second step, this assumption is verified by computing \(q\). Specifically, the solution \(\mathcal{P}_1 = \mathcal{P}_1\) is valid in that part of the domain where \(q\) contours intersect the eastern or northern boundary (region A). In regions where characteristics do not extend to the boundary (region B), like in the situation with a local extremum of \(q\), it follows that the circulation must be determined by different dynamics (Luyten et al. 1983).

Figure 4 shows the Rossby wave characteristics (black curves) for solutions with different types of surface densities [(2)], which all include regions A and B, respectively. A necessary condition for a local extremum of \(q\) (and hence region B) to occur is that \(q_y = 0\) somewhere in the domain, which means that the density gradient term balances the planetary \(\beta\) term in the zonal Rossby wave speed (22). Since the meridional density gradients are largest at \(x = 0\), \(q_y = 0\) does occur there first, and substitution of the solution in region A, that is \(h = H\) and (7), gives
\[
\left(1 - \frac{H_1 R_{21}}{H}\right)^\frac{g_{21}}{g_{21}} = 0. \tag{24}
\]
Because \(f\) increases with \(y\), \(q_y < 0\) at \(r > R_2\), where \(g_{21}\) is constant, and \(q = 0\) can only occur when the numerator in (24) increases with \(y\), which requires \(g_{21} > 0\). Furthermore, since \(g_{21} \to 0\) as \(y \to R_1\), the second term in the numerator must eventually get sufficiently large for \(q_y = 0\) somewhere at \(y > R_1\). Finally, \(q_y \geq 0\) for the entire interval \(R_1 \leq y \leq R_2\) at \(x = 0\) when
\[
\frac{2\beta \Delta R H_2}{f} \frac{|\Delta \rho|}{\Delta \rho + \delta \rho} \leq 1, \tag{25}
\]
which is obtained by substitution of \(h_1 = H_1\) and (2) into (24) (although \(\delta \rho = 0\) here, it is kept for later reference when \(\delta \rho > 0\)). The standard parameters used in this study satisfy condition (25), which somewhat simplifies the following discussion, where this property is implied. Note, however, that the results are valid more generally if \(R_2\) is replaced by \(R'_2\), which is the distance from the origin where (24) is satisfied first (see upper-right panel of Fig. 4), and \(H_1\) by \(H_1'(R'_2)\) from (7). Further note that although we consider solutions with \(\Delta \rho > 0\), (24) and (25) are also valid in case \(\Delta \rho < 0\), as discussed in appendix C.

Because characteristics follow curves of constant \(q\), region B is enclosed by the characteristic that extends westward from the eastern boundary at \(y = R_2\) (thick black curve in the upper-left panel of Fig. 4). This characteristic also separates characteristics that proceed north and southward of region B, respectively. To calculate the boundary of region B, that is, its distance from the origin \(\mathcal{R}\), we use that \(q = \hat{q} = H_1(R_2)\) is constant along that boundary, which can be rewritten as
\[
h_1(\mathcal{R}, y) = H_1 + H_1^2 \frac{\beta(R_2 - y)}{f' + \beta R_2}. \tag{26}
\]
The second constraint is that \(\mathcal{P}_1 = \mathcal{P}_1\) throughout region A, which together with (2) requires
\[
(R - R_y)h_1^2 = \Delta RH_1^2. \tag{27}
\]
Now we use both constraints to eliminate $h_1$ and solve for $\tilde{R}$. With $f_2 = f_o + \beta R_2$, we get

$$\tilde{R} = R_1 + \frac{\Delta R}{\left[1 + \frac{H_1}{H_2}\beta(R_2 - y)/f_2\right]^2}. \quad (28)$$

Relation $y = \tilde{R}\cos\theta$ can be used to transform $\tilde{R}(y)$ to $\tilde{R}(\theta)$, which is a third-order polynomial in $\tilde{R}$.

3) REGION B

Because region B is characterized by closed contours, it is convenient to derive the solution in polar coordinates $r$ and $\theta$. Specifically, the radius $r$ is stretched such that it is constant along characteristics, and we assume that solutions are (to the highest order) independent of the angle $\theta$, that is, $(V_r, V_\theta)_\theta = 0$ and $h_{R_2} = 0$. Furthermore, it is assumed that any terms associated with that stretching of the coordinate $r$ do not enter the highest-order equations, which then are

$$fV^\theta_r = -\frac{h_2}{H} \frac{\partial}{\partial r} \phi - \nu V^\theta_r, \quad (29a)$$
$$-fV^r_r = -\nu V^\theta_r \quad \text{and} \quad (29b)$$
$$(r V^\theta_r)_r = 0. \quad (29c)$$

A physical interpretation of (29) is (in particular with $f \gg \nu$) that the radial pressure gradient drives a cyclonic geostrophic current around the outcropping region [(29a)] and an eddy mass flux toward the center.
Because water can only transfer to layer 2 across \( R_1 \), the eddy flux per angular unit across the closed characteristics has to be independent of \( r \) \([29c]\).

Solutions to \((29c)\) have the form
\[
V_1^r = \frac{c_1}{r},
\]
where the constant of integration \( c_1 \) is related to the overturning strength \( \mathscr{M} \) by integrating along a closed circle around the outcropping region:
\[
\mathscr{M} = -\oint_{R_1} V_1^r \, d\theta = -2\pi c_1.
\]
Using \((29b)\) to eliminate \( V^\theta \) from \((29a)\) and substitution of \((30)\) gives
\[
\mathcal{P}_1 = \mathcal{P}_1(\tilde{R}),\end{equation}
where typically \( f^2/\nu^2 \gg 1 \), and the small term in the parentheses will be neglected. Further assuming that the factor \( h/h_2 \) varies slowly and can be replaced by \( H/H_2 \), \((31)\) is integrated from \( R_1 \) to \( r \), and we obtain the solution
\[
\mathcal{P}_1 = \frac{f^2}{\nu} \frac{H}{H_2} \frac{\mathscr{M}}{2\pi} \ln \frac{r}{R_1} - \frac{\mathcal{P}_1}{\mathcal{P}_1(\tilde{R})}.
\]
We define \( \tilde{\mathcal{P}}_1 = \mathcal{P}_1(\tilde{R}) \), and evaluate \((32)\) at \( \tilde{R} \), before solving for \( \mathscr{M} \), which gives
\[
\mathscr{M} = \frac{2\pi}{\zeta} \frac{H_2}{\tilde{r}_v} \frac{\mathcal{P}_1}{\mathcal{P}_1(\tilde{R})},
\]
where \( \zeta^{-1} = \int_{\tilde{r}_v} \ln(\tilde{R}/R_1) \, d\theta = 2\pi/\ln(\tilde{R}/R_1) \) is an average value of \( \tilde{R} \). Equation \((33)\) now relates the overturning strength to the pressure at the boundary of region B. Since both are undetermined at this point, the solution must generally be found by matching with the outer solution in region A. In the limit \( \nu \to 0 \), we note that with finite \( \mathcal{P}_1, \mathscr{M} \to 0 \) in \((33)\). Consequently, \( V^r = 0 \) in that limit, and the solutions in regions A and B are matched by setting \( \tilde{\mathcal{P}}_1 = \mathcal{P}_1 \).

With \( \mathcal{P}_1 \) and \( \mathscr{M} \) known, the pressure field in region B \((32)\) is determined, and a map of the \( \mathcal{P}_1 \) with standard parameters is shown in the upper-left panel of Fig. 4. All other variables can be derived from \( \mathcal{P}_1 \) using \((29)\), and hence the solution is complete. For example, the model also allows for an estimate of convection depth \( h_c = h \) at \( r \leq R_1 \); because the depth-integrated barotropic pressure remains constant throughout the domain \( \mathcal{P} = \mathcal{P}_1 \) and \( \mathcal{P}_1 = 0 \) in the region \( r \leq R_1 \), it follows directly from the definition of \( h_c \)
\[
h_c = \sqrt{\frac{2}{g_{32}^{\mathcal{P}_1}}} = H \sqrt{1 + \frac{g_{32}^{\mathcal{P}_1}}{g_{32}^{\mathcal{P}_1} H^2}},
\]
that is, the layer interface between layers 2 and 3 is depressed in the outcropping region. Another consequence of constant \( \mathcal{P} \) is that the combined flow of layers 1 and 2 around the outcropping region \( V^\theta = V_1^r + V_2^r \) vanishes. Because \( \mathcal{P}_1 \) goes to zero in the center, it then follows from \((29a)\) that the flow in the surface layer is always cyclonic, whereas the layer-2 flow \( V_2^r \) is anticyclonic and exactly balances \( V_1^r \).

In the limit \( \nu \to 0 \), the overturning circulation vanishes in the 2.5-layer model, whereas it remains finite in the 1.5-layer model, pointing toward fundamental dynamical differences; a finite \( \mathcal{M} \) \((9)\) and formation of a \( \beta \) plume (Fig. 2) are possible in the 1.5-layer model because the width of the current encircling \( R_1 \) becomes infinitely thin, such that its associated pressure gradient becomes indefinitely large, and the eddy–mean transport \((\nu \nabla \mathcal{P}_1)\) remains finite. In the 2.5-layer model, on the other hand, the linkage between the surface density field and the layer-1 thickness generates region B with closed PV contours around the convection region, which cannot be crossed by the geostrophic Eulerian-mean flow. Because the width of region B is finite, the pressure gradient across region B must remain finite, and the eddy–mean transport must vanish in the limit \( \nu \to 0 \).

b. Finite \( \nu \)

Solutions to the numerical 2.5-layer model with standard parameters and two different values of \( \nu \) are shown in the middle and right panel of Fig. 5. In the solution with weaker eddy mixing \((\nu = 0.25 \times 10^{-6} \text{ s}^{-1})\), the solution near the center is close to the analytic one in the limit \( \nu \to 0 \) (cf. upper-left panel in Fig. 4). To the west of \( \tilde{R} \), the pressure field indicates zonal currents extending to the western boundary, with a westward current roughly north of \( y = R_2 \) and a broader eastward flow farther to the south. While the general pattern remains similar when \( \nu \) increases (right panel in Fig. 5), the interior pressure gradients get stretched, and the zonal circulation intensifies markedly.

The persistence of solution \((32)\) with finite \( \nu \) in region B is not surprising since the solution itself does not assume \( \nu \to 0 \). Because \( \mathcal{M} \) is finite now, however, the model must adjust to a state with nonzero circulation outside of region B to close the mass balance. One possible way to construct such an analytic solution is to
assume that the factor $h_2/h = H_2/H$ in the pressure terms [(4b)]. Then, a solution analogous to (18) can be derived and by matching $\tilde{P}_1$ and $\mathcal{M}$. On the other hand, that the zonal flow outside region B is centered around $R_2$ suggests that variations in layer-2 thickness remain dynamically important. For this reason, we take a different path, aiming to construct a conceptual, dynamical picture of the circulation that takes that into account (see left panel of Fig. 5) rather than a complete solution.

We start by assuming that the pressure difference $\Delta \mathcal{P}_1 = \tilde{\mathcal{P}}_1 - \bar{\mathcal{P}}_1$ drives a $\beta$-plume type of circulation, which is geostrophic in its main direction, that is, first around $\tilde{R}$, and later on zonally to and from the western boundary. At the “starting point,” that is, where the eastern boundary characteristic encircling region B first extends to $\tilde{R}$ (cf. Fig. 4), we expect that the current has the width of a Stommel layer $L_S$. From thereon, characteristics are directed mainly normal to $\tilde{\mathcal{P}}_1$, and hence we expect the current to further broaden like a zonal Stommel layer. Furthermore, because characteristics extending poleward and equatorward around region B are separated by $y = R_2$, this latitude separates the east and westward branches of the flow to the west of region B. Finally, because the equatorward pathway is longer than the poleward one, we expect that the equatorward branch is somewhat broader than the poleward one.

Now we will use this conceptual picture to derive a relation between the pressure difference $\Delta \tilde{\mathcal{P}}_1$ and the overturning strength. This relation will then allow us to eliminate $\tilde{\mathcal{P}}_1$ from (33) and hence to estimate the overturning strength for a given set of parameters. Since the flow is assumed to be geostrophically balanced in its main direction, its transport is

$$\tilde{\mathcal{V}}(\theta) = \frac{1}{f} \frac{H_1}{H} \Delta \tilde{\mathcal{P}}_1,$$

where we have approximated the factor $h_2/h$ by $H_2/H$, as in the solution for region B. Since the pressure difference is constant, the transport convergence is that of a $\beta$ plume [e.g., (1) and (9)]. That is, owing to the Coriolis parameter increasing poleward, the transport difference between the northernmost and southernmost points is

$$\Delta \tilde{\mathcal{V}} \approx \left( \frac{1}{f} - \frac{1}{f_0 + \beta \mathcal{X}} \right) \frac{H_2}{H} \Delta \tilde{\mathcal{P}}_1 \approx \frac{2 \beta \mathcal{X}}{f_0} \frac{H_1}{H} \Delta \tilde{\mathcal{P}}_1,$$

where $\mathcal{X}$ is the (to be determined) meridional distance from the center of the currents to the origin of the domain as they flow across $x = 0$, and $\mathcal{X}$ is the average of $\mathcal{X}^+$ and $\mathcal{X}^-$. In the north, it follows from the dynamical considerations above that $\mathcal{X}^+ = R_2 + L_S$. In the south, on the other hand, we expect that $\mathcal{X}^-$ is equal to $\tilde{R}(-\pi/2)$ plus a contribution from the boundary layer width, which will be larger than $L_S$ because of the broadening along characteristics. The exact broadening is difficult to estimate, however, because of the curvature of characteristics and the Rossby wave speed (22) being dependent on the $\nabla g$ terms, and hence it will be neglected.

To relate the convergence on the eastern side of region B $\Delta \tilde{\mathcal{V}}$ to the total convergence $\mathcal{M}$, we make use of the symmetry across $x = 0$ in region B, that is, we assume that the radial eddy fluxes on both sides are equal, which finally gives

$$\mathcal{M} = 2 \Delta \tilde{\mathcal{V}} = \frac{4 \beta \mathcal{X}}{f_0} \frac{H_1}{H} \Delta \tilde{\mathcal{P}}_1.$$

![Fig. 5. Solutions to the 2.5-layer model with finite $\nu$. (left) Schematic map of circulation, and $\tilde{\mathcal{P}}_1$ in numerical solutions with standard parameters and (center) $\nu = 0.25 \times 10^{-6}$ s$^{-1}$ and (right) $\nu = 10^{-6}$ s$^{-1}$.](image-url)
Note that because \( f \) decreases along \( \hat{R} \) on the western side of region B, a flow convergence is only possible when the \( h_2/H \) factor in the pressure term or the pressure difference across the current varies. The latter is possible because the flow on the western side of the outcrop extends to the western boundary, and hence the pressure on the “outside” of the current may be different from \( \hat{\mathcal{P}}_1 \).

To obtain an expression of the overturning strength with a finite \( \nu \), we use (33) and (37) to eliminate \( \hat{\mathcal{P}}_1 \), which gives

\[
\mathcal{M} = \frac{ab}{a+b} \frac{H_2 \hat{\mathcal{P}}_1}{H f_0},
\]

with the two dimensionless parameters

\[
a = \frac{2\pi \nu}{\xi f_0}, \quad b = \frac{4\beta}{\xi f_0}.
\]

Figure 3 compares the overturning strength (38) as a function of \( \nu \) (solid curves) to estimates from the numerical model (× symbols). The black curve corresponds to solutions with standard parameters (\( H_1 = 1000 \) m), whereas for the gray curve \( H_2 = 2000 \) m is used. Both curves show a good correspondence to numerical results, and the difference between the black and gray curves illustrates that \( \mathcal{M} \) depends on \( H_2 \) most prominently through the factor \( H_2/H \), although \( H_2 \) also affects \( \hat{R} \) and hence enters both \( a \) and \( b \). Figure 3 also shows the two different asymptotic limits that \( \mathcal{M} \) approaches as \( \nu \to 0 \) (thin dashed curves) and \( \nu \to \infty \) (thin dashed–dotted curves). In the limit \( \nu \to 0 \), the overturning strength [(38)] approaches (33), since \( a \propto \nu \), and hence \( a \) goes to zero, whereas \( b \) remains finite, such that \( ab/(a+b) \to a \).

Note that (33) does not depend on \( \beta \), which means that although the horizontal circulation has the form of a \( \beta \) plume, its overturning strength does not scale like one [the current \( \hat{\mathcal{P}} \) still converges like a \( \beta \) plume but is associated with a weak pressure difference \( \Delta \hat{\mathcal{P}}_1 \ll \hat{\mathcal{P}}_1 \), such that the total convergence is even weaker than predicted by (1)]. The overturning strength approaches another linear relationship with \( \nu \) as \( \nu \to \infty \), which is also indicated in Fig. 3. In this limit, which implies \( \hat{\mathcal{P}}, L_5 \gg \Delta R \), the convergence is now consistent with a \( \beta \) plume, and the 2.5-layer model must become equivalent to the 1.5-layer model (except for the factor \( H_2/H \)). Indeed, both (38) and (9) approach a similar asymptote; however, slight differences in slope persist due to the approximate nature of the expressions being not entirely consistent with the limit \( \nu \to \infty \).

Finally, we note that \( \mathcal{M} \) is proportional to \( \hat{\mathcal{P}}_1 \) as in most other MOC studies (Schloesser et al. 2014, and references therein). Thus, an interesting feature of the present solution is that this linear relationship relies on the form of eddy parameterization chosen here. As demonstrated in appendix B, other relationships between \( \mathcal{M} \) and \( \hat{\mathcal{P}}_1 \) are possible with different closures.

\section{Solutions without outcropping (\( \delta \rho > 0 \))}

\subsection{Dynamics}

The situation considered next differs from the one in the previous sections in that layer 2 does not crop out to the surface [\( \delta \rho > 0 \) in (2) and Fig. 1]. As a consequence, no overturning \( \mathcal{M} = 0 \) occurs between layers 1 and 2. (One can interpret the situation such that the overturning cell is entirely confined to layer 1, and layer 2 represents the next deeper layer, that is, the passive, deep ocean in the solutions discussed above.) The basic Rossby wave dynamics associated with the solution remain unchanged under this modification as long as the surface cooling causes layer 1 to sufficiently deepen such that (24) holds, that is, \( q_1 \) changes its sign and region B with closed characteristics exists (see lower-left panel of Fig. 4).

For simplicity, we assume that (25) holds as in the previous sections, although again a more general solution can be found by replacing \( R_2 \) with \( R'_2 \). The boundary \( \hat{R} \) is then essentially determined by constraints (26) and (27). Constraint (26) remains unaffected by \( \delta \rho > 0 \); however, in (27), \( R_1 \) must be substituted by \( R'_1 = R_1 - (R_2 - R_1)\delta \rho/\Delta \rho \) due to the modified form of (2). It follows that \( \hat{R} \) is given by (28), again with \( R_1 \) replaced by \( R'_1 \).

The equilibrium state can be understood by considering the spinup from an initial state with constant layer thickness \( h_1 = H_1 \), which is a state of rest in region A, and density gradients drive a cyclonic layer-1 flow in region B. When the model is started, the Rayleigh damping (the development of baroclinic instability in an eddy-resolving model) causes a flow convergence toward the center. Since, unlike in the solutions with \( \delta \rho = 0 \), water does not detrain into layer 2 there, the convergence causes a deepening of the surface layer, which reduces the pressure gradients and continues until the radial eddy–mean flow finally vanishes in region B. That means \( \hat{\mathcal{P}}_1 = \hat{\mathcal{P}}_1 \), a state of no depth-integrated flow (a rather boring solution but nonetheless shown in the lower-left panel of Fig. 4; shading indicates \( h_1 \)).

\subsection{Cycloonic versus anticyclonic circulation}

It is instructive to examine how using eddy closures different from (A4) affect the circulation. Here, we
illustrate how the equilibrium state changes if the model is closed by (A6). The basic idea underlying (A6) is that different unstable wave modes are associated with subsurface isopycnal slopes (Phillips 1954) and the density gradient in a homogenous mixed layer (Stone 1966; Fukamachi et al. 1995), and two different eddy coefficients are included to represent each of these processes: \( \mu \) is associated with \( \nabla h \), and \( \lambda \) is associated with \( \nabla g_{21} \) (see appendix A). In that sense, closure (A4) is a special case of (A6) with \( \mu = \lambda \).

As done previously, we assume that \( \mathcal{V}^d \) is geostrophic in (29); only (29b) is then modified compared to the standard case, which becomes

\[
\frac{\lambda h}{2} \tilde{g}_{21r} + \mu g_{21} h_{ir} = 0, \tag{39}
\]

where we have set \( V' = 0 \), which follows from (30) with \( \mathcal{M} = 0 \). Equation (39) can be integrated for specific choices of \( \lambda \) and \( \nu \) and using the boundary condition \( h_{i}(\tilde{R}) = H_{i} \). In particular, if the eddy coefficients have a form \( \mu = \mu_{o} \mathcal{F} \) and \( \lambda = \lambda_{o} \mathcal{F} \), with \( \mathcal{F} > 0 \) being an arbitrary function of variables and external parameters, integration of (39) gives

\[
h_{1} = h_{i} \left( \frac{g_{21}}{g_{21}} \right) \lambda_{o}^{(2\mu_{o})}, \quad \mathcal{F}_{1}(r) = \mathcal{F}_{0} \left( \frac{g_{21}}{g_{21}} \right)^{(\lambda_{o}/\mu_{o})^{-1}}. \tag{40}
\]

Since \( g_{21} \) decreases monotonically toward the center of region B, \( \mathcal{F}_{1} \) remains constant according to (40) only if \( \mu_{o} = \mu_{o} \). Furthermore, \( \mathcal{F}_{1}(r) \) decreases toward the center and according to (29a) drives a predominantly cyclonic circulation if \( \lambda_{o} < \mu_{o} \), whereas \( \mathcal{F}_{1}(r) \) increases toward \( R_{1} \), and an anticyclonic circulation occurs in layer 1 with \( \lambda_{o} > \mu_{o} \).

A possible solution for the case where \( \lambda_{o} > \mu_{o} \) is the Lofoten vortex, an anticyclonic, quasi-permanent eddy associated with the deepest convection in the Lofoten basin (Rossby et al. 2009). Comprehensive models and observations also support an important dynamical role of topography and the merging of anticyclonic eddies form the boundary currents (Köhler 2007; Raj et al. 2015), processes that are not included in the simple model presented here. Topography may, however, play an important role in suppressing Phillips-type instabilities, whereas it has presumably less of an effect on the surface-intensified modes.

5. Summary and discussion

In this study, we analyze the circulation in variable density 1.5- and 2.5-layer models driven by a buoyancy flux generating surface density anomalies in the ocean interior [(2) and Fig. 1]. Model transports represent the residual-mean circulation, that is, the sum of geostrophic Eulerian mean and eddy−mean flow, the latter parameterized in terms of Rayleigh damping (appendix A).

With the 1.5-layer model, we consider the circulation in response to strong surface cooling, which forces the layer interface to outcrop and the surface layer to vanish in an interior region \( r \leq R_{1} \). The pressure gradients associated with this outcrop drive a \( \beta \) plume (Fig. 2) with cyclonic flow around the outcrop, which extends zonally toward the western boundary. The convergence of that large-scale Eulerian-mean circulation is balanced by a divergent eddy−mean flow, which converges into the outcropping region, where water detains into the deep ocean.

The overturning strength \( \mathcal{M} \) in these solutions [(9)] has a form consistent with those in previously studied \( \beta \) plumes (Spall 2000). It is proportional to the background depth-integrated pressure, or available potential energy \( \mathcal{F} \), and the appropriate width scale of the \( \beta \) plume (19) depends on both the eddy mixing strength and the width of the convection region. As has been noted by a number of studies (Spall and Pickart 2001; Pedlosky and Spall 2005), \( \mathcal{M} \) is substantially weaker than in circulations where the dense water is formed near an ocean boundary, with a scaling factor of \( \mathcal{F}/R_{e} \ll 1 \), where \( \mathcal{F} \) is the meridional width of the \( \beta \) plume, and \( R_{e} \) is the radius of Earth (section 3c).

These 1.5-layer model solutions extend previous studies of similar \( \beta \)-plume circulations (Talley 1979; Davey and Killworth 1989) in that the model can internally determine \( \mathcal{M} \) in response to a prescribed background stratification [(5)], surface density field [(2)], and eddy mixing strength \( \nu \). Furthermore, the model allows for an outcropping of the layer interface, such that detrainment occurs horizontally across the outcrop line \( (r = R_{1} \) and not across the bottom of the layer.

To consider the dynamical effect of the surface density gradient requires a model with at least two active layers, such as the 2.5-layer model considered here [or alternatively a 2-layer model as in Schloesser et al. (2012, 2014)]. Because of the dynamical linkage between the surface density gradient and the tilt of the layer interface [(7)], then the PV gradient is dominated by the stretching term rather than the planetary \( \beta \) near the outcrop, where stratification is low [see the characteristic equation (22)]. As a result, Rossby wave characteristics are closed in region B around the outcrop or a sufficiently strong density maximum [(24)]. (Note that a similar region B can occur when the model is forced by an interior density minimum, although no convection and outcropping of layer 2 occurs in this case; appendix C). The dynamical difference between
the 1.5- and 2.5-layer models is emphasized in limit $\nu \to 0$, where $\mathcal{U}$ only remains finite in the 1.5-layer model. It vanishes in the 2.5-layer model because the geostrophic Eulerian-mean flow cannot cross closed $\mathcal{R}_1/PV$ contours, and the eddy–mean flow vanishes as $\nu \to 0$ (it can remain finite in the 1.5-layer model because a pressure jump develops along the outcropping line). The difference between the 1.5- and 2.5-layer models is reduced in the strong mixing limit, where the Stommel layer becomes broader than the scales of the surface density gradient, thereby reducing the “barrier” effect of the closed PV contours.

Although the overturning strength [(38)] scales only consistently with a $\beta$ plume when mixing is strong in the 2.5-layer model (alternative scalings relying on an alternative eddy closure are also discussed in appendix B), the horizontal circulation retains the form of a $\beta$ plume (Fig. 5). In region B, the pressure gradient drives a cyclonic vortex circulation in layer 1 and an opposite, anticyclonic flow in layer 2. This structure is linked to the western boundary by zonal currents, which are shifted poleward compared to the symmetric circulation in the 1.5-layer model. The equatorward zonal current is broader than the one farther to the pole because eastern boundary characteristics wrap around the region on its equatorward side, where they are subject to eddy mixing.

Because $\mathcal{R}_1$ goes to zero when the layer interface outcrops, the circulation in the surface layer is necessarily cyclonic. And because $\nabla \mathcal{R} = 0$ in the 2.5-layer model, the layer-2 flow is exactly opposite to that in the surface layer, and hence $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$ vanishes. In a model with more vertical modes, however, it is anticipated that this requirement would be relaxed, and in section 4c, it is argued that depending on the relative efficiency of surface- and subsurface-intensified unstable wave modes in driving an eddy volume flux, the total circulation of the convective cell can either be cyclonic or anticyclonic (as in the Lofoten basin).

The solution considered here can be viewed as a simplified model of the circulation in the Labrador Sea and GIN Seas; the strong stratification along the ocean boundaries, here imposed by a surface buoyancy flux, is then a crude parameterization of boundary currents, which are trapped over the continental slope and restratify the water column by advecting warm Atlantic waters (Mauritzen 1996). The boundary currents exchange heat with the interior marginal seas primarily through mixing by eddies and baroclinic instability (Spall 2004, 2010, 2011). Here, the focus is on the effect of eddies to establish a mean flow between the boundary current (represented by the rim current) and the ocean interior. Because this mean flow scales like a $\beta$ plume, it is weak compared to the boundary current itself. Nonetheless, it may have an appreciable effect on the water mass properties.

Other dynamically related circulations have been explored in previous studies. For example, regions with closed characteristics can also be caused by topography (Straub and Rhines 1990; Kawase and Straub 1991; Kida 2003; Marshall 2011). The similarity exists because topographic features like seamounts can squeeze the deep-layer thickness, causing regions with closed PV contours equivalent to those generated here by changes in the surface layer thickness in response to the surface density field. Specifically, localized surface cooling (which forces a deepening of the surface layer) causes a somewhat similar response as a seamount, as they are both associated with a local minimum in $q$. Conversely, surface heating and a bottom depression (e.g., a marginal sea enclosed by continental slopes and ridges) are both associated with a subsurface maximum of $q$ (cf. Fig. 4 here to Figs. 2a and 4a in Straub and Rhines 1990). From this perspective, the effects of the surface density field and topography are opposite in the marginal seas with continental slopes and convection in the interior and may (partially) cancel. Another similarity is that topographic features are often associated with diapycnal fluxes, for example, through hydrothermal vents (Stommel 1982; Kida 2003) or increased diapycnal mixing (Lueck and Mudge 1997). When such diapycnal fluxes occur in a region with closed PV contours, they consequently drive very similar (dynamically equivalent) $\beta$ plumes to those discussed here (Straub and Rhines 1990).

Another dynamical regime characterized by closed contours is the Antarctic Circumpolar Current (ACC) in the Southern Ocean. Because no eastern boundary exists at the Drake Passage latitudes, characteristics are closed around Antarctica (Wyrtki 1961; Toggweiler and Samuels 1995), as they are in region B. In the ocean basins adjacent to the Southern Ocean, on the other hand, the thermocline and circulation is controlled by gyre dynamics and eastern boundary Rossby waves as in region A (Samelson 2009; Radko and Kamenkovich 2011). In both cases, a boundary layer (here that is the current $\mathcal{R}$; section 4b) is required in to link the dynamically distinct regions. For the ACC, such a boundary layer is discussed in a barotropic model by Gill (1968), and the solution is extended to include baroclinicity in McCrery et al. (2015, manuscript submitted to Prog. Oceanogr.). Specifically, because no Ekman transport is considered and the layer interface is forced to outcrop by surface buoyancy fluxes rather than Ekman pumping, the solutions considered here closely correspond to the solutions forced by diffusion (solution
Because the model does not resolve fluctuations associated with the eddy field, it can at best capture the lowpass filtered, residual-mean flow version of (A1), that is,

\[-f k \times \mathbf{V}_1 = -\nabla \bar{\rho}_1 = -\nabla_2 h_1^2/\bar{g}_2, \tag{A2}\]

where the bar denotes the averaging operation. Because of the nonlinearity in the pressure term, \(\bar{\rho}_1\) generally has components due to the mean and eddy fluctuations of \(g_{21}\) and \(h_1\). Thus, it is useful to make the decomposition \(\mathbf{V}_1 = \mathbf{V}_1 + \mathbf{v}_1\), with

\[-f k \times \mathbf{V}_1 = -\nabla \bar{\rho}_1 = -\nabla_2 h_1^2/\bar{g}_2, \quad \text{and} \tag{A3a}\]

\[-f k \times \mathbf{V}_1 = -\nabla \bar{\rho}_1 = -\nabla_2 h_1^2 + \nabla_2 h_1 h_1, \tag{A3b}\]

where primes denote the deviations from the mean fields. Transport \(\mathbf{V}_1 = \bar{h}_1 \mathbf{V}_1\) represents the eddy–mean flow, which can also be expressed in terms of a bolus velocity \(\mathbf{v}_1\). The model resolves \(\bar{h}_1\) and \(g_{21}\) but not the prime variables; hence, a closure is required for \(\mathbf{V}_1\). Here, we choose to relate it to the mean depth-integrated pressure (or available potential energy) gradient:

\[\mathbf{v}_1 = -\frac{\kappa}{g_{21} h_1} \nabla \bar{\rho}_1, \tag{A4}\]

where \(\kappa\) is an eddy diffusivity similar to that of Gent and McWilliams (1990) (when \(\nabla g_{21} = 0\), it follows \(\mathbf{v}_1 = -\kappa \mathbf{v}_h\)). After substituting (A3) on the right-hand side of (A2) and then using (A4) to eliminate the terms depending on prime variables, we arrive at our final set of equations:

\[-f k \times \mathbf{V}_1 = -\nabla \bar{\rho}_1 - \nu \mathbf{V}_1, \tag{A5}\]

where \(\nu = \kappa/R_D^2\) is inversely proportional to the squared deformation radius \(R_D^2 = g_{21} \bar{h}_1/f^2\) and assumed to be constant. Equation (A5) is solved in the main text (bars and hats are omitted). Its form is mathematically convenient and generally consistent with the interpretation of eddies acting as a form drag (Greatbatch and Lamb 1990; Greatbatch 1998), allowing for a reasonable representation of processes. Nonetheless, it is instructive to consider the consequences of using different eddy closures, which is done in section 4c and appendix B. The more general form used there is based on the idea that gradients of the surface density and layer/thermocline depth may be associated with different types of baroclinic instability (Phillips 1954; Fukamachi et al. 1995).

Specifically, we choose

\[\mathbf{v}_1 = -\frac{\kappa}{g_{21} h_1} \nabla \bar{\rho}_1, \tag{A4}\]
\[ \hat{V}_1 = -\frac{\lambda \hat{h}}{2f} \nabla g_{21} = \frac{\mu_\rho g_{21}}{f} \nabla \hat{h}, \quad (A6) \]

where \( \lambda \) and \( \mu \) can be functions of the mean fields and other variables. Note that (A6) includes (A4) as a special case with \( \lambda = \mu \).

**APPENDIX B**

**Overturning Strength with an Alternative Eddy Closure**

Here, we derive an alternative form of constraint (33), using an eddy parameterization of the form (A6) with \( \lambda = \lambda_\rho h^\rho \) and \( \mu = \mu_\rho h^\rho \). We start with the equations of motions in polar coordinates, assuming that the solution is independent of the angle \( \theta \), analogous to (29) with (A6), and assume that flow around the outcropping region is approximately geostrophic:

\[ -fV'_1 = \frac{\lambda}{2} g^+_{21} - g_1 + \mu_\rho g_{21} h_{1r}, \quad \text{and} \quad (B1a) \]
\[ (\nu V'_r) = 0. \quad (B1b) \]

As in the main text, (B1b) constrains the structure of the eddy flux \([30]\), independent of the eddy parameterization used, as it solely depends on the geometry of region B. Substitution into (B1a) gives

\[ \frac{\mu_\rho g_{21}}{n+1} (h^\rho_{1r})_r + \frac{\lambda}{2} g^+_{21} h^\rho_{1r} = \frac{f \nu}{2 \pi \mu_\rho}. \quad (B2) \]

To solve that equation, we define \( \phi = (g_{21}/g^+_{21})(n+1)\mu_\rho \rho_{1r} \) and rewrite (B2) as

\[ \phi_r = \frac{(n+1)f \nu}{2 \pi \mu_\rho} \left( \frac{g_{21}}{g^+_{21}} \right)^{1/(n+1)} \phi_{1r}. \quad (B3) \]

Integration from \( r = R_1 \) to \( \hat{R} \) and then taking the \( 2/(n+1) \)th root to express \( \hat{\phi} \) in terms of \( \hat{\rho}_1 \) yields

\[ \hat{\rho}_1 = \frac{g_{21}}{2} \hat{\phi}^{2/(n+1)} = \frac{g_{21}}{2} \left( \frac{(n+1)f \nu}{2 \pi \mu_\rho} \right)^{1/(n+1)} \hat{R}^{2/(n+1)}, \quad (B4a) \]

with

\[ \gamma = \frac{1}{\hat{R}} \left( \frac{g_{21}}{g^+_{21}} \right)^{1/(n+1)} \hat{R}^{1-(n+1)} \]

(B4b)

an alternative, more general expression for (33). Note that for \( n = 1 \) and \( \lambda_\rho = \mu_\rho = \nu H_2^2 / H \), we get \( \gamma = \ln(\hat{R}/R_1) \), and (B4a) is equivalent to (33), a linear relation between \( \mathcal{M} \) and \( \hat{\rho}_1 \).

**APPENDIX C**

**Negative Density Anomalies**

When a region with lighter surface water is maintained in the ocean interior \( \Delta \rho < 0 \) and \( \delta \rho > |\Delta \rho| \) in (2), as, for example, the freshwater lens in the Beaufort Gyre, dynamics are closely related to the case with interior cooling and convection described in the main text. Because no outcropping of layers occurs, a simple solution with \( \hat{\rho}_1 = \hat{\rho}_1 \) and no depth-integrated flow exists, analogous to those discussed in section 4c. In contrast to the solutions with \( \Delta \rho > 0 \), \( g_{21} \) now increases toward the center, and hence \( h_1 \) reduces and reaches a minimum of

\[ h_1 = \sqrt{\alpha H_1}, \quad r \leq R_1, \quad (C1) \]

where \( \alpha = \delta \rho/|\delta \rho - \Delta \rho| < 1 \). The lower-right panel in Fig. 4 illustrates this state; it also shows Rossby wave characteristics (black curves). As in the case with cooling, a region exists with closed contours. Because \( h_z \) increases toward the center, region B is now encircled by the characteristic extending from the eastern boundary at \( y = -R_2 \) and bended characteristics extend around the poleward side of region B. With \( \hat{\rho}_1 = \hat{\rho}_1 \), the depth-integrated flow vanishes; however, cyclonic or anticyclonic circulations can occur when the alternative eddy closure [(A6)] is used as in section 4c.

**REFERENCES**


