Mixing, Dissipation Rate, and Their Overturn-Based Estimates in a Near-Bottom Turbulent Flow Driven by Internal Tides

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ABSTRACT

Direct numerical simulation (DNS) and large-eddy simulation (LES) are employed to study the mixing brought about by convective overturns in a stratified, oscillatory bottom layer underneath internal tides. The phasing of turbulence, the onset and breakdown of convective overturns, and the pathway to irreversible mixing are quantified. Mixing efficiency shows a systematic dependence on tidal phase, and during the breakdown of large convective overturns it is approximately 0.6, a value that is substantially larger than the commonly assumed value of 0.2 used for calculating scalar mixing from the turbulent dissipation rate. Diapycnal diffusivity is calculated using the irreversible diapycnal flux and, for tall overturns of $O(50)$ m, the diffusivity is found to be almost 1000 times higher than the molecular diffusivity. The Thorpe (overturn) length scale is often used as a proxy for the Ozmidov length scale and thus infers the turbulent dissipation rate from overturns. The accuracy of overturn-based estimates of the dissipation rate is assessed for this flow. The Ozmidov length scale $L_O$ and Thorpe length scale $L_T$ are found to behave differently during a tidal cycle: $L_T$ decreases during the convective instability, while $L_O$ increases; there is a significant phase lag between the maxima of $L_T$ and $L_O$; and finally $L_T$ is not linearly related to $L_O$. Thus, the Thorpe-inferred dissipation rates are quite different from the actual values. Interestingly, the ratio of their cycle-averaged values is found to be $O(1)$, a result explained on the basis of available potential energy.

1. Introduction

It is well known that internal tides generated at rough topography lead to near-bottom turbulence during generation and reflection. Many observational studies have found enhanced mixing near topographic boundaries (Kunze and Toole 1997; Eriksen 1998; Munk and Wunsch 1998; St. Laurent et al. 2001; St. Laurent and Garrett 2002; Moum et al. 2002; Cacchione et al. 2002; Rudnick et al. 2003; Nash et al. 2004; Aucan et al. 2006) that is modulated at tidal frequencies. Accurate quantification of the turbulence and mixing at rough topography is a prerequisite for the parameterization of tidal mixing in ocean models, inferring mixing rates from observed overturns, and ultimately to our ability to model the global ocean circulation (Vallis 2000; Park and Bryan 2000; Wunsch and Ferrari 2004).

Because of the large variation of scales ranging from $O$ (km) down to $O$(cm), simulations of oceanic internal tides that resolve the complete range of spatial and temporal scales are still beyond our capability. Nevertheless, recent simulations in an idealized setting have been helpful by showing pathways to turbulence through wave breaking near the top of topographic features (Legg and Klymak 2008; Klymak et al. 2010; Rapaka et al. 2013), in intensified boundary flows at near-critical slopes (Gayen and Sarkar 2011b) in the generation problem, and during reflection at critical and near-critical slopes (Chalamalla et al. 2013). Convective instability was found to be responsible for the
transition to turbulence at near-critical slopes through the generation of counterrotating, streamwise rolls by Gayen and Sarkar (2010), who employed direct numerical simulation (DNS) for a laboratory-scale model problem. In a scaled-up, large-eddy simulation (LES), Gayen and Sarkar (2011a) found vigorous turbulence following large convective overturns during flow reversal from downslope to upslope flow similar to the observation by Aucan et al. (2006) at a deep flank of Kaena Ridge. Convective overturns have been identified above topography, for example, at the crest of Kaena Ridge, by Klymak et al. (2008) in an observational study, in the bottom boundary layer of lakes (Lorke et al. 2005, 2008; Becherer and Umlauf 2011), in two-dimensional simulations (Legg and Klymak 2008; Buijsman et al. 2012), and in three-dimensional DNS and LES (Rapaka et al. 2013). From the prevailing literature, it is thus clear that convective instability is one likely route to mixing in internal tides. However, accurate quantification of the amount of turbulent mixing accomplished by the convective overturns, especially given the oscillatory shear and stratification of near-bottom internal tides, remains an outstanding issue that motivates the DNS and LES of the present model problem.

A popular method to infer the turbulent dissipation rate from observational data is based on the overturning length scale (Thorpe 1977) that is a measure of the vertical extent of density overturns computed by adiabatically rearranging the density profile to attain a stable configuration. The Thorpe length scale $L_T$, is defined as the root-mean-square (rms) of the parcel displacements required to attain a density profile that is statically stable. Dillon (1982) performed a detailed study of the overturn method using measurements in the upper ocean. The ratio of the Ozmidov scale $L_O = \sqrt{\Gamma / N^2}$ to the Thorpe scale $L_T$ was found to be of the same order in regions far away from the wind-driven boundary layer, both in mixed layers and in seasonal thermoclines. The turbulent dissipation rate was estimated to be $\epsilon \approx L_T^2 N^3$, where $N$ is the buoyancy frequency. Diapycnal diffusivity can also be estimated (Osborn 1980) using $K_p = \Gamma \epsilon / N^2$, where $\Gamma$ is the mixing efficiency, often taken to be 0.2. Observational studies of internal tides (e.g., Alford et al. 2006; Martin and Rudnick 2007; Levine and Boyd 2006) often infer the turbulent dissipation rate and diapycnal diffusivity from overturns, although there are notable exceptions involving direct measurements of the dissipation rate using shear probes, for example, in the Brazil basin in the study by Polzin et al. (1997) and at the Hawaiian Ridge in the study by Klymak et al. (2006).

During the mixing, near-bottom topography driven by internal tides, the evolution of Thorpe and Ozmidov scales could be different because of the strong and systematic temporal variability of turbulence. The three-dimensional, turbulence-resolving simulations performed here enable a direct calculation of the ratio of Ozmidov to Thorpe scales, as we can independently calculate both length scales from the available simulation data. The relationship between Thorpe and Ozmidov scales has been explored in the case of shear-driven turbulence in previous studies. For instance, Smyth et al. (2001) analyzed DNS of the nonlinear evolution of Kelvin–Helmholtz billows finding that the ratio of Ozmidov to Thorpe length scales $L_O/L_T$ continuously increases during the evolution from a very small value to an $O(1)$ value. The flux coefficient $\Gamma = B/\epsilon$, where $B_i$ is the irreversible buoyancy flux, was found to decrease with increasing time or, equivalently, with increasing value of $L_O/L_T$. More recently, Mater et al. (2013) examined the relationship between $L_T$ and $L_O$ using three-dimensional DNS results of decaying, statistically homogeneous, stratified turbulence. It was found that $L_T$ and $L_O$ were not linearly related. Instead, large overturns were found to be reflective of turbulent kinetic energy rather than turbulent dissipation rate in strongly stratified ($NT_L > 1$, where $T_L$ is a large-eddy turbulence time scale) situations. In the present study, the evolution of Ozmidov and Thorpe length scales is discussed in the context of convectively driven turbulence in an oscillating boundary flow, the energetics of which is quite different from shear-driven turbulence. In shear-driven turbulence, direct transfer from mean kinetic energy to turbulent kinetic energy occurs through shear production. However, in convectively driven turbulence, mean kinetic energy drives the flow to a statically unstable density configuration, leading to an increase in the available potential energy, which is then released to turbulence through the breakdown of the density overturn. The overturning length scale $L_T$ in this context is representative of the potential energy available in an overturn to be released to turbulence and eventually to dissipation and background mixing. In the present problem, the density variation returns to a statically stable state owing to the oscillatory nature of the flow.

Ocean models utilizing internal tide parameterizations (Simmons et al. 2004) suggest that the spatial and temporal variability of turbulent diffusivity need to be taken into account to properly model bottom abyssal stratification. The mixing efficiency $\Gamma$, which is used to estimate diapycnal diffusivity $K_p$ from the turbulent dissipation rate through $K_p = \Gamma \epsilon / N^2$ has often been oversimplified in ocean models by assuming a constant value of $\Gamma = 0.2$. In shear-dominated flows, a small fraction is utilized for mixing the density field (Linden 1979; Peltier and Caulfield 2003; Ivey et al. 2008). In contrast, convective instabilities show higher mixing efficiencies (Dalziel et al. 2008; Gayen et al. 2013). Also, $\Gamma$ varies substantially during a turbulent event (Smyth et al. 2001; Dalziel et al. 2008), suggesting that the systematic phasing of turbulence that is known to
occur in internal tide-driven mixing needs to be considered. In the present study, we calculate the mixing efficiency and diapycnal diffusivity from the values of turbulent dissipation, irreversible diapycnal flux, and the stratification available from the simulation data.

The paper is organized as follows: The formulation of the problem is given and the numerical method is summarized in section 2. The cyclic variation of turbulent kinetic energy and the density variance budgets is described in section 3. In section 4, the evolution of available potential energy (APE; which represents the energy that can be released to drive fluid motion) and irreversible diapycnal flux (which represents the transfer of energy from available to background potential energy) is discussed. The evolution of Thorpe and Ozmidov scales during the period of large convective overturns is described in section 5. Sections 6 and 7 contain results regarding the mixing efficiency and diapycnal diffusivity, respectively. The paper concludes with the conclusions drawn in section 8.

2. Formulation

The schematic of the problem is shown in Fig. 1. The baroclinic bottom flow has thickness $I_b$, peak velocity $U_b$, and oscillates with the $M_2$ tidal frequency. The thickness $I_b$ is defined as the slope-normal distance between the bottom and the first zero crossing of the along-slope velocity. Going beyond Gayen and Sarkar (2011a), we present new results regarding the evolution of scalar variance, mixing efficiency, and the recipe for inferring turbulent dissipation rate from overturns.

In our previous simulations of internal tide generation at laboratory scale (Gayen and Sarkar 2011b; Rapaka et al. 2013) and in two-dimensional large-scale simulations, an oscillating barotropic tide leads to a baroclinic boundary flow with amplitude $U_b$ and thickness $I_b$ that is maintained against turbulent losses by barotropic-to-baroclinic energy conversion. Turbulence-resolving, three-dimensional simulations in streamwise, inhomogeneous domains of $O(10)$ km required to realize $O(100)$ m overturns in the baroclinic bottom boundary layer are not yet possible. We therefore adopt the approach described by Gayen and Sarkar (2011a). Thus, a patch of the larger-scale internal wave is simulated in a domain that is periodic in the spanwise and along-slope directions. The velocity amplitude $U_b$ is maintained by adding a forcing term $-\sigma_f(z_s)[u_t(x,t) - u_s,f(z_s,t)]$ to the right-hand side of the along-slope momentum equation. The target velocity $u_{s,f}$ is given by
Table 1. Parameters of the simulated cases. Each case has background stratification \( N_s = 1.6 \times 10^{-3} \text{ rad s}^{-1} \), frequency \( \Omega = 1.407 \times 10^{-7} \text{ rad s}^{-1} \), and slope angle \( \beta = 5^\circ \). The Reynolds number is based on the amplitude of the bottom velocity \( U_b \) and the Stokes boundary layer thickness. The bulk Richardson number, defined as \( R_i = N_s^2 l_s^2 / U_b^2 \), is 0.59 for both the cases. The kinematic viscosity \( \nu \) is taken as \( 10^{-6} \text{m}^2 \text{s}^{-1} \) and Prandtl number \( Pr = 7 \). Reference density \( \rho_0 \) is taken as 1000 kg m\(^{-3}\) for all the cases. Grid spacing for case 1 is \( \Delta x = 0.019, \Delta y = 0.008, \Delta z_{\text{min}} = 0.002 \), and for case 2 is \( \Delta x = 0.23, \Delta y = 0.156, \Delta z_{\text{min}} = 0.0037 \), with all dimensions in meters. The term \( \Delta z^* \) is \( \leq 1.6 \) for DNS and \( \leq 10 \) for the LES case. The minimum value of Kolmogorov length scale is \( \eta_{z,\text{min}} \approx 0.002 \text{ m} \) in both the cases. Stokes boundary layer thickness \( \delta_i = \sqrt{2 \pi \nu} \) is 0.12 m in both the cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>( U_b ) (m s(^{-1}))</th>
<th>( l_b ) (m)</th>
<th>( R_i )</th>
<th>( N_s )</th>
<th>( N_r )</th>
<th>( l_{x_0} ) (m)</th>
<th>( l_t ) (m)</th>
<th>( l_{x_2} ) (m)</th>
<th>Remark</th>
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</thead>
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<td>1</td>
<td>0.0125</td>
<td>6</td>
<td>1500</td>
<td>256</td>
<td>128</td>
<td>513</td>
<td>5.0</td>
<td>1.0</td>
<td>15.0</td>
</tr>
<tr>
<td>2</td>
<td>0.125</td>
<td>60</td>
<td>15000</td>
<td>128</td>
<td>64</td>
<td>641</td>
<td>30.0</td>
<td>10.0</td>
<td>150.0</td>
</tr>
</tbody>
</table>

\[ u_{x,f}(z_s,t) = A_0 \exp(A_1 z_s) \sin(A_2 z_s) \sin(\Omega t + \phi_0), \]
\[ l_b/3 \leq z_s \leq l_b, \tag{1} \]

where \( z_s \) is the slope-normal coordinate, and \( l_b \) is the slope-normal distance between the bottom and the first zero crossing of the along-slope velocity. The frequency is \( \Omega = 1.407 \times 10^{-7} \text{ rad s}^{-1} \), and the quantity \( \sigma_r(z_s) \) is the forcing term with a value \( \sim (1/\Delta t) \) (where \( \Delta t \) is the time step of the numerical simulation) in the forcing region that tends to zero outside the forcing region. The dimensional coefficients \( A_0 = U_b/0.47, A_1 = -1.85/l_b, \) and \( A_2 = 3.2/l_b \) are chosen as a best fit of the chosen function to the shape of self-similar velocity profiles obtained for different slope lengths in the laboratory-scale internal wave generation problem of Gayen and Sarkar (2011b). The value of \( \phi_0 = -\pi/2 \), that is, the simulation starts with a phase corresponding to peak downslope initial velocity. The density is initialized with the background profile that describes the density variation at this phase.

Two cases are considered: (1) width \( l_b = 6 \text{ m} \) and peak velocity amplitude of \( U_b = 0.0125 \text{ m s}^{-1} \), leading to Reynolds number \( Re \) small enough to allow DNS, and (2) width \( l_b = 60 \text{ m} \) and peak velocity amplitude of \( U_b = 0.125 \text{ m s}^{-1} \) that is closer to oceanic conditions. Both cases have the same value of background stratification \( N_s \) and wave shear \( U_b/l_b \). Physical and computational parameters for the simulations are given in Table 1. The choice of background stratification \( N_s \approx O(10^{-3}) \text{ rad s}^{-1} \) is consistent with measurements at deep (order of 1 km) flanks of rough topography, for example, Kaena Ridge (Aucan et al. 2006) and the west ridge of the double-ridged Luzon Strait (Buijsman et al. 2012).

a. Governing equations

Direct and large-eddy simulations are performed to numerically solve the Navier–Stokes equations under the Boussinesq approximation and in a nonrotating environment. Rotation of the earth is not considered in this study since the time scale over which the convective instability evolves is too small for the rotation to have a direct effect on turbulent dissipation and mixing, the main focus of this paper. However, rotation can affect the baroclinic boundary flow, and this implicit effect of rotation is not considered in this study for simplicity. The dimensional form of the governing equations in the reference axis system \((x_s, y, z)\) rotated by an angle \( \beta \) with respect to the horizontal in the \( x-z \) plane is as follows:

\[ \mathbf{V} \cdot \mathbf{u}_s = 0, \tag{2a} \]

\[ \frac{D \mathbf{u}_s}{Dt} = -\frac{1}{\rho_0} \mathbf{V} \mathbf{p}^* - \frac{g \mathbf{p}^*}{\rho_0} (\sin \beta \mathbf{i} + \cos \beta \mathbf{k}) + \nu \nabla^2 \mathbf{u}_s - \mathbf{V} \cdot \boldsymbol{\tau}, \]

and

\[ \frac{D \rho^*}{Dt} = \kappa \nabla^2 \rho^* - \frac{d \rho^*}{dz} (u_s \sin \beta + w_s \cos \beta) - \mathbf{V} \cdot \mathbf{\lambda}. \tag{2c} \]

Here, \( u_s, v_s, w_s \) are along-slope, spanwise, and slope-normal velocities, respectively, in a rotated coordinate.
system. Equations (2a)–(2c) will be solved numerically, details of the numerical method are discussed in the following section. Here, \( p^* \) denotes the deviation from the background hydrostatic pressure, and \( \rho^* \) denotes the deviation from the linear background state \( \rho'(z) \). The term \( z \) represents the true vertical coordinate here. For LES simulations, \( u_i \), and \( \rho^* \) represent filtered quantities. The quantities \( \tau \) and \( \lambda \) represent the subgrid-scale stress tensor and density flux vector, respectively. The subgrid-scale stress tensor \( \tau \) in (2b) is computed using the dynamic mixed model (Zang et al. 1993; Vreman et al. 1997), and a dynamic eddy diffusivity model (Armenio and Sarkar 2002) is used for the density flux vector \( \lambda \):

\[
\tau_{ij} = -2\nu_T \delta_{ij} + \overline{\mu_i j} - \overline{\mu_i j}, \quad \nu_T = C \Delta^2 |\delta|, \quad \text{and} \quad (3)
\]

\[
\lambda_j = -\kappa_T \frac{\partial \rho^*}{\partial x_j}, \quad \kappa_T = C \rho \Delta^2 |\delta|, \quad (4)
\]

where \( \delta \) represents the test filter, and \( \overline{\delta} \) represents the grid filter. The term \( \Delta \) is the grid filter size, \( S_{ij} \) is the strain rate tensor, and \( |\delta| = \sqrt{2\delta_{ij}\delta_{ij}} \).

The eddy viscosity and diffusivity coefficients, \( \nu_T \) and \( \kappa_T \) defined above, are computed using current values of velocity and density. Here, \( C \) and \( C_p \) are the Smagorinsky coefficients evaluated through a dynamic procedure (Germano et al. 1991) through the introduction of an additional test filter. The coefficients are averaged over the homogeneous directions (slope-parallel plane); expressions for calculating these coefficients are described by Gayen et al. (2010). For the DNS, \( \tau \) and \( \lambda \) are zero.

### b. Boundary conditions

No slip boundary conditions are imposed at the bottom boundary for \( u_i, v_i, \) and \( w_i \). The total density can be written as the sum of the background density and the density deviation:

\[
\rho = \rho_0 + (z_s \cos \beta + x_s \sin \beta) \frac{d\rho^b}{dz} + \rho^*.
\]

The zero mass flux boundary condition is imposed at the sloping bottom, resulting in the density deviation at the sloping boundary given by

\[
\frac{\partial \rho^*}{\partial z_s} = -\cos \beta \frac{d\rho^b}{dz}.
\]

### c. Numerical method

The simulations use a mixed, spectral–finite difference algorithm. Derivatives in the streamwise and spanwise directions are treated with a pseudospectral method and derivatives in the vertical direction are computed with second-order finite differences. A staggered grid is used in the wall-normal direction. A low-storage, third-order Runge–Kutta–Wray method is used for time stepping, and viscous terms are treated implicitly with the Crank–Nicolson method. The code is parallelized using the message passing interface (MPI). Periodicity is imposed in the \( x_i \) and \( y_i \) directions. The top boundary is an artificial boundary corresponding to the truncation of the domain in the vertical direction. Rayleigh damping or a sponge layer is used to minimize spurious reflections from the artificial boundary into the computational domain.

The velocity and scalar fields are relaxed toward the background state in the sponge region by adding damping functions \( -\sigma(z_i) [u_i, v_i, w_i] \) and \( -\sigma(z_i) [\rho^*] \) to the right-hand side of the momentum and scalar equations, respectively. The value of \( \sigma(z_i) \) increases exponentially from zero at the bottom boundary of the sponge to a maximum value of \( \sigma(z_i) \Delta t \sim O(0.1) \).

### d. Turbulence diagnostics

Various averaging procedures are used in analyzing the results of this study. The Reynolds average, denoted by angle brackets \( \langle \cdot \rangle \), is computed by averaging in the slope-parallel plane. The term \( \langle \cdot \rangle_{\phi} \) represents phase averaging, \( \langle \cdot \rangle_z \) represents depth averaging, and \( \langle \cdot \rangle \) represents averaging over the extent of an overturn. Any fluctuating quantity in the flow field is defined by subtracting the Reynolds average from the instantaneous value:

\[
A' = A - \langle A \rangle.
\]

The evolution of turbulent kinetic energy (TKE) is governed by the following equation:

\[
\frac{dK}{dt} = P - \varepsilon + B - \frac{\partial T_j}{\partial x_j},
\]

where \( K = 1/2[u_i'u_i'] \), \( P = -\langle u_i'u_i' \rangle / S_{ij} \rangle - \langle \tau_{ii} \rangle / S_{ij} \rangle, \varepsilon = \nu \langle \partial u_i' \partial x_j \rangle \langle \partial u_i' \partial x_j \rangle - \langle \tau_{ij} S_{ij} \rangle, B = (g_i / \rho_0) \langle p'w' \rangle = -g \langle \partial \rho^* / \partial z_s \rangle, T_j = (1/\rho_0) \langle p'w' u_i' \rangle - \nu \langle \partial K / \partial x_j \rangle + 1/2 \langle u_i'u_i' u_i' \rangle, \) and \( \delta_T / \partial x_j \) is the divergence of the transport of turbulent kinetic energy with \( x_j' = 1/2 \langle (\partial u_i' / \partial x_j) + (\partial u_j' / \partial x_i) \rangle \) denoting the fluctuating strain rate.

The evolution of density variance is governed by the following equation:

\[
\frac{d \rho^2}{dt} = T_\rho + P_\rho - \chi_\rho,
\]
where, the scalar production
\[ P_r = -2(u'\rho')(\partial\rho/\partial x_j) - 2(\rho'w')d\rho/\partial z) - 2(\lambda_1)(\partial\rho/\partial x_j); \]
scalar transport \[ T_r = \kappa\nabla^2(\rho^2) - (\partial(u'\rho^2)/\partial x_j); \]
scalar dissipation rate \[ \chi_\rho = 2(\kappa + \kappa_T)(\partial\rho/\partial x_j)(\partial\rho/\partial x_j); \] and \( \kappa_T = 0 \) for the DNS case. Here, the molecular scalar diffusivity is defined by \( \kappa = \nu Pr \), where \( Pr \) is the Prandtl number.

### 3. Turbulence budgets

The velocity is proportional to \( \sin\phi \), where \( \phi \) is the \( M_2 \) tidal phase. The simulation starts at the phase \( \phi = -\pi/2 \) of peak downslope velocity and the linear background density profile. At this time, the density and velocity fields have no fluctuations with respect to their Reynolds average. The downward flow brings fluid in from above that is lighter than the fluid that it replaces. This causes the density gradient in the flow to progressively decrease. As shown in Fig. 7 below, the density profile spanning almost the entire region affected by the flow (except the thin boundary layer at the wall) becomes nearly uniform at \( \phi = -0.3\pi \), exhibits a large region that is convectively unstable at \( \phi = -0.15\pi \), and eventually breaks down into smaller overturns owing to turbulence at \( \phi = 0.03\pi \). Figures 2d-f show the density field at various phases during the evolution of the convective overturning event. The phase corresponding to each density snapshot is indicated by a solid circle in Figs. 2a-c. At a phase \( \phi = -0.11\pi \), just before the transition from downslope to upslope flow, lighter fluid is pushed from above which replaces the denser fluid between \( z = 10 \) and \( 40 \) m, as shown in Fig. 2d. At a slightly later phase \( \phi = 0.04\pi \), when the velocity is upslope but close to zero, the overturn breaks down into turbulence as shown in Fig. 2e. At this phase, lighter fluid moves upward, whereas the denser fluid moves downward in the process of attaining a stable stratification. This large convective event mixes up the density field as shown at a later time in Fig. 2f.

For completeness, we summarize the TKE balance that was previously discussed by Gayen and Sarkar (2011a) before presenting new results regarding the density variance balance. Figure 3a shows the cycle evolution of various terms in the TKE budget equation for case 1 along with the streamwise velocity (dotted line) to establish the phase. All the terms plotted in this
Corresponding to a convective instability, the turbulent buoyancy flux representing the transfer of available potential energy to turbulent kinetic energy starts to increase at phase $f = 0.1$ and peaks at $f = 0.2$. We refer to this event during the flow reversal from downslope to upslope as a large convective overturning event (LCOE) in our subsequent discussions. As the cascade to small scales proceeds, the turbulent dissipation rate progressively increases. The peak value of the turbulent dissipation rate occurs at a slightly later phase $f = 0.08$ compared with the buoyancy flux. The transient term $dK/dt$ follows a similar trend as the turbulent buoyancy flux throughout the LCOE, showing the dominance of the buoyancy flux in the life cycle of turbulence, a signature of convective instability. The shear production rate is small compared with the buoyancy flux. It has a small negative value just after upslope flow commences; a somewhat surprising phenomenon that is explained by Gayen and Sarkar (2011c) as due to the inclined coherent structures that form when shear distorts the convective overturns. Later, when the along-slope velocity approaches its positive maximum, the bottom boundary layer becomes turbulent because of the shear instability. During this phase of the cycle, the turbulent production is balanced by the turbulent dissipation rate, whereas the buoyancy flux is negligible. At a later time, that is, during the transition from upslope to downslope flow, no significant turbulent activity is observed. The reason is that the convective overturning region formed by heavier fluid replacing lighter fluid is close to the wall so that the eddies are restricted from growing by the bottom wall and are also subject to stronger viscous damping.

Figure 3b shows the time evolution of various terms in the density variance equation. During the LCOE ($-0.15 \leq \phi \leq 0.2$), there is an increase in scalar production and scalar dissipation. The transient term $d(\rho^2)/dt$ averaged over the LCOE is approximately zero. Thus, the main balance is between the scalar production and scalar dissipation; that is, hydrodynamic fluctuations extract energy from the background stratification to create scalar fluctuations that are dissipated at small scales by molecular effects. Scalar transport is negligible throughout the cycle. The evolution of scalar production resembles the buoyancy flux (shown in the TKE budget). Consider the scalar production term in the density variance equation:

$$P_\rho = -2\langle \rho' w' \rangle \frac{\partial (\rho^2)}{\partial \zeta_s} - 2\langle \rho' w' \rangle \frac{dp}{d\zeta},$$ and (10)

$$P_\rho \approx B \frac{\rho_0}{g} \left( \frac{\partial (\rho^2)}{\partial \zeta_s} + \frac{dp}{d\zeta} \right) \Rightarrow P_\rho \approx B \frac{\rho_0}{g} \frac{\partial (\rho)}{\partial \zeta}. \quad (11)$$

During the large convective overturning event, the density profile is unstable ($dp/d\zeta > 0$). Since the buoyancy flux and density gradient are both positive during the LCOE, the scalar production that is the product of the buoyancy flux and the density gradient is also positive and similar to buoyancy flux in its evolution. The second half of the tidal cycle in case 1 shows negligible activity in the density variance balance.

Figure 4a shows the time evolution of various terms in the TKE budget equation for case 2, simulated using LES. The turbulent dissipation is the sum of the
resolved and subgrid contributions. The flow evolution and turbulence budgets are qualitatively similar to the low Re DNS case discussed above. In the LES case, since the width of the beam is 10 times larger than in the DNS case, the overturns are bigger and the magnitudes of terms in the turbulence budgets are higher compared to the DNS case. An important qualitative difference is that, at the smaller Re, the buoyancy flux is negative for some fraction of the LCOE, whereas, at the larger Re, the buoyancy flux is almost always positive and lasts for a longer fraction of the cycle. Also, negative turbulent production is more prominent at higher Re, presumably because the inclined coherent structures suffer less damping by molecular viscosity.

Figure 4b shows the time evolution of various terms in the density variance equation. The behavior is qualitatively similar to that at lower Re. One difference is that there is higher turbulence activity during $\pi/2 < \phi < \pi$ at higher Re. The TKE budget shows positive buoyancy flux during the phase $\pi/2 < \phi < \pi$, and the density variance budget shows a slight increase in scalar production and dissipation at the same time. This activity is because of the turbulence created by combination of bottom shear (large upslope velocity) and convective overturns formed by the advection of heavier fluid from below.

4. Available potential energy

Background potential energy is calculated by reordering the density field (Winters et al. 1995; Winters and Barkan 2013) such that it has stable stratification everywhere. A volume element $dV_i$ with density $\rho_i$ at depth $z$ is assigned a new depth $z^*$ in the reordered stable density profile. Background potential energy is defined by $\text{BPE} = \int \rho_i g z^* dV_i$. Available potential energy is then defined as the difference between total potential energy (TPE) and background potential energy (BPE):

$$\text{APE} = \text{TPE} - \text{BPE} = \int \rho_i g z^* dV_i - \int \rho_i g z dV_i.$$

The background potential energy evolves according to

$$\frac{d}{dt} \text{BPE} = \int g z^* (\textbf{u} \cdot \nabla \rho) + \nabla \cdot (\zeta \kappa_{\text{eff}} \nabla \rho) \right) dV - g \int \kappa_{\text{eff}} \nabla \rho \cdot \nabla z^* dV.$$

Using the product rule for divergence $z^* \nabla \cdot (\kappa_{\text{eff}} \nabla \rho) = \nabla \cdot (\kappa_{\text{eff}} z^* \nabla \rho) - \kappa_{\text{eff}} \nabla \rho \cdot \nabla z^*$, the second term in the above equation can be rewritten as

$$\frac{d}{dt} \text{BPE} = \int g z^* \nabla \cdot (\textbf{u} \cdot \nabla \rho) + \nabla \cdot (\textbf{u} \cdot \nabla \rho) + \nabla \cdot (\kappa_{\text{eff}} z^* \nabla \rho) - \kappa_{\text{eff}} \nabla \rho \cdot \nabla z^* dV.$$

Following Winters et al. (1995), since $z^*$ is a monotonic function of the density $\rho$, and its spatial distribution depends implicitly on the density, a function $\psi$ can be defined such that $z^* \nabla \rho = \psi \nabla \rho$. Using the del operator product rule and the mass conservation equation $\nabla \cdot \textbf{u} = 0$, the first term can be written as a surface integral. The second term in the above equation can also be written as a surface integral. Since $z^*$ is a monotonic function of density, $\nabla z^*$ in the third term above can be written as $(dz^*/d\rho) \nabla \rho$, leading to

$$\frac{d}{dt} \text{BPE} = \int g \psi \textbf{u} \cdot \nabla \rho + \nabla \cdot (\zeta \kappa_{\text{eff}} \psi \nabla \rho) \nabla \rho - g \int \kappa_{\text{eff}} \nabla \rho \cdot \nabla z^* dV.$$

Finally,

$$\frac{d}{dt} \text{BPE} = F_{\text{adv}} + F_{\text{diff}} + \Phi_d.$$

In the case of LES, $\kappa_{\text{eff}} = \kappa + \kappa_T$, the sum of molecular and eddy diffusivity. In the case of DNS, $\kappa_{\text{eff}} = \kappa$ (since $\kappa_T = 0$) is a constant. The first term on the rhs of (16) represents the change in background potential energy due to advective fluxes across the boundary of the integration domain. The second term represents the change in BPE due to the diffusive fluxes across the boundary, and the third term is the irreversible diapycnal flux (Winters et al. 1995; Winters and D'Asaro 1996) given by

$$\Phi_d = -g \int \kappa_{\text{eff}} \frac{dz^*}{d\rho} |\nabla \rho|^2 dV.$$

Figure 5 shows density profiles at various time instances chosen to show the formation of the large convective overturn in case 1. The solid line represents the one-dimensional instantaneous density profile, which is constructed from the instantaneous, three-dimensional density field. In the first step, each volume element is assigned an index ranging from $i = 1$ to $(N_x N_y N_z)$. Starting with the bottom-left corner of the domain, all the volume elements in the first $x$-$y$ plane (slope-normal index = 1) are assigned indices ranging from 1 to $N_x N_y$ in the order of spanwise and along-slope directions. Then the second $x$-$y$ plane (slope-normal index = 2) is considered, with indices of volume
elements ranging from $N_x N_y + 1$ to $2N_x N_y$ and so on until the last $x-y$ plane, which has indices ranging from $N_x N_y (N_z - 1) + 1$ to $N_x N_y N_z$. Each volume element $dV_i$ is then squashed, leading to a triangular- or parallelogram-shaped, two-dimensional element of area $dA_i = dV_i/l$, and arranged in the vertical direction in the order of their indices, that is, the volume element with index $= 1$ is placed at the bottom followed by the volume element with index $= 2$, and so on. Each element is assigned a height $z^*$, which is the height of the centroid of the squashed two-dimensional element. The dashed line represents the one-dimensional reference state. The procedure employed in constructing the one-dimensional reference state is similar to that of the one-dimensional instantaneous profile, except that the density is sorted such that the stratification is stable everywhere. The densest element is placed at the bottom, and the next dense element is placed on top of it and so on until the lightest of all the elements is placed at the top. A more detailed description about the reference state construction is available in Winters and Barkan (2013).

Figures 6a and 6b show the evolution of APE in the flow. The buoyancy flux and diapycnal flux are also shown to explain the phasing of these energy rate terms relative to the APE. The APE in the DNS (Fig. 6a) starts to increase at phase $\phi = -0.3\pi$ at the same time that the density profile starts deviating from the background. At a slightly later phase $\phi = -0.15\pi$, APE is close to the maximum. The density profile at this phase (Fig. 5c) shows a density overturn that extends from 1 m above bottom to 4 m above bottom, spanning half the bottom flow thickness. APE continues to increase until $\phi = -0.1\pi$, that is, when the large overturn starts to break down into small-scale turbulence. The irreversible diapycnal flux starts to increase at phase $-0.1\pi$, after the APE has reached its maximum value. It is worth noting that diapycnal flux commences to rise at the same phase when the TKE budget plot shows an increase in the turbulent dissipation. Just after the state of zero flow, when $\phi = 0.03\pi$, many small-scale density overturns are found spanning the entire thickness, $l_b = 6$ m, of the bottom flow. The evolution of these overturns and the corresponding Thorpe scales will be discussed in the subsequent sections of this paper. Diapycnal flux, which is a measure of irreversible mixing, continues to increase until $\phi = 0.08\pi$ and then starts to decrease, following a similar evolution as the turbulent dissipation rate shown in the TKE budget (Fig. 3). The buoyancy flux begins to rise at the same time that the diapycnal flux rises and peaks at a similar time. The peak value of the buoyancy flux is significantly larger than that of the diapycnal flux. However, the buoyancy flux decreases rapidly after its peak, plummets to zero, and attains negative values while $\Phi_B$ remains always positive.

The density profiles in case 2 show similar qualitative behavior. Figure 7 shows 1D density profiles at various phases during the LCOE. In this case, the density profile at $\phi = -0.15\pi$ exhibits unstable stratification spanning 20 to 60 m above the bottom. The convective overturn...
breaks down so that, at $\phi = 0.03\pi$, small-scale density fluctuations are present all the way from the bottom up to 70 m above bottom, spanning the entire thickness of the bottom flow as was seen in case 1. In contrast to the lower $Re$ case, the available potential energy and diapycnal flux show some activity in the latter half of the cycle, as shown in Fig. 6b. The lower flank of the upslope flow in this larger-scale problem is able to produce overturns (by bringing heavy fluid from depth to overlay lighter fluid) that extend farther away from the bottom wall restraint and are more effective in mixing the density field.

5. Evolution of Thorpe and Ozmidov scales

The evolution of the Thorpe scale $L_T$ is described and contrasted with that of the Ozmidov scale in section 5a. Simulation data provide the simplification of computing the Thorpe length scale from the mean density profile (obtained by averaging the three-dimensional density field in both the horizontal directions). The mean density profile is a function of slope-normal coordinate $z_s$ and time. In observational studies, the turbulent dissipation rate is customarily inferred from overturns in instantaneous density profiles obtained at moored or towed profilers. Therefore, the simulation data are used to investigate the behavior of Thorpe scales computed from individual density profiles, that is, a virtual mooring with infinite vertical profiling speed. Both mean and instantaneous density profiles lead to the same qualitative result regarding the inability of $L_T$ to serve as a proxy for $L_O$ in the present flow. Thorpe estimates of the turbulent dissipation rate are compared to the actual values in section 5b.

To meet the overturn selection criterion, the density gradient is required to be positive over at least four contiguous grid points. The Thorpe displacement for each volume element $dV_i$ within an overturn is defined as $d_i = z_i - z$, where $z$ and $z_i$ are vertical positions of a volume element before and after reordering the density profile. The Thorpe scale $L_T$ for each overturn is defined

![Fig. 6. Evolution of APE, buoyancy flux, and diapycnal flux ($\Phi_d$). (top) Corresponds to case 1 and (bottom) case 2. The dashed line is the depth-averaged streamwise velocity ($u_i$) (m s$^{-1}$). APE has units of kg m$^2$ s$^{-2}$, whereas the units of $\Phi_d$ and $\int \rho B dV$ are kg m$^2$ s$^{-3}$.](image)

![Fig. 7. As in Fig. 5, but for case 2.](image)
as the rms of the displacements of each volume element within an overturn:

\[ L_T = \sqrt{\frac{\sum d_i^2 dV_i}{\sum dV_i}}. \]  

(18)

The overturn Ozmidov scale is defined as

\[ L_O = \sqrt{\frac{\varepsilon}{N^3}}, \]  

(19)

where \( \varepsilon \) and \( N \) are the turbulent dissipation rate and stratification averaged over each overturn.

**a. Evolution of Thorpe length scales**

Figure 8 shows the distribution of overturns, computed from the mean velocity profile, at four different time instances over the lifetime of a LCOE for case 1. The vertical extent of each bar corresponds to the overturn height, and the horizontal extent corresponds to the Thorpe length scale calculated for that overturn. At phase \( \phi = -0.19\pi \), there is a single overturn extending from 1 m above bottom to 4 m above bottom with a Thorpe length scale of \( L_T = 1.96 \text{ m} \). The sequence from Figs. 8a to 8d shows the cascade to small scales: the number of small overturns increases and the Thorpe length scales decrease. At phase \( \phi = 0.07\pi \), multiple overturns occur over \( z = 0.5 \text{ m} \) to \( z = 7 \text{ m} \) with a maximum Thorpe length scale of only 0.17 m. Figure 9 is an analogous plot for case 2. At phase \( \phi = -0.19\pi \), there is a single, large overturn as in case 1 but larger by an order of magnitude (\( L_T \simeq 20 \text{ m} \)) that subsequently breaks down into multiple smaller overturns. For both cases 1 and 2, the ratio of the Thorpe scale to the
overturn height ranges from 0.5 to 0.7 with a mean value of $0.6$.

Instantaneous density profiles at various virtual moorings have also been examined for the evolution of $L_T$. Similar to the mean density profile, the density profiles at all virtual mooring obtained by sampling the simulation data also show that $L_T$ is large at the beginning of the LCOE and progressively becomes smaller with a broader distribution as the initially large overturns disintegrate.

We now turn to a comparison of the evolution of Thorpe scales with that of the Ozmidov scales. Figure 10a shows the time evolution of Thorpe and Ozmidov length scales in case 1. As the flow transitions from peak downslope to near-zero velocity, a large region of unstable density gradient is created resulting in the increase of the Thorpe scale ($L_T \approx 2 \text{m}$). The Thorpe scale remains nearly constant for a certain fraction of the cycle (about 30 min) and then disintegrates into multiple, small overturns over a finite time (about 90 min). The Ozmidov scale starts to increase during the disintegration of the large overturn. Ozmidov scales decrease a bit around $\phi = 0.2\pi$, as the turbulence caused by convective overturns decays. The term $L_O$ starts to increase at around $0.3\pi$ because of the increased dissipation caused by enhanced near-bottom shear when the peak upslope velocity is large. In contrast, there are no significant overturns at this time, and $L_T$ is close to zero.

Case 2, with larger $l_b$, $U_b$, and Re, shows a similar dependence of the Thorpe and Ozmidov scales on phase in Fig. 10b. Nevertheless, there are two qualitative differences: the range of values taken by $L_O$ show a significantly larger spread in case 2 after the large overturn disintegrates, and both $L_T$ and $L_O$ show substantially more activity in case 2 relative to case 1 during the upslope flow (cf. Fig. 10b to Fig. 10a). The bottom panel of Fig. 10 is the analog of the top panel for an individual virtual mooring profile. Similar to the mean profile, the evolution of the Thorpe scales from an individual mooring profile is different from that of the Ozmidov scales and exhibits a similar phase lag. It is interesting to

![Fig. 9. As in Fig. 8, but for case 2 (LES). Time instances chosen are (a) $-0.19\pi$, (b) $-0.12\pi$, (c) $-0.02\pi$, and (d) $0.06\pi$.](image-url)
note that Thorpe length scales obtained from the simulation data at an individual location exhibit a wider range of values during the buildup of the LCOE and a longer time interval over which the large overturns disintegrate.

The evolution of Ozmidov and Thorpe scales, employing either mean or single location density profiles, shows that they are not linearly related. There is phase lag between their maxima because the Thorpe scale, a measure of overturning length scales, is the largest just prior to breakdown by convective instability, whereas the Ozmidov scale is computed based on the turbulent dissipation that attains its peak value only after there is sufficient time for the cascade to small scales to be established. The instantaneous value of the ratio of Ozmidov to Thorpe scale is found to vary between $O(0)$ and $O(100)$, and it is amply evident that $L_T$ cannot be used as proxy for $L_O$ at the corresponding time in the present flow.

b. Turbulent dissipation rate inferred from Thorpe scales

The turbulent dissipation rate is estimated from the Thorpe scale by assuming that $L_O$ is proportional to $L_T$, and the customary choice of $C_d = L_O/L_T = 0.8$ leads to the following Thorpe dissipation rate estimate:

$$
\varepsilon_T = C_d^2 L_T^2 N^3 = 0.64 L_T^2 N^3.
$$

The depth-averaged value of the actual dissipation rate in the simulation $\langle \varepsilon_S \rangle_z$ is compared with the depth-averaged value of the Thorpe dissipation rate estimate $\langle \varepsilon_T \rangle_z$ at the same time. Here,

$$
\langle \varepsilon_S \rangle_z = \frac{\int_0^{Z_0} \varepsilon \, dz_s}{Z_0}, \quad \text{and}
$$

![Fig. 10. Comparison of the phase evolution of Thorpe and Ozmidov scales. Both quantities are multivalued functions of phase. (top) Results calculated using the mean density profile for the (a) DNS and (b) LES cases. (bottom) Results calculated from the density profile at a single location ($x_s = l_x/2, y = 0$) for (c) DNS and (d) LES cases.](image-url)
Note that the vertical averaging distance is $Z_0 = 8$ m for the DNS case and 80 m for the LES case. Subscript $i$ denotes the $i$th overturn, and $n_{ovt}$ denotes the number of overturns in the profile.

Figures 11a and 11b compare the Thorpe dissipation rate estimated from overturns in the mean density profile with the actual value of turbulent dissipation rate. The phase range shown here is during the LCOE. Initially, when the overturn is large and the stratification is still strong, $\langle \epsilon_T \rangle_z$ is large. At a later time, when the overturn breaks down into turbulence, $\langle \epsilon_T \rangle_z$ decreases, whereas $\langle \epsilon_S \rangle_z$ increases. The term $\langle \epsilon_S \rangle_z$ peaks at around $\phi = 0.1\pi$ and subsequently exhibits a gradual decrease as the turbulence decays. The peak value of $\langle \epsilon_S \rangle_z$ is approximately 6–7 times smaller compared to the peak value of $\langle \epsilon_T \rangle_z$ inferred from the mean density profile in both DNS and LES cases.

Various locations were considered to calculate the Thorpe dissipation rate from individual density profiles. The variability among the dissipation rate estimates at different locations is small, and only two locations ($x_s = l_x/2, y = 0$ and $x_s = l_y/2, y = l_y/2$) are shown as examples. Figures 11c and 11d compare $\langle \epsilon_T \rangle_z$ at these two locations with the actual value for DNS and LES cases, respectively. The phase lag between peak values of $\langle \epsilon_T \rangle_z$ and $\langle \epsilon_S \rangle_z$ that was found when using the mean density profile remains when individual density profiles are used. It is worth noting that $\langle \epsilon_T \rangle_z$ inferred from the mean density profile has its peak value 3–4 times larger compared with $\langle \epsilon_T \rangle_z$ inferred at virtual mooring locations. Tracking each overturn obtained from both the mean density profile and virtual mooring profiles suggests that the peak value of $L_T$ is slightly larger, and the value of $N$ is noticeably larger for overturns obtained from the mean density profile, resulting in higher peak Thorpe scale dissipation ($\propto L_T^2 N^3$) inferred from the mean density profile.
Table 2. Key results for case 1 (DNS) having bottom flow with peak velocity $U_b = 0.0125$ m s$^{-1}$ and width $l_b = 6$ m. First row corresponds to results obtained by averaging over the LCOE that spans the phase range $(-0.3\pi, 0.4\pi)$, and the second row corresponds to averaging over the entire cycle $(-0.5\pi, 1.5\pi)$. The term $C_\phi$ is the dissipation coefficient defined in (23), $\langle \eta \rangle_\phi$ is the phase-averaged mixing efficiency, and $\langle K_{v\phi} \rangle_\phi$ is the phase-averaged diapycnal diffusivity.

<table>
<thead>
<tr>
<th>Averaging period ($\phi$)</th>
<th>$C_\phi$ (mean)</th>
<th>$C_\phi$ (mooring)</th>
<th>$\langle \eta \rangle_\phi$</th>
<th>$\langle K_{v\phi} \rangle_\phi$ m$^2$s$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-0.3\pi, 0.4\pi)$</td>
<td>0.5</td>
<td>1.17</td>
<td>0.55</td>
<td>$9.3 \times 10^{-5}$</td>
</tr>
<tr>
<td>$(-0.5\pi, 1.5\pi)$</td>
<td>0.7</td>
<td>1.78</td>
<td>0.4</td>
<td>$3.7 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

The phase evolution of $\langle e_T \rangle_z$ and $\langle e_S \rangle_z$ show that Thorpe length scales substantially overestimate the dissipation during the initial stages of LCOE and substantially underestimate the dissipation during the later stages of LCOE. In fact, there are time periods when one of the two dissipation values is near zero, while the other is substantial. The cycle-averaged values of the dissipation rate have also been compared, and, upon cycle averaging, the actual turbulent dissipation rate is found to be comparable to the value inferred from Thorpe scales for both the DNS and LES cases. In the DNS, the cycle-averaged value of the actual dissipation rate is $\langle e_T \rangle_z = 1.42 \times 10^{-10}$ m$^2$ s$^{-3}$, while the corresponding values of Thorpe dissipation rate are $\langle e_T \rangle_z = 1.83 \times 10^{-10}$ m$^2$ s$^{-3}$ (from mean density profile) and $\langle e_T \rangle_z = 2.85 \times 10^{-11}$ m$^2$ s$^{-3}$ (from density profile at a location). The cycle-averaged values in the LES are $\langle e_T \rangle_z = 1.36 \times 10^{-8}$ m$^2$ s$^{-3}$, $\langle e_T \rangle_z = 6.54 \times 10^{-8}$ m$^2$ s$^{-3}$ (from mean density profile), and $\langle e_T \rangle_z = 1.10 \times 10^{-8}$ m$^2$ s$^{-3}$ (from density profile at a location).

Recall that a modeling coefficient $C_d$ appears in the Thorpe dissipation rate estimate ([20]). From the preceding results, it is clear that $C_d$ at a given time is generally not an $O(1)$ coefficient. The simulation data are examined to assess if $O(1)$ values of $C_d$ may apply in an “average” sense over parts of the cycle. To do so, we define

$$C_\phi = \sqrt{\frac{\langle e_T \rangle_z}{\langle e_T \rangle_{z\phi}}}$$

(23)

Here, $\langle e_T \rangle_{z\phi}$ represents depth averaging followed by phase averaging. The value of the dissipation coefficient $C_\phi$, which indirectly represents an average value of $L_O$ to $L_T$ during the flow evolution, depends on the phase range over which averaging is done and may be less than or greater than unity.

Values of $C_\phi$ for two choices of the length of phase averaging ($\phi$ varying from $-\pi/2$ to $3\pi/2$, spanning the entire wave cycle and a shorter interval spanning the LCOE) are shown in Tables 2 and 3. The term $C_p$, computed as an average over the entire cycle, has values of 0.7 and 1.78 for the mean density profile and virtual mooring profile, respectively, in the DNS case; corresponding values for the LES case are 0.36 and 0.88, respectively. The $O(1)$ value of the phase-averaged estimate of the dissipation coefficient $C_\phi$ found in this study is not a mere coincidence. It can be explained on the basis of available potential energy as given below. As was described earlier in section 4 of this paper, available potential energy is the primary source of eventual turbulence and dissipation during the convective instability and its subsequent nonlinear evolution. The available potential energy of an overturn is

$$APE \approx \Delta \rho g L_T \approx g \frac{d\rho}{dz} L_T^2 \approx N^2 L_T^2,$$

(24)

where the density difference over an overturn $\Delta \rho$ is taken to be proportional to the product of the Thorpe displacement and the sorted density gradient of that overturn, and $N$ corresponds to the average background stratification (after sorting) of that overturn. The characteristic time scale of the overturn is given by

$$\sqrt{\frac{L_T}{g' \rho} = \frac{1}{N}},$$

(25)

where $g' = (\Delta \rho/\rho_0)g$ is the reduced gravity. Thus, energy is released at the rate of APE times $N$. Over the duration of the convective event, a substantial fraction of the

Table 3. Key results for case 2 (LES) having bottom flow with peak velocity $U_b = 0.125$ m s$^{-1}$ and width $l_b = 60$ m. Quantities defined as in Table 2.

<table>
<thead>
<tr>
<th>Averaging period ($\phi$)</th>
<th>$C_\phi$ (mean)</th>
<th>$C_\phi$ (mooring)</th>
<th>$\langle \eta \rangle_\phi$</th>
<th>$\langle K_{v\phi} \rangle_\phi$ m$^2$s$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-0.3\pi, 0.4\pi)$</td>
<td>0.27</td>
<td>0.66</td>
<td>0.39</td>
<td>$2.1 \times 10^{-3}$</td>
</tr>
<tr>
<td>$(-0.5\pi, 1.5\pi)$</td>
<td>0.36</td>
<td>0.88</td>
<td>0.31</td>
<td>$2.0 \times 10^{-3}$</td>
</tr>
</tbody>
</table>
APE cascades to turbulence to be eventually dissipated by molecular effects so that over the entire cycle,

\[ h_{\text{APE}} \sim \langle N \rangle \sim \langle \frac{L^2 N^3}{\varepsilon} \rangle. \]  

Thus,

\[ C_\phi = \frac{\langle \varepsilon_S \rangle}{\langle \frac{L^2 N^3}{\varepsilon} \rangle} = O(1). \]  

### 6. Mixing efficiency

The instantaneous and phase-averaged mixing efficiencies are as follows:

\[ \eta(t) = \frac{\Phi_d}{\Phi_d + \int_0^{Z_0} \varepsilon_V d z_s}, \quad \text{and} \]

\[ \langle \eta \rangle_\phi = \frac{\int_0^{Z_0} \Phi_d^2 dt}{\int_0^{Z_0} \left( \Phi_d + \int_0^{Z_0} \varepsilon_V d z_s \right) dt}. \]

where \( \Phi_d \) is the irreversible diapycnal flux integrated over the volume defined by \( V = l_x l_y Z_0 \) and \( \varepsilon_V = \rho_0 l_x l_y \varepsilon \). For case 1, \( Z_0 = 8 \text{ m} \) and for case 2, \( Z_0 = 80 \text{ m} \). The term \( Z_0 \) is the height up to which turbulent fluctuations are observed during a tidal cycle. Figure 12 shows the time evolution of mixing efficiency for both cases 1 and 2.

Mixing efficiency starts with a small value at peak downslope flow. For case 1, mixing efficiency increases with time during the first quarter of the cycle and reaches its peak slightly before the irreversible diapycnal flux has its maximum value. At a later time, mixing efficiency drops gradually to near-zero value. It starts to increase again at \( \phi = 0.7\pi \) until \( \phi \approx 1.3\pi \). The TKE budget during the second half of the cycle exhibits little turbulent dissipation or diapycnal flux, the mixing efficiency during this part of the cycle increases because \( \varepsilon \) becomes smaller than \( \Phi_d \). Mixing efficiency averaged over various fractions of a tidal cycle is shown in Tables 2 and 3. Mixing efficiency averaged over the LCOE ranges between 0.4 and 0.5, which is substantially higher than the commonly used value of 0.2 in both DNS and LES cases. This agrees well with the mixing efficiency computed in previous studies (Dalziel et al. 2008; Gayen et al. 2013) of turbulence driven by convective instability. The cycle-averaged mixing efficiency ranges between 0.3 and 0.4 in both the cases; this value is also high compared to the value of 0.2 used to estimate diapycnal diffusivity from the turbulent dissipation rate.

### 7. Diapycnal diffusivity

Quantification of mixing, that is, estimation of diapycnal diffusivity, needs a good understanding of the details of the turbulent event. The standard method of estimating the diffusivity from the turbulent dissipation rate (usually an inferred value) is \( K_\rho = 0.2 \varepsilon / N^2 \). This model had been derived making two important assumptions: 1) the mixing efficiency is 0.2 and constant throughout the turbulent event, and 2) turbulence is a quasi–steady state process so that the turbulent dissipation is balanced by the sum of turbulent production and buoyancy flux. Neither of these assumptions is correct in the present flow. Analysis of the TKE and scalar variance budgets in the previous sections of this paper reveal that the tendency term can be significant compared to the other terms during the tidal cycle. Also, assuming a constant value of 0.2 for mixing efficiency during the LCOE is a substantial underestimate.

We calculate the diapycnal diffusivity following Barry et al. (2001) who derived a formula to calculate the diapycnal diffusivity based on diapycnal flux \( \Phi_d \) and the density gradient averaged over the volume. Phase-averaged diapycnal diffusivity is given by

\[ \langle K_\rho \rangle_\phi = \frac{\langle \Phi_d \rangle_\phi}{V g (dp/dz)_{\text{average}}}. \]

Here, \( V \) is the volume of the fluid, and \( (dp/dz)_{\text{average}} \) is the density gradient averaged over volume and phase.
The term \((K_\rho)_b\) calculated over various fractions of a cycle is shown in Tables 2 and 3. The diffusivity averaged over the large convective overturning event is high in both cases 1 and 2 when compared with the overall cycle-averaged value. The diffusivity averaged over the entire wave period is two orders of magnitude higher in case 2 when compared with case 1, consistent with an inertial scaling of \(\sim U/l_b\). It is also worth noting that the diffusivity in the high Re case is \(2 \times 10^{-3} \text{ m}^2 \text{ s}^{-1}\), which is almost four orders of magnitude larger than the molecular diffusivity.

8. Summary and conclusions

Mixing during near-bottom convective overturns driven by internal tides is investigated with a detailed analysis of scalar mixing and turbulent dissipation rate in an oscillatory, baroclinic boundary flow. During downslope flow, light fluid from above replaces heavier fluid resulting in the formation of an unstable density gradient detached from the viscous boundary layer. The available potential energy (APE) continues to increase until the large overturn breaks up into multiple small overturns via three-dimensional convective instability during flow reversal from down to upslope. The overturn height in this large convective overturn event (LCOE) scales with the thickness \(l_b\) of the boundary flow. In the case with a small \(l_b = 6 \text{ m}\) that is amenable to DNS, the tallest overturn has a height of 3 m with the corresponding Thorpe length scale close to 2 m. For the case with \(l_b = 60 \text{ m}\) computed with LES, the tallest overturn has a height of 35 m with the corresponding Thorpe scale of 20 m. There is substantial energy transfer from APE to the turbulent kinetic energy (TKE) through the fluctuating buoyancy flux, and the shear production of turbulence is small during the LCOE.

It is found that there is a substantial phase lag between Thorpe and Ozmidov length scales during the evolution of the convective overturning event. Thorpe scales are at maximum during the initial stages of the convective instability, but the Ozmidov scale is small. At a later time, when three-dimensional turbulence and dissipation become prominent leading to an increase of the Ozmidov scale, the overturns are smaller and are spread over the most of the water column involved in the boundary flow. In brief, when \(L_T\) decreases, \(L_O\) increases. The simulations show that Ozmidov \(L_O\) and Thorpe length scales \(L_T\) are different functions of the tidal phase and \(L_T\) cannot serve as a proxy for \(L_O\) at that time. Our result is consistent with Smyth and Moun (2000), who, in the case of shear-driven stratified turbulence, found that \(R_{OT} = L_O/L_T\) varies from \(O(0.1)\) during the initial large overturn stage to \(O(1)\) during the later stage of well-developed broadband turbulence. The present case of convectively driven turbulence shows one to two orders of magnitude increase in \(R_{OT}\) during the course of LCOE in both DNS and LES cases.

Turbulent dissipation rate estimated from Thorpe scales using mean density profile and individual density profiles (virtual mooring) is compared with the actual dissipation rate computed from the simulation data. It is found that Thorpe length scales overestimate the dissipation by more than an order of magnitude during the initial stages of the overturning event and underestimates the dissipation, again by more than an order of magnitude, during the later stages. It is not possible to obtain a reliable estimate of turbulent dissipation rate at a given time by measuring overturns at that time. However, their cycle-averaged values are comparable. This is manifested in the form of the cycle-averaged dissipation coefficient \(C_\eta\) (usually taken to be 0.8) being \(O(1)\) in both DNS and LES cases. This \(O(1)\) value of cycle-averaged \(C_\eta\) is explained based on available potential energy of the large convective overturns cascading to turbulence and molecular dissipation over a cycle. Thus, it may be possible to average the turbulent dissipation obtained by Thorpe analysis of an ensemble of density profiles during an entire wave period to approximate the cycle-averaged dissipation rate. Similarly, Thorpe analysis over the entire lifetime of a LCOE could give the average dissipation associated with the LCOE. A single value of buoyancy frequency \(N \sim 10^{-3} \text{ rad s}^{-1}\), representative of deep-ocean stratification, is considered in this study. Varying the buoyancy frequency may have an effect on the formation and detailed characteristics of convective instabilities. However, once a convective instability is formed, the fundamental argument made in this paper that the Thorpe scale \(L_T\) is not linearly related to the Ozmidov scale \(L_O\) at the same time is expected to be valid, independent of the precise value of \(N\).

The instantaneous mixing efficiency \(\eta\), during the convective mixing event has its peak value close to 0.7 in DNS and 0.6 in LES. The cycle-averaged mixing efficiency of 0.3–0.4 is also substantially larger than the commonly assumed value of 0.2. High mixing efficiencies close to 0.5 have been reported (Dalziel et al. 2008; Lawrie and Dalziel 2011; Gayen et al. 2013) in situations where turbulence is driven by convective instability. A recent experimental study (Davies Wykes and Dalziel 2014) reported that mixing efficiency can be as high as 0.75 when the unstable density region is confined within a stable stratification from above and below, a configuration that occurs here (Figs. 5c, 7c) during the flow reversal mixing phase.
Thus, it is important that the mechanism leading to turbulence (e.g., shear instability or convective instability) needs to be established before choosing a value for $\eta$ to infer mixing rates. The volume-averaged diffusivity varies by two orders of magnitude between DNS and LES cases, following an inertial scaling of turbulent diffusivity: $K_\eta \propto U_{\text{rms}}^2$. The tidally modulated enhanced diffusivity of $O(10^{-3}) \text{m}^2\text{s}^{-1}$ that we find here for the high Re case is consistent with enhanced values of bottom diffusivity found near rough topography.

It is encouraging that DNS of this problem of tidally modulated, bottom-intensified boundary flow that occurs at near-critical slopes and LES of a scaled up version give consistent results regarding the mixing efficiency and Thorpe/Ozmidov scales. Nevertheless, other scenarios of tidally driven turbulence near topography, for example, breaking lee waves and down-slope jets, remain to be explored to examine the generality of the results obtained here.

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