The Energetics of Centrifugal Instability

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ABSTRACT

It has been shown recently that the California Undercurrent (CUC), and possibly poleward eastern boundary currents in general, generate mixing events through centrifugal instability (CI). Conditions favorable for CI are created by the strong horizontal shears developed in turbulent bottom layers of currents flowing in the direction of topographic waves. At points of abrupt topographic change, like promontories and capes, the coastal current separates from the boundary and injects gravitationally stable but dynamically unstable flow into the interior. The resulting finite-amplitude development of the instability involves overturnings and diabatic mixing. The purpose of this study is to examine the energetics of CI in order to characterize it as has been done for other instabilities and develop a framework in which to estimate its regional and global impacts. This study argues that CI is very efficient at mixing and possibly approaches what is thought to be the maximum efficiency for turbulent flows. The authors estimate that 10% of the initial energy in a CUC-like current is lost to either local mixing or the generation of unbalanced flows. The latter probably leads to nonlocal mixing. Thus, centrifugal instability is an effective process by which energy is lost from the balanced flow and spent in mixing neighboring water masses. The mixing is regionally important but of less global significance given its regional specificity.

1. Introduction

An important question in modern physical oceanography is how does the ocean get mixed? It has been recognized since Munk (1966) that the ocean is “turbulent,” implying that mixing occurs at rates that cannot be sustained by laminar, molecular processes alone. The primary mixing mechanism in the open ocean is thought to be Kelvin–Helmholtz (K–H) instability. However, it has long been proposed that ocean boundaries are also preferred locations of enhanced mixing followed by advection into the ocean interior. The former mechanism is undoubtedly present in the ocean, but some boundary processes have been recently discovered that might participate importantly in ocean mixing. The objective of this study is to examine the energetics of one of those processes, leading to an evaluation of its regional and global importance.

a. Background

Interest in mixing comes from a desire to understand the stratification of the ocean and the meridional overturning circulation (MOC). The connection between these two very disparate phenomena (mixing on the scale of a few centimeters and the global-scale MOC) is in the existence of a stably stratified abyssal ocean. Buoyancy is preferentially added at the near-equatorial, low-latitude air–sea interface. Mixing in whatever form is responsible for conducting this buoyancy downward into the deeper ocean, again in the low latitudes. Then, along geopotentials, density gradients occur that through gravitational mechanics drive an equatorial upwelling and high-latitude downwelling. The accompanying buoyant flux balances the ocean global heat budget. As fluctuations in the MOC are thought to participate in multidecadal to centennial climate variability, small-scale mixing is of interest to global climate.

As suggested by the above, a convenient language for the quantitative discussion of mixing is that of energetics. Buoyant fluids are mixed downward and heavy fluids are mixed upward in the scenario, and this requires energy.

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Recent reviews of the topic are given by Ferrari and Wunsch (2010, 2009) and Wunsch and Ferrari (2004). They argue that 0.4 TW (1 TW = 10^{12} W; Munk and Wunsch 1998; St. Laurent and Simmons 2006) is supplied to the potential energy field of the ocean in order to sustain the observed stratification and that this energetic requirement can be approximately accounted for by the winds and the tides. Hughes et al. (2009) argue that buoyant forcing also plays an important role.

The sequence leading to wind-driven and tidally driven mixing involves the excitation of internal waves by topographic scattering or the propagation of near-inertial energy from the mixed layer into the interior. Once in the internal wave field, the accepted idea is that a variety of mechanisms (e.g., induced diffusion, resonant triad interactions, and stratified turbulence; see Lindborg 2006) drive a forward cascade to small scales where shears promote classical K–H instability. Of the total energy cascading to small scales and mixing, some is dissipated into heat via viscous processes and the remainder is dissipated into potential energy. The ratio of these two energetic fates, potential energy creation divided by viscous dissipation, is referred to as the efficiency of the mixing. The K–H mixing occurring in the ocean is typically assigned an efficiency of 0.2 (Osborn 1980). Peltier and Caulfield (2003) provide backing for this number in the case of low Reynolds number flows. Recent higher Reynolds number computations (Re \sim 10^4) suggest higher efficiencies because of the onset of finescale, three-dimensional instabilities (Mashayek and Peltier 2013; Mashayek et al. 2013), so there does remain some ambiguity. We will employ the classical efficiency value of 0.2 in this paper where it is needed.

It is estimated that the tides and the high-frequency winds each provide about 1 TW to mixing, so if all the energy entering the internal wave field were to develop K–H overturnings and we adopt the classical efficiency estimate, potential energy would be created at the rate of approximately

\[
\text{dissipation} + \text{pot energy} = 6 \times (\text{pot energy}) = 2 \text{ TW},
\]

pot energy = 0.3 TW. This is similar to the abyssal requirement, although the errors on net source and net requirement are large. There is observational support for this chain of events (Gregg and Sanford 1988; Gregg 1998, and references therein).

b. Large-scale flow

Wunsch (1998) argues that approximately 1 TW of low-frequency, large-scale wind energy enters the large-scale circulation that, assuming an approximately steady large-scale circulation, must also exit the flow. The primary means by which this occurs is via geostrophic, baroclinic instabilities, the end result of which is the global mesoscale eddy field. This step in the global energetic budget is well studied and supported by several decades of observation, numerical study, and theoretical study.

What is poorly understood is the fate of the 1 TW entering the mesoscale. The mesoscale flows themselves are unable to dissipate energy viscously because of their tendencies toward upscale, inverse energy cascades (Charney 1971). Other possibilities are that the mesoscale is unstable to gravity wave perturbations or loses energy via dissipative interactions with the boundaries. Spontaneous gravity wave generation is possible, but observational and theoretical evidence of it is not strong. Dissipative interactions with the boundaries, which include the ocean surface, are a somewhat more promising pathway. Turbulent, and thus dissipative, boundary layers naturally develop against lateral and bottom boundaries when perturbed by flow. Arbic et al. (2010) estimates that perhaps 20% of the needed dissipation (0.2 TW) could occur by this mechanism. D'Asaro et al. (2011) report some very interesting observations in the Kuroshio in which it appeared some of the mixing was driven by the submesoscale. A catalyst for this was downfront wind that forced the surface mesoscale flow into an unstable profile. Nikurashin and Ferrari (2010) argue mesoscale flow over rough topography energizes the internal wave field at rates similar to those observed by Naviera-Garabato et al. (2004) in the Southern Ocean.

Of direct relevance to this study, Dewar et al. (2015) argued that eastern boundary currents appeared capable of directly driving mixing by centrifugal instability (CI). The sequence of events leading to mixing were that poleward-flowing currents, like the California Undercurrent (CUC; see Fig. 1), develop strips of strong negative relative vorticity in their bottom boundary layers. Often, the vertical component of the vorticity was well in excess of the local Coriolis parameter so that the absolute vertical vorticity \( \nu_z \) was negative. Free of the boundary, the unstable flow drove overturnings of potential density surfaces (Fig. 2). Interest in the associated mixing comes partly from the fact that the source of the mixing was different than K–H instability.

c. This study

K–H instability has been studied for many years (Peltier and Caulfield 2003; Mashayek et al. 2013; Mashayek and Peltier 2013, and references therein). In
contrast, little is known about the general energetic development of CI, thus motivating this study. Specifically, we wish to know how much of the energy initially in the balanced, but unstable, flow is lost during the finite-amplitude development of the instability and how that compares to the increase in background potential energy driven by the mixing. Equivalently, we provide an estimate of the mixing efficiency of CI for comparison with that estimated for K–H instability. We also characterize the evolution more broadly by addressing other energy reservoirs that may be affected by the unstable flow.

We examine an idealized setting of a jet meeting the conditions for CI, compute the solution using the MITgcm, and analyze the results. We have examined a couple of profiles but will here focus on one modeled after the CUC. Accordingly, we argue the classical efficiency of CI appears to be relatively high compared to the value of 0.2 traditionally assigned to K–H instability and comparable to the higher values estimated in more recent high Reynolds number simulations. In fact, the CI efficiency as defined in the turbulence literature approaches the theoretical maximum efficiency for stratified flows. The implication of this is that CI is an effective means for directly transferring balanced energy to local mixing. This agrees with the analysis by Arobone and Sarkar (2013) who studied horizontally sheared, stratified flow in an idealized setting. The local effective diffusivity can grow to $10^3$ m$^2$ s$^{-1}$ or values approximately 100 times those typically seen in the open ocean. Last, we find the unstable flow is an effective generator of nonturbulent, unbalanced phenomena (e.g., internal waves). In the example here, we estimate approximately 6% of the energy available to the initial balanced, but unstable, flow scatters into unbalanced flows. We do not follow the long-term evolution of this energy, but expect it follows a forward cascade with eventual mixing by classical mechanisms. Hence, CI should drive both local mixing by overturning and non-local mixing via nonturbulent, unbalanced generation.

The next section reviews some theoretical background and introduces several concepts used in later analysis. Our numerical model is introduced in section 3 and the solutions are described in some detail. We present the results of the energetics analysis in section 4 and conclude with a summary.

2. Theoretical background

a. Centrifugal and symmetric instability

An insightful instability study was due to Hoskins (1974), who considered a unidirectional jet with constant vertical shear, constant horizontal shear, and uniform stratification. The mean flow $V = -af + \beta z$ with constant $\alpha$, $\beta$ was balanced geostrophically:

$$fV_z = f^2 \beta = B_z,$$

where $B_z = N^2$ represents the uniform buoyancy frequency. Linear analysis shows the flow $V$ is unstable provided

$$q = (f + V_x)N^2 - f V_z^2 = (1 - \alpha) f N^2 - f \beta^2 < 0,$$

that is, if the Ertel potential vorticity (PV) of the basic state is negative. Ooyama (1966) showed local regions of negative PV in vortices exhibit comparable instabilities.
There are three ways PV can be negative: The most familiar is if the buoyancy frequency is negative, resulting in gravitational instability and in this study is relevant to the later stages of CI. Second, for positive vertical absolute vorticity \((1 - \alpha)f > 0\), a strong enough vertical shear can drive the PV negative. Third, negative absolute vorticity \((1 - \alpha)f < 0\) guarantees negative potential vorticity.

It is instructive to examine the linearized energetics of the Hoskins instability problem for the latter two cases. The linearized energy equation

\[
\frac{\partial}{\partial t}(K + P) = -\nabla \cdot (pu) - uv\nabla_x - wu\nabla_y - ub\frac{B_z}{B^2},
\]

where \(P\) is the potential energy, \(P = b^2/(2N^2)\), and \(K = (\alpha^2 + v^2 + w^2)/2\) is kinetic energy (KE), shows that the sources of perturbation energy are horizontal shear (HS), vertical shear (VS), and available potential energy (APE), respectively. Evaluating HS, VS, and APE from the Hoskins solutions shows for negative horizontal shears \((1 - \alpha)f < 0\), HS dominates the release from the mean flow. We will adopt the terminology of Thomas et al. (2013), who named such instabilities as CI to differentiate them from symmetric instabilities (SI), where the vertical shear was the primary perturbation energy source.

b. Energy in Boussinesq systems

The full Boussinesq equations are

\[
\begin{align*}
\frac{\partial u}{\partial t} + uu_x + u u_y + w u_z - f v &= -p_x + \nu_h \nabla^2_h u + \nu u_{zz}, \\
\frac{\partial v}{\partial t} + uv_x + u v_y + w v_z + f u &= -p_y + \nu_h \nabla^2_h v + \nu v_{zz}, \\
\frac{\partial w}{\partial t} + uw_x + u w_y + w w_z &= -p_z + b + \nu_h \nabla^2_h w + \nu w_{zz}, \\
b_x + ub_x + vb_y + wb_z &= \kappa_h \nabla^2 b + \kappa b_{zz}, \\
u_x + v_y + w_z &= 0,
\end{align*}
\]

where notation is standard, and \(\nu, \kappa\) represent coefficients of viscosity and diffusivity, respectively. We assume a linear equation of state and hence consider only buoyancy \(b\).

The kinetic energy equation formed from the momentum equations is

\[
K_t + \nabla \cdot [u(K + p)] = wb + \nu_h \nabla^2_h u + \nu u \cdot u_{zz},
\]

where \(K\) is the usual kinetic energy per unit mass, and conversion to potential energy appears as \(wb\). The latter quantity is computed by multiplying the buoyancy equation by \(-z\):

\[
\pi_t + \nabla \cdot (u\pi) = -wb - \kappa_h \nabla^2_h b - \kappa b_{zz},
\]

where \(\pi = -zb\) is the potential energy per unit mass. Summing Eqs. (5) and (6) gives

\[
(K + \pi)_t + \nabla \cdot [u(K + \pi + p)] = \nu_h \nabla_h^2 u + \nu u \cdot u_{zz} - \kappa_h \nabla^2_h b - \kappa b_{zz}.
\]

We will primarily be interested in volume-integrated forms of this equation. Assuming no dependence on \(y\), stress-free boundaries and insulating boundaries

\[
\begin{align*}
\frac{d}{dt} \left[ \iint_A (K + \pi) dA \right] &= -\varepsilon + \kappa \int^0_{-H} [b(z = 0) - b(z = -H)] dx = -\varepsilon + D, \\
\varepsilon &= \int_A (\nu_h \nabla_h^2 u \cdot \nabla_h u + \nu u \cdot u_{zz}) dA
\end{align*}
\]

represents the dissipation due to viscous processes.

Clearly dissipation is not positive definite. In contrast, surface buoyancies are greater than bottom buoyancies in a stably stratified fluid, implying the effect of diffusion \(D\) is to increase the energy. This can be understood as being due to the energy implicitly associated with the coefficient \(\kappa\) and is not of interest in this study. We will frequently correct for its input to the energetics in what follows.

c. Available potential energy

The total potential energy of a fluid parcel at height \(z\) is given by \(-bz\), but a more practical measure of potential energy is APE (Lorenz 1955; Winters et al. 1995). APE is the potential energy relative to a state where the fluid is adiabatically levelled. The key here is the energy inherent in sloping buoyancy surfaces is recognized as the important dynamical component. In an incompressible, Boussinesq fluid, volume is conserved. In an adiabatic fluid, buoyancy is also conserved, and the combination requires that in a closed domain, the volume of fluid beneath a surface of constant buoyancy is invariant. If this amount of fluid is levelled out, it achieves the depth \(z^*\) defined by

\[
\begin{align*}
z^*(b) + H = \frac{1}{L} \int_{x_w}^{x_e} \int_{-H}^{z(b)} dz \ dx = \frac{1}{L} \int_{x_w}^{x_e} [z(b) + H] \ dx, \\
\end{align*}
\]

where \(H\) is the (assumed constant) depth of the fluid, the zonal boundaries of the domain are given by \(x_w\) and \(x_e\), \(L = x_e - x_w\), and we have neglected dependence on \(y\). The local potential energy of this levelled fluid is given by \(-bz^*\) and in an incompressible fluid is at its minimum value. Note that the concept of available potential
energy assumes implicitly the existence of some predefined volume over which leveling takes place.

Introducing \( z^* \) in the integrated energy [Eq. (7)] yields

\[
\frac{d}{dt} \left( \int_A \left[ K - b(z - z^*) \right] dA \right)_{\text{DE}} + \int_A (-b z^*) dA \right)_{\text{BPE}} = -\varepsilon + D, \tag{9}
\]

showing that dissipation and diffusion can affect both the total “dynamic” energy (DE) and the background stratification represented by the potential energy of the leveled state [background potential energy (BPE)]. In an adiabatic fluid, the background potential energy is time invariant but can change if the fluid is diabatically active. Finally, moving \( D \) to the left-hand side of Eq.(9) isolates dissipation on the right. The origins of \( D \) are found in the potential energy [Eq. (6)], so subtracting \( D \) from BPE yields the change in background potential energy correcting for the effects of the parameterized, subgrid-scale mixing.

Winters et al. (1995) equated BPE to the potential energy of a sorted buoyancy profile and also that the local BPE tendency can be computed directly from the model data according to the formula

\[
-\frac{\partial}{\partial t} b z^* = \frac{\nabla b \cdot \mathbf{K} \cdot \nabla b}{b_z^*}, \tag{10}
\]

where \( \mathbf{K} \) is the (spatially anisotropic) diffusivity tensor, and \( b_z^* \) is the buoyancy frequency of the statically stable sorted buoyancy profile. The right-hand side of Eq. (10) is positive definite; the area integrating it over the domain yields the global BPE change. We will refer to Eq. (10) when discussing BPE rates.

d. Potential vorticity in a buoyancy coordinate system

The hydrostatic momentum equations expressed in a coordinate system whose vertical coordinate is buoyancy are (see Vallis 2006)

\[
\begin{align*}
u_1 + \nu u_x + \nu v_y + Hu_y - f v &= -M_x + X \\
u_1 + \nu u_x + \nu v_y + Hu_x + f u &= -M_y + Y \tag{11}
\end{align*}
\]

We have used the shorthand notation \( H \) to denote the diabatic effects acting on buoyancy,

\[
\frac{d}{dt} b = H,
\]

and \((X, Y)\) to denote “frictional” effects along isopycnals. The quantity \( M \) is the so-called Montgomery potential \( M = p - b z \), and all horizontal derivatives are taken on surfaces of constant buoyancy. From the horizontal momentum equations alone, the equation for Ertel’s potential vorticity can be formed:

\[
(z_b q)_t + \nabla \cdot (u z_b q) = -\nabla \cdot \mathbf{N}, \tag{12}
\]

where \( q = (u_x - u_e + f)/z_b \) is potential vorticity, and \( \mathbf{N} \) involves only the nonconservative effects \( H, X, \) and \( Y \). Integrating this equation on a surface of constant buoyancy over a reentrant channel with fixed, vertical walls relates net PV change to boundary fluxes:

\[
\frac{d}{dt} \int_A z_b q \, dx = -N(x_e) + N(x_w), \tag{13}
\]

where \( x_e \) and \( x_w \) are the locations of the walls. If we assume the boundary fluxes can be ignored, as we will do in this paper, the net PV on a buoyancy surface is conserved:

\[
\int_A z_b q(t) \, dx = \int_A z_b q(0) \, dx = \Pi. \tag{14}
\]

It then follows for fixed lateral boundaries that the time-averaged net PV, when spatially integrated, also conserves this value:

\[
\int_A \frac{1}{T} \int_0^T z_b q(t) \, dt \, dx = \Pi, \tag{15}
\]

where \( T \) is an averaging interval.

Although PV is always related diagnostically to vorticity and stratification, its instantaneous form is not terribly useful. However, as in Rossby adjustment problems, if we assume the system eventually settles to a steady, geostrophic state, the PV equation becomes a simple relation between PV and the Montgomery potential, that is,

\[
-f q = \frac{(M_{xx} + f^2)}{M_{bb}}, \tag{16}
\]

The rationale for the steady-state assumption in Rossby adjustment problems is that wave activity generated by the adjustment radiates to the far field and can be ignored. We shall see later that internal gravity wave activity is a natural result of the evolution of our initial-value problem. Since we work in a finite computational domain, internal waves remain in the domain, so we will assume that time averaging yields a reliable estimate of the steady PV distribution.
In general,
\[ \zeta + f = z_b q, \]  
(17)
where \(\zeta = v_x - u_y\), so
\[ \zeta + f = \overline{z_b q} = \frac{M_{xx}}{f} + f, \]  
(18)
where the overbar is as in Eq. (15). Defining
\[ \hat{q} = \frac{\overline{z \rho q}}{\zeta}, \quad q = \hat{q} + q', \]  
(19)
it is simple to show \(\overline{q' z \rho} = 0\). The above procedure is often called thickness-weighted averaging and is a standard method of averaging on density surfaces (Greatbatch 1998; Young 2012). Thus,
\[ \frac{M_{xx}}{f} + f = -\hat{\dot{q}}, \]  
(20)
which can be rearranged to
\[ M_{xx} + f \hat{q} M_{bb} = -f^2. \]  
(21)
Equation (21) is a linear elliptic equation as long as \(\hat{q}\) is nonnegative and can be solved to yield the mean Montgomery potential. The balanced dynamic fields can then be deduced according to
\[ f \overline{\omega} = \overline{M_x}, \quad z = -\overline{M_b}. \]  
(22)
This will be used later to estimate the balanced state that should reside in the domain, on top of which is an active internal wave field. The point of the above discussion is that the proper way to temporally average for the mean PV is to use thickness weighting.

Last, from Eq. (1), it is simple to show in the two-dimensional, zonal–vertical case that angular momentum \(v + fx\) is conserved on fluid parcels up to (weak) viscous effects.

3. Model description and results

We employ the MITgcm, a well-known and documented ocean general circulation model with a nonhydrostatic capacity (Marshall et al. 1997). We have computed solutions in several zonal–vertical channel configurations but will focus here on a two-dimensional, 16-km-wide, and 1000-m-deep case. We have also computed a small number of solutions allowing for three active spatial dimensions and have consistently found for meridionally independent initial conditions that the evolution up to and through the finite-amplitude evolution of the instability is two-dimensional to an excellent approximation. Hence, here we focus on the two-dimensional case with grid sizes of 5 m in the horizontal and 2.5 m in the vertical. In addition, we are working in the regime of moderate rotation as identified by Arobone and Sarkar (2012), where 2D instabilities at small scales were dominant. We performed a small number of convergence tests and parameter sensitivity runs using various grids, including some with twice the above resolution, and found our results are robust.

The lateral boundary conditions imposed on the model were free slip on all \((x, z)\) boundaries, no flux on all \((x, z)\) boundaries, no normal flow on the zonal and bottom boundaries, a rigid lid on the top, and periodicity in the meridional direction. The nonhydrostatic option was turned on. We have studied several viscous and diffusive coefficients and here focus on results obtained using 0.05 m² s⁻¹ in the horizontal and 0.5 × 10⁻⁴ m² s⁻¹ in the vertical for both. Again the results appear to be qualitatively insensitive to these values. The Coriolis parameter was set to \(f = 10^{-4}\) s⁻¹, and the runs were performed on a Cartesian \(f\) plane.

The initial condition for this run was patterned after the jet seen in the CUC simulations of Molemaker et al. (2015) and Dewar et al. (2015). A plot of the model mean current downstream of Point Sur appears in Fig. 3. The maximum speed is \(\approx 0.3\) m s⁻¹, and the eastern flank of the jet is considerably sharper in shear than the offshore side. This is the imprint of the viscously generated inshore shear zone that just downstream of the point of

![Fig. 3. The mean current averaged over 6 days downstream of the CUC separation from Point Sur. The flow is roughly unidirectional and has a maximum speed slightly larger than 0.3 m s⁻¹. The marked spot is in the anticyclonic shear.](image-url)
separation was often seen to be in excess of $-3.5f$. Analysis of overturning events in Dewar et al. (2015) and of turbulent mixing argued the supercritical jet structure was spread vertically over a few hundred meters. We have thus modeled the jet with the velocity profile

\begin{align}
    v(x, z) &= 0; \quad x < -4 \text{ km} \\
    v(x, z) &= 0.35\left(\frac{(x + 4000 \text{ m})}{4000 \text{ m}}\right) \exp\left[-(z - 500 \text{ m})^2/(80 \text{ m})^2\right] \text{ m s}^{-1}; \quad -4 \text{ km} < x < 0 \text{ km} \\
    v(x, z) &= 0.35\left(\frac{(z - 500 \text{ m})}{500 \text{ m}}\right) \exp\left[-(z - 500 \text{ m})^2/(80 \text{ m})^2\right] \text{ m s}^{-1}; \quad x > 0 \text{ km}
\end{align}

(see Fig. 4, top). The maximum jet speed is 0.35 m s$^{-1}$, and the minimum inshore vertical vorticity was $-1.8f$. The biggest distinction between our model jet and the CUC is the isolation of our jet from the vertical boundaries, whereas the CUC jet is close to the surface. The temperature profile $T$ associated with the jet was obtained according to geostrophy:

\begin{equation}
    T_x = f v_x / (g \alpha),
\end{equation}

where $g = 9.81$ m s$^{-2}$ is gravity, and $\alpha = 2 \times 10^{-4} \text{C}^{-1}$ is the linear temperature expansion coefficient. The western edge boundary condition for the above was

\begin{equation}
    T(0, z) = 25^\circ \text{C} + 25^\circ \text{C} \frac{z}{1000 \text{ m}}
\end{equation}

(see Fig. 4 middle).

Figure 4 (bottom) shows the Ertel PV associated with these conditions. A region of negative PV is found between 8 and 11 km and centered on 500-m depth. The experiments were conducted for 6 days of model time with time steps of 3 s. Outputs were archived at intervals of 10 min.

**Numerical results**

The very early evolution, approximately 1.5 h after the onset of the instability, is shown in Fig. 5 (top). The tendency is for the jet structure to shift zonally in the region where the PV is negative; little happens elsewhere. Associated with this is a distortion in the isotherms, as seen in Fig. 5 (bottom).

Soon after this state, the isotherms steepen and overturn. This is due to the structure of the CI cells; they are very long and thin and hence wrap the isotherms up as they develop. An example 5 h later appears in Fig. 6, which demonstrates this structure. The upper plot is of angular momentum that is roughly conserved following the fluid. Comparing Fig. 6 to Fig. 5 shows the anisotropic nature of the instability. Associated with this is the development of convective cells. Examples in both $v$ and $T$ appear in Fig. 7, where we show magnified views in the areas of the overturns. These plots are shown as a function of the grid counts and demonstrate that the convective rolls with about 15 points per wavelength are well resolved in these calculations.

The evolution of the system involves two main overturning events, as suggested by the time series of $w$ for hours 15 through 60 in Fig. 8. These are taken from a point in the middle of the convecting zone of the jet. The maximum of 0.05 m s$^{-1}$, occurring about 23 h out, is followed by a series of smaller, more irregular oscillations and the second, smaller organized event around 35–40 h. Once the primary overturning is completed, the system is very slow to change, as suggested by the comparison of the velocity fields from 72 and 96 h in Fig. 9. Differences between the fields are at small scales, consistent with a transition of the system to a stable, balanced state in the presence of unbalanced phenomena.

The interval from 72 to 96 h has short periods where very weak unstable density distributions appear, but their vertical velocity is quite weak, and they are almost inert.

**4. Analysis of model data**

We now proceed to the analysis of the model data. The first question to address is what is the nature of the instability occurring early in the evolution? It is clear that the initial conditions meet the necessary constraint for the fluid to be SI or CI unstable; that is, regions of negative PV and stable stratification exist. Further, the horizontal shear is sufficiently strong in the anticyclonic side of the jet that absolute vorticity is negative.

**a. What type of instability dominates?**

A signature of CI is the dominance of horizontal shear over vertical shear as an energy source for the growing instability. In Fig. 10, we compare the sizes of the horizontal ($-u'v'V_x$) and vertical ($-w'v'V_z$) sources integrated zonally over the domain from early during the onset of the instability. Both processes are positive, both showing release energy from the mean, but the horizontal source is routinely at least an order of magnitude larger than the vertical.
We also show in Fig. 11 a plot of the Richardson number $R_i$ in the immediate vicinity of the jet at the same time as the plot in Fig. 10. The instability is underway at that time, as seen from the energy releases in Fig. 10, but nowhere is the Miles (1961) and Howard (1961) condition for K–H instability of $R_i < 0.25$ met. The minimum for the field is $R_i = 0.90$. Later in the evolution, $R_i$ drops below 0.25, but that is well after the finite-amplitude effects develop. The lack of a subcritical $R_i$ allows us to eliminate the possibility of K–H instability during the linear stages of the growth.

From these bits of evidence, we conclude that CI is the dominant instability.

**Fig. 4.** Jet initial condition in (top) $v$ and (middle) $T$. The maximum downstream speed is $0.35 \text{ m s}^{-1}$, and the inshore edge is characterized by a minimum relative vorticity of $-1.8f$. Salinity is neglected, and the density is computed according to geostrophy. (bottom) The potential vorticity field of this initial condition. A region of negative potential vorticity appears at middepth between 8 and 10 km. Away from the jet, potential vorticity is constant.
b. Energetics

We plot in Fig. 12 two measures of the net energy in the flow during the 4 days of model calculation. We area integrate, rather than volume integrate, so the resulting quantities are energy per unit distance \((J \cdot m^{-2} \cdot m^2) = (J \cdot m^{-1})\). As is often the case, the important aspect of energy is its change, rather than its absolute value, so all the curves are relative to their initial values.

Time-integrated, or net, dissipation (blue curve in Fig. 12) is given by

\[
- \int_0^T \rho_o e \, dt = \Delta \left[ \iint_A \rho_o \left( \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2} - b z \right) \, dA \right]_0^T - K_v \rho_o \int_0^T \int_x [b(z = 0) - b(z = z_b)] \, dx \, dt.
\]

(26)

Note that this is essentially the change in the total energy of the flow, corrected for the change in potential energy because of the implicit diffusion. Shifting the diffusive effect to the right-hand side isolates dissipation on the left and focuses on changes driven by the modeled overturning.

There is an overall decline in total energy as time proceeds, which is expected in this unforced and viscous problem. Clearly, however, there is a qualitative change in the rate of decrease starting at about 1 day relative to the remainder of the plot. This is the time at which the jet goes unstable and develops its primary overturning and mixing events. Later in the record, the curves are seen to asymptote toward the slopes prior to the onset of overturning.

The potential energy contributions from the explicit diffusivity [i.e., the second term on the right-hand side in Eq. (26)] that would occur in a stable system are not of interest to this study. Similarly, we will correct for the background viscous and diffusive processes that would occur for a stable flow. We identify those with the dissipation and diffusion seen in the first day prior to the onset of instability and correct by removing the early slopes seen in Fig. 12.

The red curve in Fig. 12 is a measure of the change in the BPE caused by instability-driven mixing. This is essentially the measure of potential energy change
associated with $z^*$, the adiabatically leveled state of the model, correcting for the contribution due to the parameterized diffusion:

$$\Delta \text{BPE} = \rho_o \Delta \left[ \iint_A (-b z^*) \, dA \right]_0^T$$

$$- \rho_o \int_0^T \int_{x_n}^{x_f} K_v [b(z = 0) - b(z = -H)] \, dx \, dt.$$  

(27)

As with dissipation, the period of unstable evolution stands out in the growth of the BPE curve.

The time tendencies of the curves in Fig. 12 are the instantaneous dissipation and the rate of change of BPE and appear in Fig. 13.¹ We have further normalized the result by the area of the channel, thus yielding the more oceanographically familiar units of $\text{W m}^{-2}$. The more violent phase of mixing occurs at roughly 24 h, as indicated by the sharp spikes in both curves. The system then retreats to a gentler phase that holds to roughly hour 60, followed by an interval in which energy dissipation is smaller than diffusive energy gain. The maximum growth in BPE is roughly $2.7 \times 10^{-7} \text{W m}^{-2}$, and the following phase is approximately $1.0 \times 10^{-7} \text{W m}^{-2}$. Maximum dissipation is roughly $1.1 \times 10^{-6} \text{W m}^{-2}$, followed by a retreat to $2.4 \times 10^{-7} \text{W m}^{-2}$. For comparison, Ledwell et al. (2000) report enhanced dissipation levels in the Brazil basin of $O(5 \times 10^{-6}) \text{W m}^{-2}$, that would imply a rate of potential energy change of $O(1 \times 10^{-6}) \text{W m}^{-2}$. The numbers computed here are thus sizable, especially considering that the mixing takes place in a region that is roughly 3% of the total area. Locally, the values are considerably larger.

c. Mixing efficiency

A well-known parameter associated with Kelvin–Helmholtz instability is its “efficiency”:

¹ We actually show the results of computing Eq. (10) and integrating it for the BPE rate and an integral of the explicit viscous loss in Fig. 13. The curves are smoother than taking the time derivatives of the curves in Fig. 12.
$\Gamma = -\frac{\overline{w'b'}}{\epsilon}$ \hspace{1cm} (28)

(Osborn 1980), where $\epsilon$ is turbulent momentum dissipation, and $-\overline{w'b'}$ is the turbulent buoyancy flux or, equivalently, the irreversible exchange between potential and kinetic energies. The overbar implies an appropriate averaging. Although prevalent in the oceanographic literature, Eq. (28) is often replaced by

$\gamma = \frac{-\overline{w'b'}}{-\overline{w'b'} + \epsilon}$ \hspace{1cm} (29)

in the turbulence literature (Davies Wykes and Dalziel 2014). The difference is that $\gamma \leq 1$, whereas $\Gamma$ is not.

Efficiency can be used as a descriptor for any turbulent process that mixes (see Davies Wykes and Dalziel 2014). The natural connection between efficiency and centrifugal instability comes from the change in background potential energy as compared to dissipation (i.e., the two curves in Fig. 13). The ratio of these quantities $\Gamma$ as a function of time appears as the blue curve in Fig. 14 (upper), where we only show it after the onset of the instability. In addition, both the buoyancy flux and dissipation are corrected for the background inputs by subtracting their values.
occurring prior to the onset of turbulence. This curve is placed in a temporal context by comparisons with dissipation (red) and the number of unstable density distributions (green; both normalized to fit within the plot). During the violent dissipative phase, the ratio is about 0.25, mostly because the dissipation grows so aggressively. Shortly thereafter, growth in BPE picks up and the $Γ$ efficiency grows to values greater than 1. This literally implies potential energy is building in the system in the relative absence of dissipation. Note that much of this occurs during intervals where inversions are absent. The preceding convection, having removed inversions, has left well-mixed zones in the interior of the fluid bounded above and below by shoulders of rapidly varying density. It is in those locations where the bulk of the conversion takes place during these intervals. The remnant currents do not generate much dissipation in these regions.

Thus, it is seen that the efficiency of CI is highly time dependent and difficult to encapsulate as a single number. Mashayek and Peltier (2013, and references therein) and Mashayek et al. (2013, and references therein) find much the same thing for K–H instability. The cautionary tale here is that a temporally short observation of dissipation may be subtle to relate to BPE generation, as is routinely done. The early stages show a relatively low efficiency, but this is at a time prior to the convective mixing associated with the overturns. As convection sets in, the efficiency becomes inordinately high. Equivalently, the energy of the instability is effectively transferred to BPE. The reason for the relatively high efficiency is likely because of the scales of CI. The vertical extents of the overturns are relatively large, several tens of meters, creating a large-scale unstable density distribution where convection...
can mix easily in the absence of much dissipation. Eddy diffusivities/viscosities smaller than the standard values by a factor of 2 yielded comparable results, but a careful parameter study has not been performed.

It is counterintuitive to think in terms of efficiencies greater than unity, but their existence reflects the efficiency definition in Eq. (28). Equation (29), which is employed in the turbulence literature, is always less than one. That efficiency measure is compared to the inversion and dissipation time series in Fig. 14 (lower). Here, it is seen that the efficiency monotonically increases, growing to values of 0.35 at 6 days, consistent with the rapid development of BPE relative to KE loss. Davies Wykes and Dalziel (2014) argue for similarly elevated efficiencies in the Rayleigh–Taylor instability that, as here, involves gravitational convection. They further suggest that 0.5 is the maximum efficiency possible in a turbulent flow. After 6 days, our efficiency at 0.35 is still short of that value but exhibits no tendency to level off, suggesting CI is heading toward a state of maximum efficiency. We do not carry the calculations further in time because in the real ocean, it is likely time dependency in the background would present prior to this.

Standard diffusive parameterizations equate the power provided to potential energy to an eddy diffusivity times the buoyancy frequency:

\[ -w\beta = K_v N^2. \]  

Equation (6) shows the potential energy change is the time integration of the vertical buoyancy flux:

\[ \Delta \left[ \int_A \sigma \, dA \right]^T_0 - \int_T \int_A \kappa [b(z = 0) - b(z = -H)] \, dx \, dt \]

\[ = -\int_T \int_A w b \, dA \, dt; \]  

(31)

The important interval of potential energy change occurs between roughly day 1.2 and day 2, a period of about 20h. This is also the interval of density inversions. The spatial distribution of turbulent flux is computed using Eq. (10) and averaging over the above temporal interval. A similar temporal averaging is applied to the buoyancy fields. The temporal mean fields are then...
averaged over a 1.75 km by 250-m-deep region centered on the overturnings, and Eq. (30) is used to infer an effective vertical diffusivity. The result in Fig. 15 (lower) indicates values like $K_y \sim (10^{-2} - 10^{-3}) m^2 s^{-1}$. The former is anomalously large relative to typical open-ocean thermocline values of $K_y = 2 \times 10^{-6} m^2 s^{-1}$ (Ledwell et al. 1993), and the latter is 10 times the typically quoted value of $K_y = 10^{-4} m^2 s^{-1}$ needed for global balances of heat and salt (Munk and Wunsch 1998).

d. Dynamical energy

The remainder of the energy equation aside from BPE consists of what might be called the dynamical energy and is composed of the kinetic and available potential energies [see Eq. (9)]. This is the energy that the fluid flow can draw upon, and it is interesting to examine how it evolves over the period of the instability. The dynamical energy (green line) is compared to the total energy (red line) in Fig. 16; note that the drop in dynamical energy is greater than that in total energy. This is because of the increase in the background potential energy.

There are a couple of possible reservoirs for the dynamic energy. Ocean flow is routinely divided up into “balanced” flow and “unbalanced” flow. Balanced flow applies to the slowly evolving dynamics, while the unbalanced flow typically involves time derivatives at
leading order in the momentum equations. Balanced flow is also associated with potential vorticity, which is a field variable that under stable conditions is close to being conserved on fluid parcels and succinctly captures the state of the system. In classical Rossby adjustment problems, initial conditions are connected to eventual steady states through PV conservation. It is common practice in such problems to compare initial and final energies in order to comment on how effective balanced flow is in controlling the evolution of the system. A key here is that it is assumed that the system eventually relaxes to a steady state, the usual explanation being that the lost energy goes into unbalanced phenomena that radiate to the far field.

A similar decomposition into balanced and unbalanced modes is of value in the present case, but there are some important differences. First, the initial PV distribution is unstable and so cannot immediately be linked to any eventual steady state. Instead, the PV distribution must change in time in order to disable centrifugal/symmetric instability. This is what the finite-amplitude evolution does, removing the initially negative regions by dilution with the much more massive positive regions. When the PV field has become effectively nonnegative, the distribution becomes stable and we anticipate the eventual emergence of a balanced state. The tendency of the negative PV regions to mix away appears in Fig. 17, which compares the minimum in PV as a function of depth from days 1 and 4. The initial distribution appearing in red shows the regions of strong negative PV associated with the unstable profile. PV is also shown at hour 96, and it is seen that the negative PVs have retreated to nonnegative values consistent with a stable state.

There are also clear indications in several of the earlier plots of high-frequency variability, which are likely unbalanced and cannot radiate away in our closed domain. To ask how much energy is transferred to the unbalanced flow, it is necessary to isolate the balanced component.

We here estimate the balanced component of the flow in our simulations in two ways. A simple approach is to average velocity and density in time. The internal waves are fast, and the balanced flow is slow, so averaging over several internal wave time scales promises to minimize their presence. This averaged flow can then be analyzed from the above perspective. While possible, the procedure presents many practical issues. The energy in the internal waves, if sizeable, requires long averaging intervals to reduce their presence. But in this calculation with viscosity and diffusivity, the mean state will also be degraded in a way that clouds the meaning of the result. As a compromise, we average over a 24-h period from hours 72 to 96.

We have also estimated the balanced flow in a second, almost independent way by exploiting potential vorticity. By hour 60, negative PV is almost completely absent, so we declare the flow “stable” with respect to centrifugal/symmetric instability. The potential vorticity of the underlying balanced flow can then be estimated and the elliptic equation discussed in section 2 inverted to find the associated dynamical fields.

Model data from hours 72 to 96 were used for the analysis. The density fields were found to have a small number of weak, unstable density gradients dispersed throughout the domain and throughout the interval in time. These were removed by a single sweep of a running
five-point boxcar filter in the vertical. The meridional velocity from this period was then similarly filtered to make it consistent with the density. The effects on the structure of both filterings were very small and not visible to the eye. From these smoothed fields, PV was computed. We then projected that PV onto surfaces of constant buoyancy, also estimated from the smoothed data, and computed time series of thicknesses for those buoyancies. The PV was then thickness averaged according to the prescription in section 2, and the resulting profile was found to be non-negative. We then inverted the elliptic Eq. (21) to find the Montgomery potential of the underlying balanced flow.

The boundary conditions on this equation are fixed buoyancy at the top and bottom and vanishing velocity at a large distance. These reduce to the specification of Neumann conditions $M_b = (-z_x, -z_b)$ at the top and bottom, a Neumann condition $M_x = 0$ on the east boundary, and a Dirichlet condition $M = 0$ on the western domain edge. The solution of the elliptic was done using a fourth-order accurate multigrid method for nonseparable, elliptic equations from the MUDPACK collection (https://www2.cisl.ucar.edu/resources/legacy/mudpack). Differentiating the Montgomery potential in $x$ and buoyancy yielded estimates of the velocity and depths of the buoyancy interfaces where the velocities were located. These were interpolated back onto geopotential surfaces.

The velocity fields from the two estimates of the balanced flow are compared in Fig. 18, and estimates of the buoyancy field are compared in Fig. 19 (note the difference in the domain sizes). This comparison is largely successful in that the fields resemble each other closely in the structure and amplitude of the anomalies. We thus accept them as reasonable estimates of the underlying balanced flow. From them we can compute the dynamical energy in the balanced flow contained within the full fields. As the data from hours 72–96 were used, the balanced energy is representative of that interval. We compare the dynamical energy of the full solution to the dynamical energy of the inverted PV solution in Fig. 20. The dynamical energy computed from the averaged fields was larger in value by about 12%, reflecting probably some remnant unbalanced energy in the averaged fields. The PV inversion, in principle, excludes the unbalanced components.

The dynamical energy that is unaccounted for in the balanced flow, represented by the distance between the lower and upper curves in Fig. 20, must be resident in the unbalanced component, and is an estimate of the transfer to unbalanced dynamics generated by CI. Of the total dynamical energy in the final state, approximately 6% is in the internal wave field ($0.15 \times 10^7/2.26 \times 10^7$ J). Although not calculated here, presumably this energy, once in unbalanced phenomena, follows a
classical sequence leading to K–H instability and mixing. Thus, centrifugal instability may contribute to nonlocal mixing, as well as local mixing, in a manner similar to the generation of baroclinic tides by the barotropic tide.

The reason for the excitation of high-frequency, unbalanced phenomena has to do with the rapid time scales of the centrifugal instability. The initially balanced, unstable field changes on time scales that the fluid can react to in the form of gravity waves or vortical modes.

5. Summary

The objective of this work was to examine the energetics of centrifugal instability. This problem is motivated by some recent numerical work that argues CI should be a commonplace event anywhere there is a current flowing along a coast in a direction aligned with that of topographic waves (i.e., poleward on eastern boundaries, equatorward on western boundaries). An example, as argued in Molemaker et al. (2015), is the California Undercurrent where extreme anticyclonic vorticities generated in the bottom boundary layer separate from the coast and inject unstable fluid into the interior. Numerical calculations show for the CUC case that CI dominates the evolution, with the result that potential density surfaces overturn and “mix.” This study examines the energetics of CI.

The study is largely numerical and process oriented. A two-dimensional (x-z plane) meridional jet, modeled after the separated CUC, with regions of negative PV is studied. An initial-value problem is computed in a

![FIG. 19. As in Fig. 18, but for density. Again the comparison is quite favorable.](image)

![FIG. 20. Dynamical energy evolution. The strongest change occurs at the onset of the instability roughly 24 h into the experiment. The energy loss retreats to a rate consistent with the background through the last part of the experiment. The red line is the energy of the inferred balanced state. The difference between that and the dynamic energy at the end of the experiment, indicated by the arrow, is the part of the initial energy transferred to unbalanced motions.](image)
nonhydrostatic model and analyzed for its energetics’ evolution. The results are robust to reasonable parametric modifications. While an exhaustive examination of jet structures has not been performed, we have looked at a stronger jet profile, and many of the quantitative and qualitative results are comparable to those reported. The study is offered as an example of generic behavior for centrifugally unstable flows.

We observe that the evolving system is dominated by CI, which at finite-amplitude drives direct overturns of potential density surfaces. A comparison of the energy dissipation during the overturning to the growth in mean potential energy argues this process is very efficient according to the Osborn definition:

$$\Gamma_{CI} > 1. \quad (32)$$

It is also relatively high according to the efficiency in Eq. (29) and appears to be heading to its theoretical maximum value. There is considerable variability of instantaneous estimates of this measure, reflecting that CI mixes essentially as a multistage process. The number above should be compared to the classical (but probably too low) 0.2 efficiency of Kelvin–Helmholtz instability. The reason for the overturning is that the instability erases the negative potential vorticity values along isopycnals by stirring laterally. In a fluid with tilted isopycnals, like any vertically sheared rotating flow, this results in a wrapping up of the isopycnals. The relatively high efficiency implies the overturning can occur with little energy loss to heat.

We have also argued that CI naturally excites unbalanced phenomena. We have estimated the partitioning between balanced and unbalanced energy in the “final” state of the system and found that about 6% of the final dynamical energy is unbalanced. The specific value of this number is probably of less value than the recognition of the transfer of energy to unbalanced phenomena by a balanced flow.

A rough accounting of the energy budget is as follows: Of the total initial dynamical energy of the flow (initial kinetic plus initial available potential energy), about 3% \( (8 \times 10^{7}/2.4 \times 10^{7}) \) is directly dissipated as a result of the finite-amplitude instability and another 1% \( (3 \times 10^{6}/2.4 \times 10^{7}) \) goes into the background potential energy. Of the remaining 96% of the energy, approximately 6% \( (0.15 \times 10^{7}/2.4 \times 10^{7}) \) scatters into the unbalanced and predominately internal gravity wave field, and the remainder (~90% of the initial) remains in the balanced flow. All told, roughly 10% of the initially balanced energy is lost from the balanced state.

The transfer to the local, mean, state potential energy represents local mixing, and the generation of internal waves is likely connected to remote mixing. Assuming the latter occurs with a classical efficiency (0.2), approximately 1% more of the initial energy eventually ends up in the mean potential energy profile. In total, something like 2% of the initially balanced energy ends up in the mean state stratification.

### Regional and global impact

Now we speculate on the question of the regional and global impact of centrifugal instability. As a metric, the estimated 0.4 TW needed to sustain the stratification translates to an energy consumption per unit area of

$$4 \times 10^{11} \text{ W} / [0.7\pi (6.4 \times 10^{6} \text{ m})^2] = 4 \times 10^{-3} \text{ W m}^{-2}, \quad (33)$$

or 4 mW m\(^{-2}\). Figure 13 suggests an average transfer to mean stratification by a centrifugal event as \( 1 \times 10^{-7} \text{ W m}^{-3} \) when averaged over the domain; locally the values are considerably larger. Integrating the power density supplied to BPE, and removing the background contribution, returns values of \( O(1.0) \text{ mW m}^{-2} \). This is roughly 25% of the average global requirement. Further, the time series shown in section 1 suggests overturns at a rate of once per day for the CUC, so this number might represent a reasonable average.

The conditions favorable to CI are cyclonic type flows against topography, translating into poleward flows on eastern ocean boundaries and equatorward flows on western ocean boundaries. The strongest currents in the world (i.e., the Gulf Stream, the Kuroshio, and the Antarctic Circumpolar Current) do not meet this requirement while higher-latitude western boundary currents do (the Oyashio, the Labrador Current, and the Malvinas). There are several near-equatorial western boundary currents flowing equatorward, like the Mindanao Current and the North Brazil Current. Many of the deep currents in the Indian and Pacific Oceans flow equatorward on the west as well. Many eastern ocean boundary currents are poleward, examples include the often-mentioned California Undercurrent and the Leeuwen Current off western Australia. In summary, several boundary regions in the ocean have currents that meet the criteria for centrifugal instability. In view of the relatively common occurrence of promontories and capes on continental coastlines, CI should represent a widely distributed mixing mechanism that contributes importantly to regional mixing. In contrast, the areas of active CI are confined to strips along the coasts of ~10-km
width. Its contributions to the global budgets are probably relatively limited.

These are clearly very uncertain estimates with very large error bars, but it is not unreasonable to suggest CI can contribute importantly to regional mixing in eastern boundary coastal areas. Also, the locations where much of the associated mixing should take place are relatively near surface and in the permanent ocean thermocline. Mixing in such areas is also important to the maintenance of near-surface ecosystems through the delivery of deep nutrients to the photic zone.

To summarize, CI appears to be able to drive vigorous mixing in oceanlike, stratified fluids. While its global impact appears limited, it is likely an important regional contributor to the mixing climatology. The estimates suggesting this are based on what has been learned from this highly idealized study, and it would be of interest to look for observational support.

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