A New Mechanism for Mode Water Formation involving Cabbeling and Frontogenetic Strain at Thermohaline Fronts

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ABSTRACT

A simple analytical model is used to elucidate a potential mechanism for steady-state mode water formation at a thermohaline front that involves frontogenesis, submesoscale lateral mixing, and cabbeling. This mechanism is motivated in part by recent observations of an extremely sharp, density-compensated front at the North Wall of the Gulf Stream. Here, the intergyre, along-isopycnal, salinity–temperature difference is compressed into a span of a few kilometers, making the flow susceptible to cabbeling. The sharpness of the front is caused by frontogenetic strain, which is presumably balanced by submesoscale lateral mixing processes. The balance is studied with the simple model, and a scaling is derived for the amount of water mass transformation resulting from the ensuing cabbeling. The transformation scales with the strain rate, equilibrated width of the front, and the square of the isopycnal temperature contrast across the front. At the major ocean fronts where mode waters are found, this isopycnal temperature contrast decreases with increasing density near the isopycnal layers where mode waters reside. This implies that cabbeling should result in a convergent diapycnal mass flux into mode water density classes. The scaling for the transformation suggests that at these fronts the process could generate 0.01–1 Sverdrups (Sv; 1 Sv = 10^6 m^3 s^-1) of mode water. These formation rates, while smaller than mode water formation by air–sea fluxes, should be independent of season and thus could fill select isopycnal layers continuously and play an important role in the dynamics of mode waters on interannual time scales.

1. Introduction

In every ocean basin, on the equatorward side of major ocean fronts, layers of weakly stratified waters with nearly homogeneous properties are found. These so-called mode waters are thought to play an important role in the ocean–atmosphere climate system by sequestering and releasing heat and carbon dioxide on interannual time scales (Dong et al. 2007; Bates et al. 2002) and by affecting the large-scale circulation through shaping the potential vorticity field (Dewar et al. 2005).

Mode waters are identified as a local maximum in a volumetric census in a temperature–salinity diagram (Hanawa and Talley 2001). They are distinguished from other water masses by their defining characteristic—weak stratification and anomalously low potential vorticity. Mode water formation requires a convergent diapycnal mass flux that can fill isopycnal layers and reduce the stratification. The mechanism that selects the density class where a diapycnal mass flux converges and mode water forms is not well understood. Nor is it fully known how fronts are involved in the process. The classical explanation involves air–sea fluxes (e.g., Speer and Tziperman 1992). In this paradigm, mode waters form on isopycnal layers that outcrop in regions where the net buoyancy loss to the atmosphere decreases with increasing density. Fronts and their associated currents play a role in this process by shaping the outcrop locations and by bringing relatively warm waters in contact with cold air during the winter. The whole process relies critically on the geometry of the isopycnal outcrop areas. These areas are, to a large degree, determined by the distribution of the mode waters

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themselves, as their weak stratification preconditions the ocean for the convective mixing that lifts isopycnals to the surface. Thus, the classical formation mechanism has a causality dilemma: air–sea fluxes generate mode water where outcrop areas are expansive because of the underlying pool of mode water they renew.

In this article, we consider a mode water formation mechanism that does not present such a dilemma and that has a clear connection to fronts. It involves cabbeling, a process where water masses with different temperatures and salinities yet the same density becomes denser when mixed because of nonlinearities in the equation of state for seawater (Foster 1972; McDougall 1987). Cabbeling is most effective for large water mass contrasts; hence, it is most pronounced at fronts. Cabbeling has been invoked to explain the formation of intermediate waters in the North Pacific (Talley and Yun 2001) and deep and bottom waters in the Southern Ocean (Foster 1972; Klocker and McDougall 2010). Recent mesoscale-resolving simulations of the Southern Ocean suggest that the process could contribute significantly to the transformation of water masses throughout the water column, including 4 Sverdrups (Sv; 1 Sv = 10^6 m^3 s^-1) of surface/thermocline waters to Subantarctic Mode Water (Urakawa and Hasumi 2012). Urakawa and Hasumi noted that the water mass transformation ascribable to cabbeling in their simulations was strongly influenced by numerical diffusion. This implies that processes that cascade temperature and salinity variance from the mesoscale to the grid scale of their model are essential to the mechanism. One process of particular relevance is frontogenesis, that is, the intensification of horizontal tracer gradients by a strain field. In the next section, we describe observations that suggest that frontogenesis could play an important role in facilitating cabbeling at observed fronts and motivate a theoretical study. While theories for strain-driven frontogenesis and the influence of cabbeling on frontal dynamics have been investigated in isolation (e.g., Hoskins and Bretherton 1972; Garrett and Horne 1978; Bowman and Okubo 1978), the combined effects of the two processes, and their role in mode water mass formation, have not been explored and will be the focus of this article.

2. Motivating observations

Hydrographic measurements from the Kuroshio, Gulf Stream, and Subantarctic Front suggest that cabbeling might play an important role in the formation of their associated mode waters (Fig. 1, left panel). These data are from WOCE line PR03 in the North Pacific, the CLIVAR Mode Water Dynamics Experiment (CLIMODE) experiment near the Gulf Stream (Marshall et al. 2009), and the OISO-01 cruise at the Subantarctic Front to the northeast of Kerguelen Island (Lo Monaco and Metzl 2007). A common feature to all of the fronts is that their temperature and salinity difference decreases with increasing density for isopycnal surfaces surrounding the mode water layer. Here, the temperature–salinity (θ–S) relation from profiles straddling the fronts takes a

![Fig. 1. (left) θ–S diagrams from CTD casts straddling the Kuroshio (magenta), Gulf Stream (red), and Subantarctic (blue) Fronts. The isopycnal layers where the North Pacific Subtropical Mode Water $\sigma_{\text{NSPMW}}$ (dashed magenta), Eighteen Degree Water $\sigma_{\text{EDW}}$ (dashed red), and Subantarctic Mode Water $\sigma_{\text{SAMW}}$ (dashed blue) reside are indicated. Around these layers, the temperature and salinity contrast across the fronts decreases with density. As schematized to the right, mixing of these water masses and cabbeling would result in a convergent diapycnal mass flux that would increase the volume of mode water.](image-url)
Wishbone-like shape. Consequently, in this part of $\theta$–$S$ space, isopycnal mixing would drive a convergent diapycnal mass flux via cabbeling that would fill isopycnal layers (Fig. 1, right panel). Since this is where mode waters reside, we posit that this mechanism contributes to selecting their density class.

In physical space, the distance over which the $\theta$–$S$ relation traces a wishbone shape across a thermohaline front can be quite narrow. A prime example of this is the front at the North Wall of the Gulf Stream that marks the boundary between the subtropical and subpolar gyres. As revealed by recent observations taken as part of the Lateral Mixing Experiment (e.g., Shcherbina et al. 2013), this boundary is extremely sharp and manifests itself as a compensated salinity front where the intergyre salinity difference is compressed into a span of a few kilometers (Fig. 2).

These hydrographic measurements were made with a Moving Vessel Profiler (MVP) profiled from the R/V Atlantis. The MVP (Rolls Royce MVP 200) is a weighted conductivity–temperature–depth sensor (CTD) that free-falls at approximately 3 m s$^{-1}$ and is returned to the surface by a winch. Casts to 200 m are recorded approximately every 800 m as the ship steams at 8 knots (kt; 1 kt = 0.51 m s$^{-1}$), and only downcasts are used. The casts were made with nominally 1-km resolution in the horizontal and less than 2 m in the vertical. Velocity measurements were made with both 300- and 75-kHz underway acoustic Doppler current profilers (ADCPs). Vertical sampling of the ADCPs spanned the range between 15 and 87 m with 4-m bin size for the 300-kHz ADCP and between 21.5 and 570 m with 8-m bin size for 75-kHz ADCP. The 1-min ensemble averages were used, producing along-track resolution for the velocity of about 0.2 km. The sections

Fig. 2. The setting where cabbeling likely takes place in the Gulf Stream. (top left) Sea surface height, (top right) sea surface temperature, (bottom left) salinity and density cross section, and (bottom right) vertical vorticity and density cross section. The North Wall of the Gulf Stream is a region where the speed of the current drops off precipitously, resulting in a flow that tends to be confluent (e.g., sea surface height) and that is characterized by strong, cyclonic vorticity. The resultant collision of water masses generates a strong front in temperature. Below the surface boundary layer (i.e., $z < -100$ m), an extremely sharp, compensated salinity front is found that coincides with a strip of intense, cyclonic vorticity, as shown in sections of salinity, potential density (contours), and vertical vorticity (normalized by $f$). The observations are from a survey conducted 5–9 Mar 2012 (ship track shown in gray, top-right panel) with a horizontal resolution of $\sim$1.25 km.
transecting the Gulf Stream were transformed into a streamwise coordinate system, where the downstream direction (with velocity component $u_d$) is defined as the speed-weighted average direction of the current on the section. The cross-stream coordinate $y_c$ is defined to be perpendicular to the downstream direction and increases from the warm to cold side of the front. Given the velocity field, the vertical vorticity was estimated as $-\frac{\partial u_d}{\partial y_c}$ using central differences and assuming that along-stream variations in the cross-stream velocity are negligible.

The compensated salinity front in Fig. 2 coincides with a sheet of cyclonic vorticity caused by the Gulf Stream’s precipitous drop in speed. The cyclonic vorticity exceeds the Coriolis parameter Stream’s precipitous drop in speed. The cyclonic vorticity with a sheet of cyclonic vorticity caused by the Gulf Stream due to the action of the gyres and/or mesoscale variations in the cross-stream velocity are negligible.

The compensated salinity front in Fig. 2 coincides with a sheet of cyclonic vorticity caused by the Gulf Stream. Other surveys of the North Wall from the western side of the Gulf Stream extension region (not shown) reveal that the thermohaline front and vortex sheet are ubiquitous features. Here, the geostrophic flow tends to be confluent, providing a strain field that is frontogenetic (Fig. 2). Since the front is of finite width and steady in some average sense, there must be processes that mix salt (and temperature) isopycnally to balance the frontogenetic strain. Such mixing would drive cabbeling and is likely associated with submesoscale processes. The link between the frontogenetic strain and mixing implies that cabbeling-induced diapycnal mass fluxes at fronts are to some degree determined by the strain field. In the next section, we explore these physics using a simple theoretical model of a strained thermohaline front configured with parameters representative of the observations. The calculation suggests the mechanism can result in a persistent injection of fluid into mode water layers.

3. Simple model for water mass transformation at thermohaline fronts

a. Formulation

As described in previous sections, thermohaline fronts are sites of large temperature and salinity contrast, often subject to significant large-scale confluence and exhibiting strong submesoscale activity. The large temperature contrasts imply that nonlinearities in the equation of state for seawater, specifically the curvature of density with respect to temperature—which is primarily responsible for cabbeling—may be important. The presence of large-scale confluence also provides the possibility of sharpening of the temperature and salinity gradients or frontogenesis. Ultimately, frontogenesis can be arrested by mixing associated with submesoscale processes, leading to a steady state exhibiting continuous water mass transformation. Here, we formulate the simplest possible model of this system as a first step toward exploring the physics of the interplay between cabbeling and frontogenesis.

We will examine the special case of a fully compensated temperature–salinity (TS) front; that is, the changes in temperature and salinity across the front are compensated such that the far-field density distribution is the same on both sides. The advantage of this approach is that variations in density at the front can only occur because of the nonlinearities in the equation of state for seawater, thus isolating the dynamics of interest. Furthermore, these density variations will be small compared to the background stratification, meaning the isopycnals remain nearly horizontal. Thus, small-scale, isopycnal mixing at the front can be sensibly parameterized by a purely lateral diffusivity $k_y$. These simplifications are consistent with prior investigations of cabbeling by Garrett and Horne (1978), the results of which are discussed below. Here, we will utilize the classical, two-dimensional frontal configuration common to both the cabbeling (e.g., Garrett and Horne 1978; Bowman and Okubo 1978) and frontogenesis (e.g., Hoskins and Bretherton 1972) literature, where the front is assumed to be infinitely long and which we choose to be oriented along the $y$ axis. Following Hoskins and Bretherton (1972), an explicit background deformation field with horizontal velocity $(\mathbf{U}, \mathbf{V}) = (-\alpha x, ay)$ will be introduced, where the strain $\alpha$ is a constant. This strain field parameterizes the confluence occurring across TS fronts such as the Gulf Stream due to the action of the gyres and/or mesoscale eddies.

With these approximations we have all the necessary ingredients to formulate an idealized model of water mass transformation via cabbeling at thermohaline fronts. The basic configuration of the model is shown in Fig. 3. The model domain is unbounded in the cross-front direction, $-\infty < x < \infty$, and semi-infinite in the vertical, $-\infty < z \leq 0$, with a single rigid lid at the ocean surface ($z = 0$). The water masses on either side have different temperature and salinity but equal density. In particular, the temperature contrast across the front has some vertical profile $\Delta T(\bar{z})$ that usually varies from a maximum $\Delta T_0$ near the surface to zero over some characteristic height scale $H$. This generates two adjoining water masses that are distinct in the near surface region but become increasingly uniform with depth until they are indistinguishable for $z \ll -H$. This is meant to represent, in an idealized fashion, the wishbone-shaped $\theta$–$S$ relation seen in the observations (Fig. 1). Since the model incorporates both horizontal mixing and confluence, we expect a steady state to exist at the front (denoted the mixing...
region” of width $L$ in Fig. 3) where a balance is achieved between these two effects. Our objective here is to determine the steady state and compute the amount of water mass transformation occurring in this state.

We will use $(u, v, w)$ to denote the velocity components in the $(x, y, z)$ directions; respectively, $P$ as the pressure, $\rho$ as the density, $T$ as the temperature, $S$ as the salinity, and $f$ as the (constant) Coriolis parameter. The variables are decomposed into background and perturbation components:

\[
\begin{align*}
    u &= \overline{U} + u'(x, z, t), & v &= \overline{V} + v'(x, z, t), \\
    w &= w'(x, z, t), & P &= \overline{P} + p'(x, z, t), \\
    T &= T'(x, z, t), & S &= \overline{S} + S'(x, z, t), \\
    \rho &= \overline{\rho} + \rho'(x, z, t),
\end{align*}
\]

(1)

where $\overline{\cdot}$ denotes a background field. The mean pressure is $P_0 \rho_0 = -\alpha^2(x^2 + y^2)/2 + f\alpha xy + \overline{P}(z)$ ($\rho_0$ is a reference density and $\overline{P} = (g/\rho_0)\int \overline{P} d\zeta$; $g$ is the acceleration due to gravity) and ensures a force balance for the background strain flow ($\overline{U}$, $\overline{V}$) involving the advection of momentum and the Coriolis and pressure gradient forces (e.g., Shakespeare and Taylor 2013). The perturbation fields are explicitly assumed to be independent of the alongfront coordinate $y$. Upon substitution of (1), the incompressible, hydrostatic, Boussinesq equations (assuming $H/L \ll 1$ and a Prandtl number of 1) become

\[
\begin{align*}
    \frac{D u'}{Dt} &= f v' + \alpha u' - \frac{1}{\rho_0} \frac{\partial \rho'}{\partial x} + \kappa_h \frac{\partial^2 u'}{\partial x^2}, \\
    \frac{D v'}{Dt} &= -f u' - \alpha v' + \kappa_h \frac{\partial^2 v'}{\partial x^2}, \\
    0 &= -\frac{\partial \rho'}{\partial x} - \frac{1}{\rho_0} \frac{\partial \rho'}{\partial z}, \\
    0 &= \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}, \\
    \frac{DT}{Dt} &= \kappa_h \frac{\partial^2 T}{\partial x^2}, \quad \text{and} \quad \\
    \frac{DS}{Dt} &= \kappa_h \frac{\partial^2 S}{\partial x^2},
\end{align*}
\]

(2)

with the material derivative defined by

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + (u' + \overline{U}) \frac{\partial}{\partial x} + w' \frac{\partial}{\partial z}.
\]

(3)

As noted earlier, the horizontal diffusivity $\kappa_h$ in (2) represents a simple parameterization for three-dimensional small-scale mixing processes that could be associated with shear dispersion (Young et al. 1982), intrusion formation (Joyce 1977), and/or submesoscale instabilities.

In addition to (2), we also require an equation of state to connect the dynamics to the thermodynamics. For the temperatures and salinities of the thermohaline fronts of interest (see previous section), the seawater equation of state may, following Garrett and Horne (1978), be approximated as

\[
\rho(T, S) = \rho_0 \left[ 1 + \gamma_T (T_0 - T) + \gamma_S (S - S_0) + \frac{c}{2} (T - T_0)^2 \right].
\]

(4)
to the first order for thermal expansion coefficient $\gamma_T$, saline contraction coefficient $\gamma_S$, and "cabling parameter" $c = \partial_T\beta/\rho_0 < 0$. The quadratic temperature term in (4) is the vital nonlinear term and is responsible for cabling; Garrett and Horne (1978) show via a full seawater equation of state are of secondary importance. Taking the material derivative of (4) and substituting (2e) and (2f), it is readily shown that the density evolves according to

$$\frac{D\rho}{Dt} = \kappa \frac{\partial^2 \rho}{\partial x^2} - \rho_0 c \kappa \left( \frac{\partial T}{\partial x} \right)^2. \quad (5)$$

The second term on the right-hand side of (5) describes the densification of water associated with cabling (since $c < 0$) and is proportional to the diffusivity $\kappa$; that is, the amount of densification is controlled by the strength of the irreversible mixing.

The equations introduced above [(2), (4), and (5)] provide a fully nonlinear, time-evolving model for cabling at a two-dimensional, strained thermohaline front with lateral Laplacian mixing. Given the system of interest, substantial simplifications are possible:

1) Here, we will only consider weak confluence, $\alpha \sim 0.1f$ or smaller, consistent with the strain field associated with gyres and mesoscale eddies. Following Hoskins and Bretherton (1972), in this limit $u' \sim \alpha u/f$, and thus (2a) reduces to the geostrophic balance (see the appendix for the detailed scaling analysis).

2) We seek a steady state for the system $\partial_t \rightarrow 0$, characterized by a balance between confluence and lateral diffusion.

3) We consider a uniformly stratified background state $\bar{\rho} = \rho_0[1 - N^2/(g\bar{z})]$. The stratification is assumed to be associated with the temperature field, that is, $\bar{T} = T_0 + N^2/(\gamma_Tg)z$. (This latter assumption is not dynamically significant but is necessary in order to plot the temperature field in the solutions.)

4) As noted earlier, we consider only fully compensated fronts. The perturbation flow is therefore driven entirely by the small density gradients $\partial \rho'/\partial x$ associated with cabling and is very much weaker than the background strain flow. Nonlinear terms describing advection of perturbation fields by the perturbation flow may thus be neglected: $u'\partial_t - w'\partial_z \ll \bar{U}\partial_t$. The neglect of these terms is valid (see the appendix) when the Froude number defined by $Fr = g\Delta \rho_{cab}/(\rho_0 N^2 H)$ is small (i.e., $Fr \ll 1$), where $\Delta \rho_{cab}$ is the density difference associated with cabling [defined below; see (13)].

As shown in the appendix, under the above four assumptions, a complete set of steady-state equations for the system is

$$\kappa \frac{\partial^2 \rho'}{\partial x^2} + \alpha \frac{\partial \rho'}{\partial x} - \frac{\rho_0 N^2}{g} \frac{\partial \psi}{\partial x} - \kappa \rho_0 c \left( \frac{\partial T'}{\partial x} \right)^2 = 0, \quad (6a)$$

$$\kappa \frac{\partial^2 T'}{\partial x^2} + \alpha \frac{\partial T'}{\partial x} = 0, \quad (6b)$$

$$\kappa \frac{\partial^2 u'}{\partial x^2} + \alpha \frac{\partial u'}{\partial x} - \alpha u' - f \frac{\partial \psi}{\partial z} = 0, \quad (6c)$$

$$\frac{\partial u'}{\partial z} + \frac{g}{\rho_0} \frac{\partial \rho'}{\partial x} = 0, \quad (6d)$$

where we have introduced a streamfunction $\psi$ for the secondary circulation that is defined in the usual way, that is, $u' = \partial_\psi \psi$ and $w' = -\partial_z \psi$. A major result of the assumptions made above is that the perturbation temperature $T'$ defined by (6b) is independent of the other fields. Solving (6b), the perturbation temperature is

$$T' = \frac{\Delta T(z)}{2} \left( 1 + \text{erf} \frac{x}{L} \right), \quad (7)$$

for a specified vertical profile of cross-frontal temperature difference $\Delta T(z)$ (i.e., derived from observations). The horizontal scale of the front is $L = 2\kappa \sqrt{\alpha}$. Since $T'$ is known, the nonlinear cabling term in the density [(6a)] is known, and (6a), (6c), and (6d) form a linear system. We now form an equation for the perturbation density by combining (6a), (6c), and (6d) to eliminate $u'$ and $\psi$, yielding

$$\left[ \frac{N^2}{f^2} \frac{\partial^2}{\partial x^2} \left( \kappa \frac{\partial^2}{\partial x^2} + \alpha \frac{\partial}{\partial x} - 2\alpha \right) + \frac{\partial^2}{\partial z^2} \left( \kappa \frac{\partial^2}{\partial x^2} + \alpha \frac{\partial}{\partial x} \right) \rho' \right] = \kappa \rho_0 c \left( \frac{\partial T'}{\partial x} \right)^2, \quad (8)$$

with $T'$ defined by (7). While an analytic solution to (8) is possible, the high (fourth) order of the equation makes such a solution complex to write down, and we prefer to solve (8) numerically. The horizontal far-field boundary conditions are

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1 For our application in particular, thermobaric effects may be neglected since we assume that fluid parcel displacements are primarily caused by the background flow and thus are nominally horizontal and aligned with isobars.
\[ \rho' = \partial_x \rho' = 0 \quad \text{as} \quad x \to \pm \infty. \]  

(9)

The vertical boundary conditions are that the vertical velocity vanishes \((w' = 0)\) at the top of the domain, which is assumed to be a rigid lid and that all perturbation fields vanish at great depths; in particular,

\[ T' = \rho' = 0 \quad \text{as} \quad z \to -\infty. \]  

(10)

From (6a) the vertical velocity is defined in terms of the temperature and density as

\[ w' = \frac{\partial \psi}{\partial x} = \frac{g}{\rho_0 N^2} \left[ -\kappa_h \frac{\partial^2 \rho'}{\partial x^2} + \alpha_x \frac{\partial \rho'}{\partial x} + \kappa_h \rho_0 \left( \frac{\partial T'}{\partial x} \right)^2 \right]. \]  

(11)

Setting the vertical velocity in (11) to zero at the surface and substituting \(T'\) from (7), we have a density of

\[ \rho' = \Delta \rho_{\text{cab}} \left[ 1 - \left( \text{erf} \frac{x}{L} \right)^2 \right] \quad \text{at} \quad z = 0, \]  

(12)

where

\[ \Delta \rho_{\text{cab}} = \frac{\rho_0 (-c) \Delta T_0^2}{8} \]  

(13)

is the density increase due to cabbeling and \(\Delta T_0 = \Delta T\) \((z = 0)\). Once (8) is solved subject to the above boundary conditions (9), (10), and (12) to determine the perturbation density \(\rho'\), it is straightforward to determine the vertical velocity \(w'\) (and \(\psi\)) from (11) and the alongfront velocity \(v'\) via integration of the thermal wind equation [(6d)]. Below we will determine the full steady-state solution using this method for parameter values typical of the Gulf Stream.

However, before proceeding with the solution, we observe that at the ocean surface the steady state described by (12) results from a balance between lateral diffusion and the imposed large-scale confluence. In contrast, Garrett and Horne (1978) considered an interior balance between lateral diffusion and self-advection by the secondary circulation in the absence of strain. In their model, the densification due to cabbeling is balanced by the vertical advection of the background stratification or \(w = \kappa_h g c (\partial_x T)^2 / N^2\) from (6a). The vertical velocity in our model scales in the same way (see the appendix). However, in our model the width of the temperature front (and thus \(\partial_x T'\) in the scaling) is set by the balance between lateral diffusion and strain (i.e., the frontal width is \(L = \sqrt{2 \kappa_h / \alpha}\), whereas in the Garrett and Horne model the steady-state width depends on the details of the initial temperature field.\(^2\) Nonetheless, assuming the same steady temperature distribution, the magnitude of the perturbation flow associated with cabbeling is comparable in the two models. The smallness of this perturbation flow for typical parameter values \((w \sim 1 \text{ m day}^{-1})\) implies that very little water mass transformation would result by the action of lateral mixing alone. As will be shown in section 3c below, the presence of the background cross-front confluence \(-\alpha x\) in our model, which is approximately \(1/\text{Fr} \sim 25\) times larger than the cross-frontal perturbation flow \(u'\) (see the appendix) can give rise to significant transformation.

b. Solution for Gulf Stream parameters

Here, we present a solution to the simple model [(8)] for the steady-state front using parameter values typical of the Gulf Stream. We select a Gaussian vertical profile of cross-front temperature difference

\[ \Delta T(z) = \Delta T_0 \exp \left( \frac{-z^2}{H^2} \right), \]  

(14)

with decay depth \(H = 150\) m and \(\Delta T_0 = 5\) K, to approximate represent the wishbone-shaped TS profile from observations (e.g., Fig. 1). The values of some of the parameters of the model, notably the confluence and diffusivity, are not well constrained by observations of the Gulf Stream. We will discuss the sensitivity of the solution to these values in section 3c. For this example, we select a confluence of \(\alpha = 0.1 f\) and diffusivity \(\kappa_h = 18 m^2 s^{-1}\). This choice of parameter values results in a surface frontal width \(L = \sqrt{2 \kappa_h / \alpha} \sim 2\) km, consistent with the observed, subsurface thermohaline front shown in Fig. 2 found beneath 100 m and near a cross-front distance of 10 km.

The steady-state temperature and density fields are shown in Fig. 4a down to a depth of 200 m. There is a distinct temperature front around \(x = 0\), but this is compensated by salinity (not shown) such that there is only a relatively small change in the density from the uniformly stratified state. As expected, cabbeling in the frontal zone leads to an increase in the density in the near-surface region, with the corresponding upward deflection of isopycnals. Thus, a dense filament forms with a surface-intensified sheet of cyclonic vorticity transected by a double-celled secondary circulation with

\(^2\) Indeed, Garrett and Horne show that there exists a family of steady solutions to their configuration with arbitrary frontal width. Which solution is appropriate in a given situation presumably depends on the details of the formation of the initial TS front.
downwelling in its center (Fig. 4b). Below ~50 m, the downwelling decreases with depth because of the reduced cross-front temperature contrast and hence cabbeling. This drives vortex squashing that generates a strip of anticyclonic vorticity on the frontal axis. The vertical deflection of isopycnals seen in Fig. 4 implies that the background horizontal strain \( \mathbf{U} = -\alpha \mathbf{y} \) (represented by arrows in the figure) constantly fluxes water across these isopycnals in the steady state, thus driving a continuous water mass transformation. The transformed water is exported along the front by the strain flow \( \mathbf{V} = \alpha y \) and will ultimately accumulate outside of the confluence region.

c. Water mass transformation

The water mass transformation as a function of density, per unit length of the front, may be defined as

\[
F(\rho) = \int_{L_o} \mathbf{u} \cdot \frac{\nabla \rho}{|\nabla \rho|} \, dl \simeq \frac{g}{N^2 \rho_0} \int_{-\infty}^{\infty} -\alpha x \frac{\partial \rho}{\partial x} \, dx, \tag{15}
\]

where the integral is along the line \( L_o \) of an isopycnal of density \( \rho \). A schematic illustrating the control volume used to quantify the diapycnal mass fluxes is shown in Fig. 5. The expression in (15) has been simplified to be consistent with the approximations made in the model formulation: that the perturbation flow is weak compared to the background strain such that \( \mathbf{u} \simeq (-\alpha x, \alpha y, 0) \) and the change in density is small compared to the background stratification such that \( |\nabla \rho| \simeq gN^2/\rho_0 \) and \( dl \simeq dx \). With these approximations we can substitute \(-\alpha x \partial_\rho \rho \) from the density [(6a)] into (15) to yield

\[
F = \frac{g}{N^2 \rho_0} \int_{-\infty}^{\infty} \kappa_{\rho} \frac{\partial^2 \rho}{\partial x^2} - \rho_o N^2 \frac{\partial \psi}{\partial x} - \kappa_{\rho} N^2 \rho_0 \left( \frac{\partial T^\psi}{\partial x} \right)^2 \, dx, \tag{16a}
\]

\[
= \frac{\kappa_{\rho} g (-c)}{N^2} \int_{-\infty}^{\infty} \left( \frac{\partial T^\psi}{\partial x} \right)^2 \, dx, \tag{16b}
\]

\[
=F_0 \frac{\Delta T(c)^2}{\Delta T_0^2}, \tag{16c}
\]

by application of the boundary conditions on \( \rho' \) and \( \psi \) and substitution of the temperature field \( T' \) from (7). The surface (or maximum) water mass transformation \( F_0 \) in (16c) is

\[
F_0 = -\frac{caLg \Delta T_0^2}{2 \sqrt{2} \pi N^2} = \frac{4}{\sqrt{2 \pi}} \alpha L \left( \frac{g \Delta \rho_{ab}}{\rho_o N^2} \right). \tag{17}
\]

The maximum transformation depends on both the inflow at the edge of the mixing region \((-\alpha L)\) and the vertical deflection of the isopycnal height associated
The water mass transformation for the Gulf Stream example, this yields $F_0 = 0.01–0.44 \text{m}^2 \text{s}^{-1}$ for density bins of width $\Delta \rho = 0.1 \text{kg m}^{-3}$. Again, the solid line corresponds to the integral along isopycnals and the dashed line to the analytic expression, computed as $(\partial_z F) \Delta \rho$, with $-\partial_z F$ defined by (18). The formation rate peaks in isopycnal layers beneath the maximum in the transformation rate, where $F$ decreases rapidly with depth and $(\partial_z [\Delta T(z)^2])$ is maximized as predicted by (18). For the Gaussian profile of $\Delta T(z)$ used here, this depth is $z = -H/2 = -75 \text{m}$, corresponding to a peak formation rate of $0.036 \text{m}^2 \text{s}^{-1}$ in the $26.5 \text{kg m}^{-3}$ layer, which corresponds to the Eighteen Degree Water isopycnal layer. That the peak formation happens here is not fortuitous, as the parameters used in the model ($\Delta T_0, H, N^2, T_0, \rho_0$, etc.) are based on the Gulf Stream observations.

Below we use the analytic solution [(17)] to estimate and compare how much water mass transformation could result from the interplay of cabbeling and frontogenesis at the Gulf Stream, Kuroshio, and the Subantarctic Front in the southeast Indian Ocean. As noted already, the strain $\alpha$ and diffusivity $\kappa_h$ are poorly constrained by observations. However, observations from the Gulf Stream suggest that the parameters fall in the range $\alpha = 0.01f–0.1f$ and $\kappa_h = 1–100 \text{m}^2 \text{s}^{-1}$ (Thomas et al. 2013; Joyce et al. 2013). Using these values we can compute upper ($\alpha = 0.1f$, $\kappa_h = 100 \text{m}^2 \text{s}^{-1}$, and $L = 4.5 \text{km}$) and lower ($\alpha = 0.01f$, $\kappa_h = 1 \text{m}^2 \text{s}^{-1}$, and $L = 1.4 \text{km}$) limits for the transformation. For the Gulf Stream example, this yields $F_0 = 0.01–0.44 \text{m}^2 \text{s}^{-1}$. The Subantarctic Front discussed in the introduction separates water of temperature $T \sim 6^\circ C$ and $T \sim 11^\circ C$ and exists in a region with stratification of $N^2 \sim 3 \times 10^{-2} \text{K}^{-2}$. Since this water is substantially cooler than that of the Gulf Stream, the value of the cabbeling parameter $c$ changes; in this case $c = -1.0 \times 10^{-5} \text{K}^{-2}$. Substituting these values into (17) yields a water mass transformation of $F_0 = 0.03–0.79 \text{m}^2 \text{s}^{-1}$ for the Subantarctic Front. Following a similar procedure for the Kuroshio Front, which has a temperature difference of $4 \text{K}$ and

with cabbeling (i.e., the term in parentheses). The latter is proportional to the density increase caused by cabbeling, $\Delta \rho_{\text{cab}}$. At observed fronts, $\Delta \rho_{\text{cab}}$ is quite small, of order 0.01 $\text{kg m}^{-3}$ (e.g., Carter et al. 2014). This does not necessarily imply that water mass transformation by cabbeling is negligible, however, because (17) depends on the circulation as well. The dependence of the transformation (17) on the lateral diffusivity comes through the frontal width $L$. For the more realistic situation of a spatially variable diffusivity, while the frontal width will differ from the simple expression $\sqrt{2\kappa_h / \alpha}$, the dependence of water mass transformation on $L$ should be similar.

The above expression for water mass transformation [(16c)] also provides a description of where in the water column mode waters will form. The greatest mode water formation will occur at the density [or depth since $z \approx g(1 - \rho/\rho_0)/N^2$ under our approximations] where the convergence of the mass transformation is greatest. The convergence of the transformation, or formation rate, is

$$-\frac{\partial F}{\partial \rho} \approx \frac{g}{\rho_0 N^2} \frac{\partial F}{\partial z} = \frac{g F_0}{\rho_0 N^2 \Delta T_0} \frac{\partial}{\partial z} \Delta T(z)^2$$

per unit density. Thus, mode water will tend to form at the depth (density) where the vertical (diapycnal) variation in the square of the cross-frontal temperature difference is greatest.

The water mass transformation for the Gulf Stream example discussed above is displayed in Fig. 6a. The solid line is the water mass transformation obtained by directly integrating along density surfaces from the numerical solution, whereas the dashed line is the analytic expression defined by (17) (obtained by integrating in $x$ and neglecting the slope of the isopycnals). The two curves agree except near the surface where the densification due to cabbeling is maximal and the isopycnals outcrop (see Fig. 5). The outcropping of isopycnals is responsible for the reduction in the mass transformation computed from the numerical solution (solid) for the lightest density classes since it reduces the length $L_\rho$ in the integral (15). Nonetheless, both curves show that the transformation peaks at about $0.19 \text{m}^2 \text{s}^{-1}$ near the surface and decays with depth as the water mass contrast across the front diminishes. The convergence of transformation, or the formation rate, is shown in Fig. 6b for density bins of width $\Delta \rho = 0.1 \text{kg m}^{-3}$. Again, the solid line corresponds to the integral along isopycnals and the dashed line to the analytic expression, computed as $-(\partial_z F) \Delta \rho$, with $-\partial_z F$ defined by (18). The formation rate peaks in isopycnal layers beneath the maximum in the transformation rate, where $F$ decreases rapidly with depth and $(\partial_z [\Delta T(z)^2])$ is maximized as predicted by (18). For the Gaussian profile of $\Delta T(z)$ used here, this depth is $z = -H/2 = -75 \text{m}$, corresponding to a peak formation rate of $0.036 \text{m}^2 \text{s}^{-1}$ in the $26.5 \text{kg m}^{-3}$ layer, which corresponds to the Eighteen Degree Water isopycnal layer. That the peak formation happens here is not fortuitous, as the parameters used in the model ($\Delta T_0, H, N^2, T_0, \rho_0$, etc.) are based on the Gulf Stream observations.
relatively strong stratification of $N^2 \sim 10^{-4} \text{s}^{-2}$, yields a comparatively small water mass transformation of $F_0 = 0.004-0.12 \text{m}^2 \text{s}^{-1}$.

To convert these water mass transformation rates into Sverdrups, we need to know the length over which the front in question is strained by confluent flow. The Gulf Stream, for example, experiences confluent strain over much of the western half of the recirculation gyre, which covers a distance on the order of 1000 km (Fig. 2). Of course, the strain is not uniform over this distance and changes sign in different phases of the mesoscale meanders that characterize the current in this region. However, we would expect that in regions of frontolytic strain, little or no water mass transformation would occur because the lateral temperature–salinity gradients and hence mixing would be reduced. Thus, calculating the net water mass transformation would involve integrating along the front only over regions of frontogenetic strain. Presuming that these regions span a distance of order 1000 km and assuming that this length scale is also representative of the Kuroshio and Subantarctic Fronts, the analysis presented above suggests that cabling at strained TS fronts could contribute 0.01–1 Sv to the formation of their respective mode waters. These values are smaller than the 1–10 Sv of seasonal mode water formation (and destruction) by air–sea fluxes (e.g., Forget et al. 2011). That being said, diapycnal mass fluxes associated with cabling do not change sign with the seasons and thus should fill mode water layers continuously, as long as the cross-front water mass contrasts persist, the circulation is frontogenetic, and submesoscale mixing is active. Consequently, cabling at strained fronts could play an important role in the dynamics of mode waters on interannual time scales and might precondition their isopycnal layers for formation by air–sea fluxes by reducing the stratification.

4. Conclusions

Frontogenetic strain acting on a thermohaline front if equilibrated by mixing will facilitate cabling. The process naturally forms a dense filament and double-celled secondary circulation centered at the thermohaline front that is accompanied by a sheet of cyclonic vorticity, a feature reminiscent of the flow observed at the North Wall of the Gulf Stream (Fig. 2). Water parcels moving with the frontogenetic flow cross the upward-deflected isopycnals of the dense filament yielding a diapycnal mass flux. Using a simple theoretical model, we have derived an expression for the water mass transformation per unit length of front resulting from this mechanism: that is, \( F_{\rho} = \frac{1}{2 \pi} \ln \left( \frac{\rho_2}{\rho_1} \right) \). The transformation depends critically on the strain rate, equilibrated width of the front, and the square of the isopycnal temperature contrast across the front. These quantities are not influenced by the presence of mode water in any obvious way, implying that mode water formation by this
process does not suffer a causality dilemma, unlike the classical formation mechanism involving air–sea fluxes. At the major ocean fronts where mode waters are found, the isopycnal temperature contrast decreases with increasing density for the isopycnals surrounding the density surfaces where mode waters reside (Fig. 1). Consequently, the theory predicts that the interaction between frontogenesis and cabbeling should result in a convergent diapycnal volume flux that would fill mode water layers over time. Our model suggests [e.g., (18)] that the isopycnal layer selected by the cabbeling mechanism is that for which the diapycnal variation in the square of the along-isopycnal temperature contrast is greatest; that is, where \( \delta_j[\Delta T(\rho)^2] \) is maximized. In our simple model, however, the volume of isopycnal layers does not change in time. This is a consequence of the spatially uniform strain of the deformation field that flushes water that would accumulate near the frontal axis \((x = 0)\) out to \(y \to \pm \infty\). While the model illustrates how cabbeling selects the mode water isopycnal layer, since the layer never actually fills, it cannot be used to predict the properties (namely, temperature and salinity) of the mode water that results. Exploring the physics of mode water formation via frontogenesis and cabbeling, the submesoscale lateral mixing processes that allow it to occur, and the temperature–salinity relationship that it creates, in a more realistic flow with spatially variable strain and fronts that are only partially compensated in density will be the subject of future research involving high-resolution numerical simulations.

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APPENDIX

Scaling Analysis

Here, we perform a scaling analysis to demonstrate how the approximate model [6] may be obtained from the full two-dimensional Boussinesq equations for the frontal system. We scale the temperature with the surface temperature difference across the front, \(T' \sim \Delta T_0\), and the density with the change in density due to cabbeling, \(\rho' \sim \Delta \rho_{cab} = \rho_b(-c\Delta T_0)^2/8\) from (13). The cross-frontal dimension is scaled by the advective–diffusive balance length scale, \(x \sim L = \sqrt{2\kappa_d/\alpha}\) as per (7), and the vertical dimension with the frontal height scale, \(z \sim H\). The alongfront velocity is scaled geostrophically: \(u' \sim gH\Delta \rho_{cab}/(\rho_0 f L)\). The cross-front velocity is assumed to scale as \(u' \sim \alpha fL^2\nu/(N^2H^2)\) and the vertical velocity as \(w' \sim Hu'/L\).\(^3\) The perturbation pressure is assumed to scale as \(p' \sim gH\Delta \rho_{cab}\). The particular choice of scales is made such that all scaled fields take order-one values, as may be determined explicitly from the solutions. With these scales, the two-dimensional, hydrostatic, Boussinesq [(2) and (5)] equations at steady state may be written in non-dimensional form as

\[
\frac{\delta^2}{(Bu)^2} \left[ (Fr u' - x) \frac{\partial u'}{\partial x} + Fr w' \frac{\partial u'}{\partial z} \right] = u' + \frac{\delta^2}{(Bu)^2} u' \frac{\partial u'}{\partial x} + \frac{\delta^2}{(Bu)^2} \frac{1}{2} \frac{\partial^2 u'}{\partial x^2} + \frac{\delta^2}{(Bu)^2} \frac{1}{2} \frac{\partial^2 u'}{\partial x^2}, \tag{A1a}
\]

\[
(Tr u' - x) \frac{\partial u'}{\partial x} + Fr w' \frac{\partial u'}{\partial z} = -u' \frac{\delta^2}{(Bu)^2} - u' + \frac{1}{2} \frac{\partial^2 u'}{\partial x^2}. \tag{A1b}
\]

\[
0 = -\rho' - \frac{\partial p'}{\partial z}. \tag{A1c}
\]

\[
0 = \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}. \tag{A1d}
\]

\[
(Tr u' - x) \frac{\partial T'}{\partial x} + w' \left( Fr \frac{\partial T'}{\partial z} + \gamma \right) = \frac{1}{2} \frac{\partial^2 T'}{\partial x^2}, \tag{A1c}
\]

\[
(Tr u' - x) \frac{\partial p'}{\partial x} + w' \left( Fr \frac{\partial p'}{\partial z} - 1 \right) = \frac{1}{2} \frac{\partial^2 p'}{\partial x^2} + 4 \left( \frac{\partial T'}{\partial x} \right)^2. \tag{A1f}
\]

The above equations indicate that there are four relevant nondimensional numbers: the strain ratio \(\delta = \alpha/f\), the Froude number \(Fr = g\Delta \rho_{cab}/(\rho_0N^2H)\), the Burger number \(Bu = NH/(fL)\), and a nondimensional cabbeling parameter \(\gamma = -c\Delta T_0/(8\gamma f)\). This latter parameter will be small for all parameter values of interest; for instance, \(\gamma = 0.02\) for the Gulf Stream example in the main text. We now discuss the further simplification of (A1) when it is also assumed that the strain ratio and Froude numbers are small:

\(^3\) The vertical velocity scale may be rewritten as \(w' \sim (-c)\kappa_d\Delta T_0^2/(4N^2L^2)\) and so is the same scale obtained by Garrett and Horne (1978) in their unstrained model.
(i) Weak strain: \((\delta/\text{Bu})^2 \ll 1\). Assuming that the strain is weak, it is readily observed that (A1a) reduces to geostrophic balance

\[
0 = u' - \frac{\partial p'}{\partial x} .
\]  
(A2)

Combining (A2) with hydrostatic balance [(A1c)], we obtain the thermal wind equation

\[
\frac{\partial u'}{\partial z} = - \frac{\partial p'}{\partial x} .
\]  
(A3)

Thus, the maintenance of thermal wind balance for weak strain flows, as in the Hoskins and Bretherton (1972) model, is unchanged by the presence of lateral diffusion. Since we consider strains no greater that \(\alpha = 0.1\) in our model (and \(\text{Bu}\) is order one), we expect the system to remain very close to thermal wind balance:

(ii) Small Froude number: \(\text{Fr} \ll 1\). Assuming that the Froude number is small it follows that advection of the background flow in (A1b) is negligible compared with advection by the background strain flow, and thus (A1b) becomes

\[
-x \frac{\partial u'}{\partial x} = -u' - \frac{1}{(\text{Bu})^2} u' + \frac{1}{2} \frac{\partial^2 u'}{\partial x^2} .
\]  
(A4a)

A similar result applies for the density [(A1f)],

\[
-x \frac{\partial \rho'}{\partial x} - w' = \frac{1}{2} \frac{\partial^2 \rho'}{\partial x^2} + 4 \left( \frac{\partial T'}{\partial x} \right)^2 .
\]  
(A4b)

and the temperature [(A1e)],

\[
-x \frac{\partial T'}{\partial x} = \frac{1}{2} \frac{\partial^2 T'}{\partial x^2} .
\]  
(A4c)

The density [(A4b)] retains advection of the background stratification by the perturbation vertical velocity, but this term drops out of the temperature equation owing to the smallness of the \(\gamma\) parameter as discussed above. We observe that the assumption of the small Froude number is very accurate for the examples considered in the paper; for the Gulf Stream example we have \(\text{Fr} = 0.04\). The smallness of the Froude number is closely tied to our assumption that the front is compensated and therefore the frontal density variation \(\Delta \rho = \Delta \rho_{\text{bc}}\) is small.

Equations (A1d), (A3), and (A4) form a complete set of simplified nondimensional equations for the frontal system in the limit of weak strain and small Froude number. Introducing the perturbation streamfunction \(u' = \delta \psi\) and \(w' = -\delta x \psi\), we have, in summary,

\[
1 \frac{\partial^2 \rho'}{\partial x^2} + x \frac{\partial \rho'}{\partial x} - \frac{\partial \psi}{\partial x} = 0 .
\]  
(A5a)

\[
1 \frac{\partial^2 T'}{\partial x^2} + x \frac{\partial T'}{\partial x} = 0,
\]  
(A5b)

\[
1 \frac{\partial^2 u'}{\partial x^2} + x \frac{\partial u'}{\partial x} - u' - \frac{1}{(\text{Bu})^2} \frac{\partial \psi}{\partial x} = 0,
\]  
(A5c)

\[
\frac{\partial \psi}{\partial z} + \frac{\partial p'}{\partial x} = 0 .
\]  
(A5d)

The dimensional equations given in the text [(6)] may be obtained directly from (A5) by replacing the dimensional scales.

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