Stability of Baroclinic Vortices on the $\beta$ Plane and Implications for Transport

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Abstract

This paper explores stability of and transport by baroclinic vortices on the $\beta$ plane using a two-layer, quasigeostrophic model. The study adapts a wave–mean flow formalism and examines interactions between the axisymmetric flow ("the vortex") and residuals ("the waves"). Unlike baroclinically unstable vortices on the $f$ plane, such vortices on the $\beta$ plane can be also vulnerable to barotropic instability as revealed by the globally integrated energy balance analysis. The spatial structure of energy fluxes shows the energy leakage inside the vortex core when its breakdown occurs. Mixing by stable and unstable vortical flows is quantified through the computation of finite-time Lyapunov exponent (FTLE) maps. Depending on the strength of wave radiation, the upper-layer FTLE maps of stable vortices show either an annulus or volute ring of vigorous mixing inside the vortex interior. This ring region is disrupted when the vortex becomes unstable. Both stable and unstable vortices show the wavy patterns of FTLE in the near and far fields. Despite the fact that the initial vortex resides in the top layer only, significant FTLE patterns are observed in the deep layer at later times. Lagrangian analysis of the vortex-induced change of large-scale tracer gradient demonstrates significant effects of vortex instability in the top layer and the importance of the wavelike anomalies in the bottom layer.

1. Introduction

Mesoscale coherent vortices are found in every part of the World Ocean (Jochum and Malanotte-Rizzoli 2003; Chelton et al. 2011). Observational records suggest that such vortices have a baroclinic structure, and many exhibit a very long life cycle (Olson 1991). However, some of these vortices have been found to break down shortly after generation (Schonten et al. 2000). Under the influence of Earth’s rotation, these structures propagate in the northwest or southwest directions depending on the sense of their own spin (Fiorino and Elsberry 1989). Oceanic vortices can generate regions of anomalous mixing (Provenzale 1999) and contribute to large-scale transport (Dong et al. 2014). The stability and transport properties of baroclinic vortical flows on the $\beta$ plane are the main topics to be addressed in this study.

Laboratory experiments (Griffiths and Linden 1981; Thivolle-Cazat et al. 2005) and numerical simulations (Ikeda 1981; Flierl 1988; Helfrich and Send 1988) of baroclinic vortices on the $f$ plane suggest that the vortices with a radius larger than the Rossby deformation radius are vulnerable to the wavenumber 2 instability as revealed by normal-mode analysis. This goes at odds with observations, and several attempts have been pursued in order to resolve this paradox. One factor that stabilizes vortices in the upper layer is the magnitude and sense of the flow in the lower layer; a corotating flow in the lower layer strengthens the barotropic component of the flow, thereby inhibits baroclinic release of energy by its barotropic gain (Dewar and Killworth 1995; Dewar et al. 1999; Katsman et al. 2003). Benilov (2004) argues that the parametric range of baroclinic vortex
stability can be extended even further if the flow in the lower layer has uniform potential vorticity. In view of the absence of radial shear in the potential vorticity profile of the basic state, the lower-layer flow does not support a counterpropagating Rossby vortex wave that can resonate with the one in the upper layer, which is necessary for the structure of the most unstable disturbance.

On the other hand, simulations of two-layer geostrophic turbulence on the $\beta$ plane show that the vortices tend to be depth-compensated rather than having significant flow in the lower layer (Berloff et al. 2011). Within some range of parameters, not coinciding with the stability diagram on the $f$ plane, an initially depth-compensated, baroclinic vortex on the $\beta$ plane undergoes the leakage of potential vorticity out of its core (McWilliams et al. 1986). We use the term breakdown to describe this process. Nevertheless, the vortex becomes stable if the simulation is carried out in the equivalent barotropic model. This model does not have a mechanism of the conversion of the basic-state available potential energy into the kinetic energy of disturbances, and thus baroclinic instability seems at first to be the main cause of the vortex breakdown. However, the lack of the mixed-mode eddy momentum fluxes in the equivalent barotropic model partially precludes the release of the kinetic energy of the basic state, which is essential for barotropic instability.

Inhomogeneous mixing areas in the ocean can partially be explained by the fact that fluid can be trapped inside vortex cores for long times (Dewar and Flierl 1985; Provenzale 1999). Analyzing finite-size Lyapunov exponent (FSLE) maps computed from altimetry data, Lehahn et al. (2011) show that an Agulhas ring has a core with weak mixing and the spirals of FSLE encircling its core. Finite-time Lyapunov exponent (FTLE) maps of two-dimensional turbulence confirm this structure of the mixing patterns within the vortex interior but also reveal outgoing FTLE filaments from one vortex that end in the close proximity of another vortex (Lapeyre 2002). Similarly, the evolution of the tracer concentration produced by stable barotropic vortical flow on the $f$ plane shows the formation of spirals inside the vortex interior unless the initial concentration of the released tracer has the same form as potential vorticity or the streamfunction profile of the vortex (Rhines and Young 1983). These spirals eventually disappear because of the action of turbulent diffusion. The evolution of the same vortex on the $\beta$ plane exhibits spiraling and a high-gradient strip peeling off the vortex (Benilov 1999). It remains unknown, however, as to how the baroclinic structure of the vortex, its potential breakdown, and radiation of waves change these mixing patterns.

The $\beta$-induced drift of the vortices and inhibited permeability of their interior for material fluxes make these structures one of the major drivers of global meridional transport (Dong et al. 2014). Beal et al. (2011) demonstrate the importance of Agulhas rings in changing thermohaline properties of the Atlantic waters. Stable North Brazil Current rings contribute up to 9.3 $\text{Sv}\cdot\text{yr}^{-1}$ ($1\text{Sv} = 10^6\text{m}^3\text{s}^{-1}$) of water transport from tropical to subtropical gyres in the Atlantic Ocean (Johns et al. 2003; Garraffo et al. 2003). In their estimates of vortex-induced transport, these authors assume that the flows generated by vortices in the deeper layers and radiated waves do not contribute significantly to the total transport. Additionally, the transport estimates of those studies are based on an assumed shape of vortices that is assumed to be constant in time. While these assumptions seem to be reasonable for stable vortices, unstable vortices generate strong flows in the deeper layers, do not preserve their shape, and radiate waves that might also change the distribution of tracers in the vicinity of the vortex as well as in the far field.

This study employs a two-layer quasigeostrophic model on the beta plane as the simplest tool to mimic the baroclinic structure of mesoscale vortices and the effect of Earth’s rotation on their dynamics. This model possesses additional energy conversion mechanisms, not captured by the equivalent barotropic model, which can potentially give rise to both baroclinic and barotropic instabilities. It also allows us to estimate vortex-induced mixing and large-scale transport in the deep ocean. We present an analysis of globally integrated and local energy balances to find the mechanism of the instability of

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**Table 1. Model parameters.**

<table>
<thead>
<tr>
<th>Case</th>
<th>$V_{\text{max}}$ (m s$^{-1}$)</th>
<th>$R_{\text{max}}$ (km)</th>
<th>$\beta$ (m$^{-1}$ s$^{-1}$)</th>
<th>$R_d$ (km)</th>
<th>$\delta = H_1/H_2$</th>
<th>$\nu$ (m$^2$ s$^{-1}$)</th>
<th>$\tau$ (s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stable vortex</td>
<td>0.8</td>
<td>60</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>Stable vortex</td>
<td>0.8</td>
<td>70</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>Stable vortex</td>
<td>0.8</td>
<td>80</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>Unstable vortex</td>
<td>0.8</td>
<td>90</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>Unstable vortex</td>
<td>0.8</td>
<td>100</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>Unstable vortex</td>
<td>0.8</td>
<td>110</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>Unstable vortex</td>
<td>0.8</td>
<td>120</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
</tbody>
</table>
baroclinic vortices on the $\beta$ plane. We then utilize FTLE maps to compare mixing regimes of stable and unstable vortical flows. Finally, we estimate the large-scale transport produced by vortices from the latitudinal change of Lagrangian particle concentration.

2. Numerical model

The numerical model used in this study is based on a traditional quasigeostrophic approximation (Pedlosky 1982) supplemented with lateral viscosity and bottom friction. Despite some shortcomings of studying ocean vortices in the context of quasigeostrophic dynamics, such as the effects of cyclone–anticyclone asymmetry on the stability properties (Benilov and Flanagan 2008; Dewar et al. 1999), it is instructive to start with a relatively simple model that still captures the essential elements of the baroclinic vortex dynamics.

The governing equations are the conservation of potential vorticity in each layer:

![Fig. 1. Evolution of an unstable vortex with $R_{\text{max}} = 100\text{ km}$ at (a) 1, (b) 20, (c) 37, (d) 45, (e) 59, and (f) 100 days. Shown is streamfunction ($10^4\text{ m}^2\text{s}^{-1}$) in the upper layer.](image)
J\left(\psi_1, q_1\right) - S_1(\psi_1 - \psi_2), \quad \text{and} \quad J\left(\psi_2, q_2\right) = S_2(\psi_2 - \psi_1).

The stratification parameters $S_n$ are given by $S_n = \frac{f_0^2}{(H_n g')^2}$. Here, $H_n$ stands for the thicknesses of the layers at rest, $g' = g(\Delta \rho/\rho)$ is a reduced-gravity acceleration. The first baroclinic Rossby deformation radius $R_d = S_1 + S_2 = (1/f_0)\sqrt{g'[(H_1 H_2)/(H_1 + H_2)]}$ is an important length scale that determines properties of baroclinic flow. The model is initialized with $\psi_1 = a \exp(-r/b)^2$ and $\psi_2 = 0$, where $r = \sqrt{x^2 + y^2}$. The parameters $a$ and $b$ are defined as $a = V_{\text{max}} R_{\text{max}} \exp(2)$ and $b = \sqrt{2} R_{\text{max}}$, where $V_{\text{max}}$ is the maximum swirling velocity and $R_{\text{max}}$ is the radius of the maximum swirling velocity. A Gaussian profile is arguably not the best approximation for oceanic vortices (Castelão and Johns 2011), but it simplifies the comparison of our results with previous studies and guides us in choosing the corresponding parameters. The governing equations are solved on the square domain of 2560 km in each direction with the spatial resolution of 5 km. The values of all parameters used in this paper are summarized in Table 1 and chosen from the stability diagram in McWilliams et al. (1986).

The snapshots of the upper-layer streamfunction for the unstable vortex with $R_{\text{max}} = 100$ km are shown in Fig. 1. Initially propagating in the northwest direction and slowly fading because of wave radiation, the vortex becomes contorted at day 37. It tends to restore its shape after this day (see Fig. 1d). At day 59 the vortex deforms again and spawns two vortices at day 100. This is an example of vortex breakdown. The same vortex with $R_{\text{max}} = 70$ km also propagates in the northwest direction.

![Fig. 2. Tracks of vortices with different radii of maximum swirling velocity within 140 days.](image)

![Fig. 3. Time series of globally integrated (a) barotropic and (b) baroclinic vortex energy.](image)
with the waves observed in its wake (not shown), yet it remains coherent during the entire period of its evolution.

The tracks of stable and unstable vortices also reveal the effects of the instability (Fig. 2). While stable vortices ($R_{\text{max}} = 60$ and $70 \text{ km}$) move to the northwest along a straight line, except for the vortex with $R_{\text{max}} = 80 \text{ km}$, which develops small oscillations along its trajectory, the unstable vortices ($R_{\text{max}} = 90–120 \text{ km}$) have looping trajectories.

### 3. Energetics

#### a. General discussion of vortex energetics

The problem of vortex instability on the $f$ plane is typically approached by linearizing the governing equations around an initial, vortical profile and subsequently calculating eigenvalues and eigenvectors of a time-independent, linearized, dynamical operator. The eigenvalues and corresponding eigenvectors (e.g., normal modes) provide information about the growth rate and structure of infinitesimal disturbances initially impressed on the vortex. On the $\beta$ plane, however, the same initial vortical profile is not a stationary solution of the governing equations; the linearized dynamical operator becomes time dependent, and the concept of normal modes does not apply (Farrell and Ioannou 1996). A more comprehensive technique to tackle the instability problem is to analyze the energy exchanges between the basic-state flow and disturbances. We analyze globally integrated and local energy balances for unstable vortical profiles with parameters listed in Table 1. Before proceeding to the discussion of the energy balance and the results of calculations, several definitions are required. First, the term “vortex” farther down in this section stands for the azimuthally averaged (vortex) flows in the comoving system of reference. Finally, we adopt the term instability to describe the process of vortex breakdown, although no linearization has been performed. Thus, we do not impose any prior assumption on the smallness of the wave field magnitude compared to the magnitude of the basic state.

The energy balance derived in the appendix shows that the energy of the vortical flow is partitioned between the barotropic and baroclinic modes. The barotropic mode contains only kinetic energy, while the baroclinic mode has both kinetic and available potential energies. The vortex releases its energy when the integrated energy tendency is negative or gains wave energy when the integrated energy tendency is positive. Except for the frictional dissipation, the energy redistribution is given by the energy fluxes that are on the right-hand side of the energy balance equations. To facilitate the analysis, we combine all energy flux terms associated with the $\beta$ effect, comoving reference frame, and dissipation with the left-hand side, so that our energy balance takes the following form:
Fig. 4. Integrated energy balance for the vortices with $R_{\text{max}} = (a) 90, (b) 100, (c) 110, \text{and} (d) 120 \text{ km}. Shown are time series of energy balance components for the baroclinic mode. Two vertical dashed lines denote two minima in energy tendency.

\[
\int_0^\infty \frac{\partial (E_{\text{BT}})}{\partial t} r \, dr = \int_0^\infty \left[ \frac{1}{\alpha} \langle \psi_{\text{BT}} \rangle \langle \psi'_{\text{BT}} \rangle \langle J(\psi_{\text{BT}}, q_{\text{BT}}) \rangle + \langle \psi_{\text{BT}} \rangle \langle J(\psi_{\text{BC}}, q_{\text{BC}}) \rangle \right] r \, dr, \quad \text{and} \quad (3)
\]

\[
\int_0^\infty \frac{\partial (E_{\text{BC}})}{\partial t} r \, dr = \int_0^\infty \left[ \langle \psi_{\text{BC}} \rangle \langle J(\psi_{\text{BT}}, q_{\text{BC}}) \rangle + \langle \psi_{\text{BC}} \rangle \langle J(\psi_{\text{BC}}, q_{\text{BC}}) \rangle \right]

- (S_1 + S_2) \langle \psi_{\text{BC}} \rangle \langle J(\psi_{\text{BC}}, \psi'_{\text{BT}}) \rangle + \alpha C \langle \psi_{\text{BC}} \rangle \langle J(\psi_{\text{BC}}, q_{\text{BC}}) \rangle \right] r \, dr, \quad (4)
\]

where the left-hand sides of these equations are defined by

\[
\int_0^\infty \frac{\partial (E_{\text{BT}})}{\partial t} r \, dr = \int_0^\infty \langle \psi_{\text{BT}} \rangle \frac{\partial \langle q_{\text{BT}} \rangle}{\partial t} r \, dr - \int_0^\infty \left\{ \langle \psi_{\text{BT}} \rangle \beta \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) \langle \psi_{\text{BT}} \cos \theta \rangle

+ \langle \psi_{\text{BT}} \rangle \left[ \frac{1}{r} + \frac{\partial}{\partial r} \right] \left[ U(t) \langle q_{\text{BT}} \rangle \cos \theta + V(t) \langle q_{\text{BT}} \rangle \sin \theta \rangle - \langle \psi_{\text{BT}} \rangle \langle D_{\text{BT}} \rangle \right] r \, dr, \quad \text{and}
\]

\[
\int_0^\infty \frac{\partial (E_{\text{BC}})}{\partial t} r \, dr = \int_0^\infty \frac{\partial \langle E_{\text{BC}} \rangle}{\partial t} r \, dr - \int_0^\infty \left\{ \langle \psi_{\text{BC}} \rangle \beta \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) \langle \psi_{\text{BC}} \cos \theta \rangle

+ \langle \psi_{\text{BC}} \rangle \left[ \frac{1}{r} + \frac{\partial}{\partial r} \right] \left[ U(t) \langle q_{\text{BC}} \rangle \cos \theta + V(t) \langle q_{\text{BC}} \rangle \sin \theta \rangle - \langle \psi_{\text{BC}} \rangle \langle D_{\text{BC}} \rangle \right] r \, dr.
\]

The impact of the energy fluxes associated with the $\beta$-effect, comoving reference frame, and dissipation can be understood by comparing how linear versus nonlinear dynamics affect the vortex energy evolution. The time series of globally integrated energy of barotropic and baroclinic modes for different vortical profiles are shown in Fig. 3. Baroclinic energy exceeds barotropic energy by approximately two orders of magnitude and
thus dominates the energy content of the vortex. After initial decline, barotropic energy increases dramatically reaching its first maximum at around day 45. The second dip in the magnitude of barotropic energy is observed between days 60 and 70 for different cases. For the cases of vortices with \(R_{\text{max}} = 90\) and 100 km, barotropic energy rises again and then subsequently decreases by the end of the vortex evolution. Barotropic energy of the bigger vortices (\(R_{\text{max}} = 110\) and 120 km) shows more irregular behavior with several sharp jumps. In contrast, the initial and final trends of baroclinic energy remain nearly flat, suggesting that the vortices remain in a nearly stationary state. The intermediate development of baroclinic energy for the vortices with \(R_{\text{max}} = 90\) and 100 km consists of two stages: it slightly declines up to day 55 and then decreases more steeply up to day 70. The baroclinic energy decays steadily for the vortex with \(R_{\text{max}} = 110\) km, and two periods of rapid decrease are observed for the vortex with \(R_{\text{max}} = 120\) km before the energy levels off for both cases. The linear dynamics of vortices, in which all nonlinear energy fluxes are absent, shows that radiation from the barotropic mode occurs very efficiently, while the baroclinic mode remains relatively stiff (Flierl 1977; McWilliams and Flierl 1979). This prompts us to claim that the hollow parts of the curves of barotropic vortex energy and the flat parts of the curves of baroclinic vortex energy are explained by the energy drain caused mostly by the \(\beta\) effect and partially by friction. The decrease of baroclinic vortex energy during the intermediate period of vortex evolution and sharp peaks in the evolution of barotropic vortex energy can therefore be attributed to the action of nonlinear energy fluxes. We, therefore, argue that the energy fluxes associated with the \(\beta\) effect, comoving reference frame, and dissipation do not affect the rate of change of baroclinic vortex energy. The energy production of the barotropic mode can be affected at the times when the nonlinear forcing is small. This, however, does not change the conclusions of this study.

The main goal of this section is to address the role of the kinetic and available potential energy fluxes as potential drivers of vortex instability. In Table 2, we summarize all terms in energy balance, their acronyms, and physical meaning. The acronyms follow the convention that EC indicates energy change, W indicates work, REY indicates the Reynolds stress forcing, FORM stands for the form stress forcing, and BT, BC, and MX are the shorthand forms for the words barotropic, baroclinic, and mixed mode, respectively. The terms WREY_BT, WREY_BC, WREY_MX, and WREY_BC1 convert kinetic energy of the vortex and are associated with wave momentum fluxes; the term WFORM
is responsible for the transformation of available potential energy of the vortex and associated with wave buoyancy fluxes. If the vortex loses its energy, these two different conversion mechanisms are typically associated with barotropic and baroclinic instabilities, respectively. The vortex can also strengthen at the expense of the wave energy.

b. Global energy balance analysis

We first present the time history of globally integrated energy balance based on (3) and (4). We use similar notations for all terms of energy balance as listed in Table 2 but add the subscript $g$ that means the globally integrated quantity. During the entire period of vortex evolution (100 days), all cases indicate that the baroclinic mode energy experiences two periods of significant loss marked by the dashed lines (Fig. 4). The first major loss of baroclinic energy is caused by the WFORM$_g$ term, meaning that the vortex loses energy through baroclinic exchanges. The WREY_MX$_g$ and WREY_BC1$_g$ terms act to increase baroclinic energy at this time, but their contribution is almost negligible. However, all vortices stabilize after this time, as indicated by the ascending trend of EC_BC$_g$. For the vortices with $R_{max} = 90$ and 100 km, the loss of energy gradually reaches zero, while for the vortices with $R_{max} = 110$ and 120 km, the energy tendency becomes positive followed by a smooth decline. A more interesting behavior is observed during the time of the vortex breakdown, when the second significant energy loss is observed. While the vortices with $R_{max} = 90$ and 120 km release their energy due to the WFORM$_g$ term, the vortices with $R_{max} = 100$ and 110 km lose their energy through the WREY_MX$_g$ term.

The barotropic mode gains energy during the entire period and has two well-defined maxima barring the case of the vortex with $R_{max} = 120$ km (Fig. 5). The energy gain is primarily supported by WREY_BT$_g$ energy flux, while WREY_BC$_g$ flux plays a secondary role.

In sum, we provide a physical interpretation of the calculations above. Since the dominant part of the vortex energy is concentrated in the baroclinic mode, the energy production by this mode sets the energy production of the entire vortex. The baroclinic energy production of every unstable vortex becomes strongly negative twice, and it is during the second period that the actual vortex splitting occurs. At this time, both energy fluxes associated with form stress forcing (WFORM$_g$) and Reynolds stress forcing (WREY_MX$_g$) can cause this strong negative maximum in energy tendency. For the cases of vortices with $R_{max} = 90$ and 120 km, the main contribution to this energy loss is provided by

![Fig. 6. Local energy balance for the vortices with $R_{max} = $ (a) 90, (b) 100, (c) 110, and (d) 120 km; $R_{max}$ is denoted by the dashed line. Shown are radial profiles of energy balance components for the barotropic mode at the time of vortex deformation (see the text).](image-url)
energy fluxes associated with wave buoyancy (WFORM$_g$), thereby the vortices are baroclinically unstable. For the cases of vortices with $R_{\text{max}} = 100$ and 110 km, this second negative maximum of energy change is supported by energy fluxes associated with the part of Reynolds stress forcing (WREY$_{Mx}$), so that the vortices are barotropically unstable. We also note a compensating role by the barotropic component, as previously suggested in Dewar and Killworth (1995). For moderate and small ratios of $R_{\text{max}}$ to Rossby deformation radius, the content of energy in barotropic and baroclinic modes is comparable in magnitude; thereby, the loss in baroclinic energy is balanced by the gain in barotropic energy keeping the vortex stable. Increasing this ratio makes the baroclinic mode stronger, thereby leading to the vortex breakdown.

c. Local energy balance analysis

The integrated energy balance analyzed above provides information on total energy tendency and energy fluxes at a particular time. Such an analysis, however, does not show the location of maximum energy growth or decay nor does it give insight into the way the energy fluxes set the energy rate distribution. We next present the analysis of local energy balance at the time of the two most significant energy transformations marked by the vertical dashed lines in Fig. 4. We show the radial structure of the tendency and flux terms in Figs. 6–9. To quantify the relative role of each energy flux in the redistribution of energy, we compute the spatial correlation between the local energy fluxes and energy tendency. Table 3 summarizes the results of these calculations. The notations $cr_{fr}$, $cr_{bc}$ stand for the correlation coefficient between WREY$_{BT}$ and EC$_{BT}^*$ and between WREY$_{BC}$ and EC$_{BT}^*$, respectively. The notations $cr_{f}$, $cr_{mx}$, and $cr_{bc1}$, respectively, stand for the correlation coefficient between each of WFORM, WREY$_{MX}$, WREY$_{BC1}$, and EC$_{BC}^*$.

Figure 6 shows the radial profiles of energy fluxes and energy rate for the barotropic mode at the time of vortex deformation. For all cases, the barotropic mode gains energy within the vortex core with the maximum gain at the vortex center. The correlation coefficients between EC$_{BT}^*$ and energy fluxes indicate that the gain of the barotropic energy is primarily supported by WREY$_{BT}$ flux, while a supportive role of the WREY$_{BC}$ flux is of less importance (Table 3). A similar result is observed at the time of the vortex breakdown for all cases, with the exception of the vortex with $R_{\text{max}} = 120$ km that releases its energy at the center and receives it at around the distance of 90 km away from the center (Fig. 7). WREY$_{BT}$ flux once again dominates the development of EC$_{BT}^*$, while WREY$_{BC}$ flux resists it (Table 3).

The shape of the EC$_{BC}^*$ curve at the time of vortex deformation indicates almost uniform energy loss within the vortex core (Fig. 8). All cases show that the WREY$_{MX}$...
and WFORM fluxes correlate positively with EC_BC*, whereas the WREY_BC1 has a negative correlation with EC_BC* (Table 2). For the cases of vortices with $R_{max} = 110$ and 120 km, the correlation of WFORM and EC_BC* is the largest (0.8825 and 0.9209, respectively), meaning that this flux dominates the development of EC_BC*.

On the contrary, the correlation coefficients for the case of vortex with $R_{max} = 920$ km show the prevailing role of WREY_MX (0.7982) in the modification of the baroclinic energy, while the change of the baroclinic energy of the vortex with $R_{max} = 100$ km is set by both WFORM (0.6493) and WREY_MX (0.6269).

At the time of the vortex breakdown, except for the case of the vortex with $R_{max} = 120$ km, the shape of the EC_BC* profile is set by the WREY_MX flux, as can be seen by comparing the blue and green lines in Fig. 9, and also by the correlation coefficient in Table 2. The profile of EC_BC* exhibits a sharp drop inside the $r = R_{max}$ for the cases of vortices with $R_{max} = 110$ and 120 km; the development of similar local minima near $R_{max}$ for smaller vortices with $R_{max} = 90$ and 100 km is also present (Fig. 9). This is consistent with the definition of vortex breakdown as a rupture of the vortex core with the outflow of energy to the vortex exterior.

Comparing the results from the globally integrated and local energy analyses, we note that the distributions of the local energy tendency and energy fluxes for the barotropic mode show the expected effect of the dominance of local WREY_BT flux in changing the barotropic energy. The results of computations for the baroclinic mode can be counterintuitive: the leading role of the globally integrated flux of wave buoyancy (WFORM) at the time of vortex deformation is not always consistent with the same flux locally (see the cases of vortices with $R_{max} = 90$ and 100 km). During the time of vortex breakdown, the predominant role of the local WREY_MX flux for the case of vortex with $R_{max} = 90$ km goes at odds with the secondary role of the globally integrated WREY_MX flux. Additionally, as opposed to the globally integrated analysis, the impact of the local WREY_MX in the case of vortex with $R_{max} = 120$ km is also considerable. It suffices to say that the local energy analysis can provide a deeper and more complete view of the mechanism of instability.

4. Transport
   a. Overview of kinematics

In this study, we use a Lagrangian approach to characterize the transport by vortices. The equations governing the trajectories of neutrally buoyant Lagrangian particles are

$$\frac{dx(t)}{dt} = u(x, t), \quad x(t_0) = X, \quad t \geq t_0, \quad (5)$$
where $\mathbf{x}(t)$ is the vector of particle positions at time $t$, $\mathbf{X}$ is a vector of particle positions at the initial time $t = t_0$, and $\mathbf{u}$ is the geostrophic velocity. This is a set of non-autonomous, ordinary differential equations. Since the geostrophic velocity may be expressed through the streamfunction we can rewrite the system of (5) as

$$\frac{dx_i}{dt} = -\frac{\partial \psi_i(x,y,t)}{\partial y}, \quad \frac{dy_i}{dt} = \frac{\partial \psi_i(x,y,t)}{\partial x},$$  \hspace{1cm} (6)

where $i$ is a number of the layer, and $\psi_i$ is the corresponding streamfunction. This system represents the so-called Hamiltonian system, where the streamfunction is the Hamiltonian. The fourth-order Runge–Kutta method is used to solve (6) with the streamfunction spatial derivatives calculated at each particle location using linear interpolation. The application of higher-order interpolation does not change the final position of the particles but does slow down the model integration considerably.

\textbf{b. Mixing}

Two methods are typically invoked in quantifying mixing by turbulent flows: the advection of tracer concentration by turbulent velocities and computation of the stretching rate between two nearby trajectories of Lagrangian particles advected by the velocity field. While both methods have their own advantages and disadvantages, the advection of tracer concentration requires the introduction of some type of diffusion. The apparent choice of diffusion largely remains uncertain, not to mention the fact that the mixing by vortical flows

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$R_{\text{max}}$ (km) & Process & $c_{\text{tr}}$ & $c_{\text{bc}}$ & $c_{\text{f}}$ & $c_{\text{mix}}$ & $c_{\text{bc1}}$ \\
\hline
90 & Vortex deformation & 0.9984 & 0.8601 & 0.4637 & 0.8093 & -0.8674 \\
90 & Vortex breakdown & 0.9831 & -0.8495 & 0.0963 & 0.9812 & 0.1904 \\
100 & Vortex deformation & 0.9946 & 0.8044 & 0.6493 & 0.6269 & -0.8726 \\
100 & Vortex breakdown & 0.9953 & -0.6373 & -0.9409 & 0.9516 & 0.6337 \\
110 & Vortex deformation & 0.9824 & 0.7692 & 0.8825 & 0.4109 & -0.8602 \\
110 & Vortex breakdown & 0.9896 & -0.3019 & -0.7789 & 0.8369 & 0.8798 \\
120 & Vortex deformation & 0.9652 & 0.7288 & 0.9209 & 0.4189 & -0.8798 \\
120 & Vortex breakdown & 0.9667 & -0.3245 & 0.9628 & 0.6752 & -0.3534 \\
\hline
\end{tabular}
\caption{Contribution of local energy fluxes.}
\end{table}
strongly depends on it (Rhines and Young 1983; Flohr and Vassilicos 1997). Furthermore, numerous studies have shown that the effective turbulent friction coefficient is spatially inhomogeneous and anisotropic; the models usually assume the opposite. The stretching rate of two Lagrangian particle trajectories can be measured either by FTLE or FSLE; however, the latter is ill posed and produces spurious ridges in its maps (Karrasch and Haller 2013). We therefore use the FTLE metric to characterize mixing.

The FTLE can be introduced (see, e.g., Haller 2001) by noting that the system of (5) admits a flow map \( F_{t_0}^{t} (x_0) := x(t; t_0, x_0) \), where \( x(t; t_0, x_0) \) is a trajectory that starts at \( x = x_0 \) at time \( t = t_0 \). We then can define the Cauchy–Green strain tensor as

\[
C_{t_0}^t (x_0) = [\nabla F_{t_0}^t (x_0)]^T [\nabla F_{t_0}^t (x_0)].
\]

The tensor \( C_{t_0}^t (x_0) \) is positive definite and symmetric, so it admits two positive real eigenvalues \( \lambda_{\min} [C_{t_0}^t (x_0)] \) and \( \lambda_{\max} [C_{t_0}^t (x_0)] \) and corresponding eigenvectors. FTLE is then defined as

\[
\sigma_{t_0}^t (x_0) = \frac{1}{2(t - t_0)} \ln \lambda_{\max} [C_{t_0}^t (x_0)].
\]

The FTLE maps in the upper layer for stable and unstable vortices are shown in Fig. 10. We show the maps only at day 100 because the patterns at earlier times do not provide any additional information. The highest values of FTLE for stable vortices are observed within the ring region that is almost continuous for \( R_{\max} = 60 \) km and has a volute structure for the vortex with \( R_{\max} = 80 \) km. In contrast, the cases of unstable vortices with \( R_{\max} = 100 \) and 120 km show the development of additional eddies and filaments connecting these eddies and indicating an increasingly complex picture of the mixing pattern. Additionally, we also observe the wavy patterns of the FTLE field in the close proximity of the vortex as well as outside of it.

To separate the role of the primary vortex from waves and secondary vortices, we run an idealized experiment in which the FTLE map is computed from the synthetic flow produced by the initial vortical profile moving along the same trajectory as in the full experiment. As shown in Fig. 11, the wavy patterns in the near and far

![Image](https://example.com/image.png)
fields are completely absent. However, the bigger vortices also have trailing spirals, although the spirals are broader than in the case of the full experiment.

Lower-layer FTLE maps do not show any presence of a coherent structure (Fig. 12). Instead, the wavelike pattern is observed for the stable vortical flows ($R_{\text{max}} = 60$ and $80\,\text{km}$), while the development of irregular filaments is seen for unstable cases ($R_{\text{max}} = 100$ and $120\,\text{km}$). These patterns indicate the presence of significant mixing in the lower layer associated with asymmetric waves that are forced by motions in the upper layer through quasigeostrophic dynamics.

To conclude, we discuss the differences between the structure of FTLE on the $f$ and $b$ planes based on the considerations derived from dynamical system theory. Consider a frictionless, $f$-plane case when an axisymmetric vortex is a stationary solution of the equations of motion and it remains so for all time. Such a flow is known to be not chaotic (Ottino 1989) with FTLE maps having perfectly continuous ring structure (not shown). The introduction of the $\beta$ effect makes the initial profile not a stationary solution of the governing equations. The dynamical system governing the evolution of particle trajectories has a right-hand side that depends on time and is, therefore, exposed to some degree of chaoticity. The vortex with $R_{\text{max}} = 60\,\text{km}$ is weakly affected by the $\beta$ effect, and so the spiraling within the ring region is insignificant (Fig. 10a). As the radius of the vortex becomes larger, more intense radiation leads to higher variability of the streamfunction; the dynamical system becomes more chaotic, and the volute structure of the FTLE ring becomes more manifest.

c. Large-scale transport

In this section, we quantify large-scale meridional transport produced by stable and unstable vortices as the change in time of the gradient of a specified tracer. The tracer distribution is represented by Lagrangian particles as follows: The domain is divided into $N \times N$ ($N = 40$) square cells, each of which has meridional $k (k = 1, \ldots, N)$ and zonal $l (l = 1, \ldots, N)$ indices. At initial time, 40 particles are released into each cell, and each $i$th particle within a particular $k/l$th cell is assigned a marker value $c_i = k, k = 1, \ldots, N$. We then define the tracer concentration

![Fig. 11. FTLE maps of idealized experiment at day 100 for the vortices with $R_{\text{max}} = (a) 60$, (b) 80, (c) 100, and (d) 120 km.](image-url)
$C_{kl}(t)$ in each $k$th square cell at time $t$ as an average marker value of all particles within this cell:

$$C_{kl}(t) = \frac{\sum c_i n_i}{\sum n_i},$$

where $n_i$ is a number of particles with marker value of $c_i$, and the summation is taken over the number of particles within a particular $k$th cell at time $t$. The initial distribution of $C_{kl}$ is equal to $k (C_{kl} = k, k = 1, \ldots, N)$ and is equivalent to the distribution of the tracer with a constant meridional gradient. An example of a composite map consisting of $C_{kl}$ values of each subdomain at day 100 for stable ($R_{max} = 60$ km) and unstable ($R_{max} = 120$ km) vortices is shown in Fig. 13. The initial gradient is clearly modified in both upper and lower layers as a result of vortex propagation and disintegration. We quantify the large-scale meridional transport at any arbitrary time $t$ as the difference between a zonally averaged tracer concentration at time $t$ and initial time, that is, $\sum [C_{kl}(t) - C_{kl}(0)]$.

As in the previous section, we preform full and idealized experiments to distinguish the relative role of vortex and asymmetric motions. The results of calculations are shown in Fig. 14. The general trend present in all cases in the upper layer is the increase of the average concentration change in the wake of the moving vortex and its decrease ahead of the vortex. In the case when the vortex is small ($R_{max} = 60$ km) and wave activity is weak, the upper-layer average concentration change in the full and idealized experiments bear little differences. However, for the larger vortex corresponding to stronger radiation ($R_{max} = 80$ km), the magnitude of the average concentration change behind the vortex becomes smaller while the area swept by the vortex-induced flow becomes wider. The instability of the vortices gives rise to even stronger differences between full and idealized experiments. For example, pinched off vorticity patches in the full simulation cause either a considerable reduction of the average concentration between $y = 0$ and 300 km ($R_{max} = 100$ km) or the change of its sign at around 600 km ($R_{max} = 120$ km). The dips in the shape of the average concentration profile in idealized experiments with large
vortices ($R_{\text{max}} = 100$ and 120 km) are because of the looping trajectories of these vortices.

The structure of the average concentration change in the lower layer is characterized by two distinct features: First, the magnitude in the lower layer is roughly half of that in the upper layer. Second, the distribution for the unstable cases is more irregular than the distribution for the stable ones. Since FTLE maps and tracer concentration field in the lower layer show no presence of a coherent structure, we conclude that this large-scale transport is entirely caused by the waves and turbulent wavelike eddies excited when the vortex breaks apart.

5. Summary and conclusions

Motivated by the significance of coherent vortices as essential elements of the ocean circulation, we have studied their stability and transport on the $\beta$ plane using the two-layer, quasigeostrophic model.

Globally integrated and local energy balance allows the examination of energy exchanges between the basic-state flow (vortex) and perturbations (waves). We find that the time evolution of globally integrated energy exhibits two significant transformations: one associated with the deformation of the vortex and another with its fragmentation. At the time of the first minimum of energy change, the distorted vortex is able to regain its shape without spawning further eddies; the breakdown of the vortex occurs during the second significant energy loss and, unlike on the $f$ plane, both baroclinic and barotropic instabilities can be the cause. Analyzing the energy balance locally, we found that the maximum barotropic energy gain occurs at the vortex center and the baroclinic energy loss takes place either uniformly within the core when the vortex deforms or decreases rapidly just before $R_{\text{max}}$ while disintegrating. The former explains the vortex breakdown as a fissure of the vortex core. It is also interesting that the results of local energy balance can give different insight into the problem of instability than globally integrated energy balance.

We also compare mixing induced by stable and unstable vortical flows. Utilizing FTLE maps, we have shown that the most active mixing in the upper layer occurs within the
ring region, which can be either continuous or volute depending on the strength of the wave radiation. During the vortex breakdown, the ring region is destroyed and several FTLE filaments connect the main vortex with pinched off smaller vortices. This indicates the exchange of water within and outside of the vortex core as a result of instability. As revealed by wavelike FTLE patterns, a clear signature of wave-induced mixing is observed in the far field. We also found that the lower-layer FTLE patterns for the cases of stable vortices have a wavy form, while the unstable cases show irregular filaments of FTLE.

Finally, both stable and unstable vortices affect the large-scale distribution of a tracer. We claim that these impacts are both due to the vortex propagation and wave radiation and, in the case of unstable vortices, to the vortex fragmentation. The upper-layer transport is dominated by coherent structure with the contribution of waves being less important. The instability of the vortices results in the reduction of particle concentration at some latitudes. In the lower layer, the transport is entirely produced by waves and wavelike eddies. Although the change in the large-scale tracer gradient is weaker than in the upper layer, this effect is still substantial and indicates an overall importance of the surface-intensified vortices for the material transport in the deep ocean.

The findings of this paper emphasize a significant role of the \( \beta \) effect and stratification in modifying stability as well as transport properties of oceanic vortices compared to the \( f \)-plane and barotropic studies. The newly found effects should be reassessed in more realistic settings with multiple vortices and in the presence of the background flow.

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**APPENDIX**

**Energy Balance**

We first derive dynamical balance for barotropic and baroclinic modes. Following Reznik and Dewar (1994), we switch to the system of reference moving with the vortex center \( \bar{x} = x - X(t) \), \( \bar{y} = y - Y(t) \), where \( \sim \) denotes the new system of reference, and \( X(t) \) and \( Y(t) \) indicate the center of vortex defined as an extremum of the baroclinic potential vorticity field. The modified equations for barotropic and baroclinic modes in the new reference frame are as follows:

\[
\frac{\partial q_{BT}}{\partial t} + \nabla \cdot (\mathbf{q}_{BT}) + \alpha T \nabla^2 \mathbf{q}_{BC} + \beta \frac{\partial q_{BT}}{\partial x} - U(t) \frac{\partial q_{BT}}{\partial x} - V(t) \frac{\partial q_{BT}}{\partial y} = D_{BT},
\]

with

\[
\frac{\partial q_{BT}}{\partial t} + \nabla \cdot (\mathbf{q}_{BT}) + \alpha T \nabla^2 \mathbf{q}_{BC} + \beta \frac{\partial q_{BT}}{\partial x} - U(t) \frac{\partial q_{BT}}{\partial x} - V(t) \frac{\partial q_{BT}}{\partial y} = D_{BT},
\]

and

\[
\frac{\partial q_{BT}}{\partial t} + \nabla \cdot (\mathbf{q}_{BT}) + \alpha T \nabla^2 \mathbf{q}_{BC} + \beta \frac{\partial q_{BT}}{\partial x} - U(t) \frac{\partial q_{BT}}{\partial x} - V(t) \frac{\partial q_{BT}}{\partial y} = D_{BT},
\]

(A1)
\[ \frac{\partial q_{BC}}{\partial t} + J(\psi_{BT}, q_{BC}) + J(\psi_{BC}, q_{BT}) + \alpha_f J(\psi_{BC}, q_{BC}) + \beta \frac{\partial \psi_{BC}}{\partial x} \bigg|_X - U(t) \frac{\partial q_{BC}}{\partial x} - V(t) \frac{\partial q_{BC}}{\partial y} = D_{BC}, \]

where
\[
q_{BT} = \nabla^2 \psi_{BT} - \frac{H_1^2 + H_2^2}{H_1 H_2} \psi_{BT},
\]
\[
q_{BC} = \nabla^2 \psi_{BC} - (S_1 + S_2)(\psi_1 - \psi_2)
\]
\[= \nabla^2 \psi_{BC} - (S_1 + S_2)\psi_{BC} = \xi_{BC} - (S_1 + S_2)\psi_{BC}, \]
\[\alpha_T = \frac{H_1^2 H_2}{(H_1 + H_2)^2}, \]
\[\alpha_C = \frac{H_2^2 - H_1^2}{(H_1 + H_2)^2}. \]

\[
\frac{\partial \langle q_{BT} \rangle}{\partial t} = -\langle J(\psi_{BT}', q_{BT}') \rangle - \alpha_T \langle J(\psi_{BC}', q_{BC}') \rangle - \beta \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) \langle \psi_{BT} \cos \theta \rangle + U(t) \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) \langle q_{BT} \cos \theta \rangle + V(t) \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) \langle q_{BT} \sin \theta \rangle + \langle D_{BT} \rangle. \]

Analogously, we get the budget for the baroclinic mode:
\[
\frac{\partial \langle q_{BC} \rangle}{\partial t} = -\langle J(\psi_{BC}', q_{BC}') \rangle - \langle J(\psi_{BC}', q_{BC}') \rangle - \alpha_C \langle J(\psi_{BC}', q_{BC}') \rangle - \beta \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) \langle \psi_{BC} \cos \theta \rangle + U(t) \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) \langle q_{BC} \cos \theta \rangle + V(t) \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) \langle q_{BC} \sin \theta \rangle + \langle D_{BC} \rangle. \]

The energy balance is derived by multiplying the (A3) and (A4) by azimuthally averaged barotropic and baroclinic streamfunctions, respectively, and then taking the integral of both right- and left-hand sides over the area of the domain.

\[
\int_0^\infty \int_0^\infty \frac{\partial \langle E_{BT} \rangle}{\partial t} r \, dr = \int_0^\infty \left\{ \frac{1}{\alpha_T} \langle \psi_{BT}' \rangle \langle J(\psi_{BT}', q_{BT}') \rangle + \langle \psi_{BT}' \rangle \langle J(\psi_{BC}', q_{BC}') \rangle + \frac{1}{\alpha_T} \langle \psi_{BT}' \rangle \beta \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) \langle \psi_{BT} \cos \theta \rangle \right. \]
\[\left. - \langle \psi_{BT}' \rangle \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) [U(t) \langle q_{BT} \cos \theta \rangle + V(t) \langle q_{BT} \sin \theta \rangle] + \frac{1}{\alpha_T} \langle \psi_{BT}' \rangle \langle D_{BT} \rangle \right\} r \, dr, \quad \text{and} \]

\[
\int_0^\infty \int_0^\infty \frac{\partial \langle E_{BC} \rangle}{\partial t} r \, dr = \int_0^\infty \left\{ \langle \psi_{BC}' \rangle \langle J(\psi_{BC}', \xi_{BC}') \rangle + \langle \psi_{BC}' \rangle \langle J(\psi_{BC}', q_{BC}') \rangle - (S_1 + S_2) \langle \psi_{BC}' \rangle \langle J(\psi_{BC}', q_{BC}') \rangle + \alpha_C \langle \psi_{BC}' \rangle \beta \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) \langle \psi_{BC} \cos \theta \rangle - \langle \psi_{BC}' \rangle \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) [U(t) \langle q_{BC} \cos \theta \rangle \right. \]
\[\left. + V(t) \langle q_{BC} \sin \theta \rangle] + \langle \psi_{BC}' \rangle \langle D_{BC} \rangle \right\} r \, dr, \]

where