Modulation of Small-Scale Superinertial Internal Waves by Near-Inertial Internal Waves

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ABSTRACT

Dynamics of small-scale (<10 km) superinertial internal waves (SSIWs) of intense vertical motion are investigated theoretically and numerically. It is shown that near-inertial internal waves (NIWs) have a pronounced influence on modulation of SSIW strength. In convergence zones of NIWs, energy flux of SSIWs converge and energy is transferred from NIWs to SSIWs, leading to rapid growth of SSIWs. The opposite occurs when SSIWs enter divergence zones of NIWs. The underlying dynamics can be understood in terms of wave action conservation of SSIWs in the presence of background NIWs. The validity of the theoretical finding is verified using realistic high-resolution numerical simulations in the Gulf of Mexico. The results reveal significantly stronger small-scale superinertial vertical motions in convergence zones of NIWs than in divergence zones. By removing near-inertial wind forcing, model simulations with identical resolution show a substantial decrease in the small-scale superinertial vertical motions associated with the suppression of NIWs. Therefore, these numerical simulations support the theoretical finding of SSIW–NIW interaction.

1. Introduction

Near-inertial internal waves (NIWs) are a ubiquitous feature in the ocean (Garrett 2001). As a natural resonant frequency of fluids on a rotating planet, NIWs are efficiently generated by time-varying wind stresses associated with the passage of atmospheric fronts, tropical cyclones, and synoptic storms (D’Asaro 1985). Global estimates of the wind work on NIWs range from 0.5 to 1.4 TW (e.g., Alford 2003; Jiang et al. 2005; Rimac et al. 2013), comparable to the wind work on large-scale ocean general circulations (Wunsch 1998).

NIWs are of central importance to a variety of ocean processes. Numerical model studies reveal an increase of vertical velocity variance if the temporal resolution of wind fields is high enough to resolve the near-inertial frequency (Danioux et al. 2008; Cardona and Bracco 2012). This increase is evident throughout the water column and is even more pronounced in the deep ocean (Danioux et al. 2008). The deep extension of near-inertial vertical velocity is mainly ascribed to the refraction of NIWs by the vorticity of mesoscale and submesoscale eddies, which significantly shortens the horizontal scale of NIWs and accelerates their downward propagation (Gill 1984; Young and Ben Jelloul 1997; Lee and Niiler 1998; Klein and Smith 2001; Klein et al. 2004). Furthermore, previous theoretical studies suggest that NIWs can transfer energy to their harmonics through triad wave resonance (Niwa and Hibiya 1997; Danioux and Klein 2008; Danioux et al. 2011). This is confirmed by simulated vertical velocity frequency spectra that exhibit pronounced peaks not only at the inertial frequency but also its harmonic frequencies (e.g., Niwa and Hibiya 1997; Komori et al. 2008; Danioux et al. 2011). As the horizontal/vertical aspect ratio of internal waves becomes smaller with the increasing wave...
frequency, this resonance mechanism provides an efficient way in intensifying small-scale vertical velocity. A recent high-resolution numerical model study by Zhong and Bracco (2013) reported energetic small-scale (<10 km) vertical velocity in the deep Gulf of Mexico. Their appearance tends to be collocated with the energetic NIWs, suggesting a dynamic linkage between them. However, their simulated small-scale vertical velocity variance does not only reside in the inertial frequency and its harmonic frequencies but also spreads over all the superinertial frequencies. This broadband feature cannot be explained by the triad wave resonance proposed in previous studies (Niiwa and Hibiya 1997; Daniaux and Klein 2008). Instead, it implies that the underlying dynamics might be understood from the perspective of wave–mean flow interactions, where the “wave” and “mean flow” correspond to the small-scale superinertial internal waves (SSIWs) and relatively large-scale NIWs, respectively. Unlike the triad wave resonance where NIWs can only interact with their harmonics, the wave–mean flow interactions allow NIWs to exert an influence on SSIWs with a broad range of frequencies. In this study, we analyze the validity of this conjecture based on theoretical and numerical models.

The paper is organized as follows. An idealized high-resolution numerical simulation designed to illustrate the relationship between SSIWs and NIWs is first presented in section 2. Theoretical solutions governing SSIW–NIW interactions are developed in section 3 and are validated against the idealized numerical simulation. A further validation of the theoretical results using a realistic simulation of circulation in the Gulf of Mexico is made in section 4. Conclusions and a discussion are finally given in section 5.

2. An idealized high-resolution numerical simulation

a. Model description and experiment design

To explore the underlying dynamics for SSIW–NIW interaction, an idealized high-resolution numerical experiment is performed using the Regional Ocean Modeling System (ROMS), which is a free surface 3D primitive equation model based on hydrostatic and Boussinesq approximations (Shchepetkin and McWilliams 2005). The model is configured over a 20° × 20° domain with a uniform depth of 2000 m. Fifty vertical layers are used, with 19 layers concentrated in the upper 100 m. The horizontal grid size is set at 1 km × 1 km. The nonlocal K-profile parameterization (Large et al. 1994) is used to parameterize vertical mixing, but no horizontal eddy viscosity and diffusivity are used since at 1-km resolution the model should be able to explicitly resolve mesoscale and submesoscale eddies. The horizontal diffusivity and viscosity are thus taken as their molecular values. Finally, a radiation boundary condition is used at lateral boundaries of the model domain.

It should be noted that the hydrostatic approximation made by ROMS tends to overestimate the vertical velocity of small-scale flow because of its much reduced horizontal/vertical aspect ratio (Vitoresek and Fringer 2011). This tendency is counterbalanced to some extent by numerical dispersion that mimics the missing physical dispersion due to nonhydrostaticity. In fact, previous numerical studies indicated that the numerical dispersion can be tuned to replicate the nonhydrostatic dispersion not resolved in a hydrostatic model (Shuto 1991; Burwell et al. 2007). Therefore, the hydrostatic approximation is acceptable for our qualitative analysis here.

The simulation starts with a quiescent and horizontally homogeneous ocean, in which the temperature and salinity profiles are representative of those in the Gulf of Mexico during summer (Figs. 1a,b). This stratification is characterized by a sharp thermocline roughly at 65 m with a maximal buoyancy frequency of 2.1 × 10^{-2} rad s^{-1} (Fig. 1c). An idealized west-to-east moving hurricane at a constant translation speed of 7 m s^{-1} is applied at the center of the model domain as wind forcing. As shown in the following section 2b, both energetic SSIWs and NIWs emerge in the wake of a hurricane. This provides an opportunity to analyze SSIW–NIW interactions. The center of the hurricane is initially located at the middle of the western boundary and moves to the eastern boundary after 4 days and 1 h (4d1h). The wind field associated with the hurricane is constructed following Price (1983):

\[
U_\theta = \begin{cases} 
U_M r/R & \text{if } r < R \\
U_M (1.2 - 0.2r/R) & \text{if } R \leq r \leq 6R, \text{ and} \\
0 & \text{if } r > 6R
\end{cases}
\]

\[
U_r = -0.3U_\theta,
\]

where \(U_\theta\) and \(U_r\) are tangential and radial wind components, \(r\) is the radial distance from the hurricane center, \(U_M = 52\text{ m s}^{-1}\), and \(R = 40\text{ km}\). Outside the hurricane, the surface heat flux and freshwater flux are set to zero. We refer to this experiment as the idealized hurricane simulation (IHS) and will use it to examine the influence of NIWs on the evolution of SSIWs.

b. IHS result

Both large-scale and small-scale vertical motions emerge in the wake of the idealized hurricane (Fig. 2). Here the large-scale (small-scale) signals are attained by a low-pass (high-pass) filter with a cutoff wavelength of 30 km (15 km). The large-scale vertical velocity is primarily associated with the hurricane-generated NIWs, as
evidenced by the pronounced peak around the Coriolis frequency $f$ in its frequency spectrum (Fig. 3a). There is a marked cross-track asymmetry for the large-scale vertical velocity with stronger amplitude on the right of the moving hurricane (Fig. 3a). This is mainly because the wind stress vector rotates anticyclonically on the right of the track and thus is able to resonate with the inertial oscillations there (Price 1981, 1983). Far away from the hurricane track, the vertical structure of large-scale vertical velocity is characterized by the first baroclinic mode (Fig. 2c). The projection coefficient of the first baroclinic mode is an order of magnitude larger than those of higher modes. The dominance of first baroclinic mode in the far field is mainly due to its fastest group velocity. The vertical structure of large-scale vertical velocity becomes more complex near the hurricane track where both the first and second baroclinic modes make important contributions.

![Fig. 1. The initial profiles in IHS for (a) potential temperature, (b) salinity, and (c) buoyancy frequency.](image)

![Fig. 2. (a) Vertical section of large-scale (>30 km) vertical velocity (m s$^{-1}$) along the longitude 7.4° on 6d0h in IHS and (c) its projection coefficients on different vertical modes. (b),(d) As in (a) and (c), but for the small-scale (<15 km) vertical velocity. The dotted vertical line denotes the center of the hurricane.](image)
The small-scale vertical velocity signals are dominated by a few isolated wavelike fronts associated with vigorous vertical velocity of $O(0.01) \text{m s}^{-2}$ (Fig. 2b). The wavelength of the fronts is on the order of 10 km, and their vertical structure is well represented by the first baroclinic mode (Fig. 2d). The frequency spectrum of small-scale vertical velocity exhibits a pronounced peak at frequencies much higher than $f$ (Fig. 3b), suggesting that the fronts of strong small-scale vertical velocity are SSIWs. Indeed, their wavenumber–frequency relationship agrees well with the short internal gravity wave dispersion relationship, $\omega = c_1 k$, where $c_1 = 2 \text{m s}^{-1}$ is the gravity wave speed of the first baroclinic mode (Fig. 4).

There are several possible mechanisms responsible for the generation of the SSIWs in IHS. One possible candidate is disintegration of large-scale NIWs into SSIWs because of nonlinear steepening of NIWs (Vlasenko et al. 2005). Other candidates include direct small-scale superinertial wind forcing by the hurricane, geostrophic adjustment, or even perturbations resulting from the numerical discretization. However, the focus of this study is not on the generation mechanism of SSIWs. Instead, we are interested in how SSIWs are energized by NIWs after their generation. To illustrate the evolution of SSIWs in the presence of NIWs, we show a series of zoom-in snapshots following the strongest SSIW front in IHS (Fig. 5). It is found that the SSIW front propagates at a speed close to $c_1$ and its phase contour is roughly aligned with that of NIWs. When entering a convergence (downwelling) zone of NIWs, the SSIW front is intensified rapidly before it becomes saturated. In contrast, the intensity of the SSIW front rapidly decreases when it goes into a divergence (upwelling) zone of NIWs. This suggests that NIWs are able to modulate the strength of SSIWs. We will examine the underlying dynamics of this modulation in the next section.

3. Analytical analysis

In this section, we present analytical analyses to understand the mechanism governing the SSIW–NIW interaction revealed by IHS described above. The fact that both the SSIWs and NIWs project strongly onto the first baroclinic mode allows us to reduce the complexity of the problem by considering only one vertical mode. In the...
following, we first present the analysis in a reduced gravity model framework represented by the first baroclinic mode and then in a more generalized dynamic model framework.

a. SSIW–NIW interaction in a reduced gravity model

The linearized equations for SSIWs in the presence of background NIWs can be expressed as

\[
\frac{Du}{Dt} = fu - g \frac{\partial \eta}{\partial x} - u \frac{\partial U}{\partial x} - v \frac{\partial U}{\partial y},
\]

(1a)

\[
\frac{Dv}{Dt} = -fu - g \frac{\partial \eta}{\partial y} - u \frac{\partial V}{\partial x} - v \frac{\partial V}{\partial y},
\]

(1b)

\[
\frac{D\eta}{Dt} + \frac{\partial (H + Z)u}{\partial x} + \frac{\partial (H + Z)v}{\partial y} = -\eta \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right),
\]

(1c)

where \( g' \) is the reduced gravity; \( H \) is the undisturbed layer thickness; \( U(u), V(v) \), and \( Z(\eta) \) the horizontal currents and interface elevation produced by NIWs (SSIWs); and \( D/Dt = \partial/\partial t + U\partial/\partial x + V\partial/\partial y \). Here the nonlinear terms associated with SSIWs have been dropped, as the focus is on the analysis of the modulation of SSIWs by NIWs. We use \( g' = 0.01 \text{ m s}^{-2} \) and \( H = 400 \text{ m} \), so that the gravity wave speed of the reduced gravity model matches that of the first baroclinic mode in IHS.

The beta effect is unlikely to play a major role here because of the small horizontal scale of the SSIWs. Therefore, an \( f \)-plane approximation is applied. Furthermore, the system is isotropic horizontally under \( f \)-plane approximation, so that the Cartesian coordinate can be rotated to make the \( y \) axis parallel with the SSIW front. As both the SSIW and NIW exhibit much less variation in the alongfront direction than in the cross-front direction (Fig. 5), the system at hand can be reduced to a unidirectional wave equation by setting all the \( y \) derivatives to zero. Note that the neglect of \( y \) derivatives does not cause a loss of generality for the following derivations. The conclusions in this section are also valid in a more generalized case where the phase contours of SSIWs do not necessarily align with those of NIWs (see appendix A for details).

To proceed with the perturbation analyses, it is essential to introduce nondimensional parameters. Let \( L_N \) and \( U^* \) represent the horizontal scale and horizontal...
velocity magnitude of NIWs and $L_S$ and $u^*$ for SSIWs. A reasonable measure of a wave’s horizontal scale should be the reciprocal of horizontal wavenumber, that is, horizontal wavelength divided by $2\pi$. In IHS, the horizontal wavelength of NIWs away from the hurricane’s track ranges from 100 to 250 km (Fig. 2a). This is comparable to $2\pi L_D$, where $L_D \approx 30$ km is the deformation radius of the first baroclinic mode. Therefore, it is reasonable to choose $L_N = L_D$. We then use these characteristic scales to nondimensionalize both dependent and independent variables, denoted by primes, as follows:

$$\eta = \frac{c}{g} u^* \eta', \quad (2a)$$

$$Z = \frac{c}{g} U^* Z', \quad (2b)$$

$$(u, v) = u^* (u', v'), \quad (2c)$$

$$(U, V) = U^* (U', V'), \quad (2d)$$

$$x = L_S x', \quad (2e)$$

$$t = \frac{L_S}{c} t', \quad (2f)$$

where $c = \sqrt{g/H} = 2 \, \text{m/s}$ is the gravity wave speed. Note that $x$ is scaled based on SSIWs, so that any $x$-derivative term related to NIWs should be multiplied by a scaling factor $\varepsilon = L_S/L_D$ to account for the difference in the horizontal scales between NIWs and SSIWs. Substituting (2) into (1) yields the following nondimensionalized equations:

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} = e \nu - \eta \frac{\partial U}{\partial x}, \quad (3a)$$

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = -\varepsilon \nu - \delta \nu \frac{\partial V}{\partial x}, \quad \text{and} \quad (3b)$$

$$\frac{\partial \nu}{\partial t} + U \frac{\partial \nu}{\partial x} + (1 + \delta Z) \frac{\partial \nu}{\partial x} + \delta \eta \frac{\partial \nu}{\partial x} = -\eta \frac{\partial U}{\partial x}, \quad (3c)$$

where $X = \varepsilon x$ is a slowly changing coordinate and $\delta = U^*/c$ is a nondimensional parameter measuring the nonlinearity of NIWs. For the sake of neatness, we drop the primes for the nondimensional variables here and hereinafter. The nondimensional energy equation corresponding to (3) is given by

$$\frac{\partial e}{\partial t} + \frac{\partial U e}{\partial x} = -\frac{\partial F}{\partial x} + P_b + P_s, \quad (4)$$

where $e = (1 + \delta Z)(u^2 + v^2)/2 + \eta^2/2$ is the energy density of SSIWs, $F = (1 + \delta Z)u\eta$ is the energy flux of SSIWs due to pressure work, and $P_b (P_s)$ is the potential (kinetic) energy exchange between NIWs and SSIWs:

$$P_b = \frac{1}{2} \delta \varepsilon \eta \frac{\partial U}{\partial x}, \quad \text{and} \quad$$

$$P_s = -\varepsilon \delta (1 + \delta Z) \left( w u \frac{\partial U}{\partial x} + w v \frac{\partial V}{\partial x} \right).$$

The $\delta$ and $\varepsilon$ in (3) are two small ($\ll 1$) nondimensional parameters, and their relative magnitude can be measured as $\chi = \varepsilon/\delta$. Here we assume that $\chi$ is on the order of unity. In fact, $\chi$ ranges from 0.5 to 2 for $U^* = 0.05 \sim 0.2 \, \text{m/s}$, $c = 2 \, \text{m/s}$, $L_S = 10(2\pi)$ km, and $f = 6 \times 10^{-5} \, \text{s}^{-1}$. As the temporal and horizontal scales of SSIWs and NIWs are well separated, a multiple-scale method (e.g., Johnson 2005) is used to obtain the solutions to (3).

We first introduce the following fast and slow variables:

$$\frac{\partial \phi}{\partial t} = -\omega, \quad \frac{\partial \phi}{\partial x} = k, \quad T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t, \quad X = \varepsilon x, \quad (5)$$

where $\omega$ and $k$ are the local wave frequency and wavenumber of SSIWs. Here we assume that the SSIWs can be approximated as a plane wave locally, which is justified by the numerical solutions in IHS (Fig. 2d). The temporal and spatial derivatives can then be expanded, in virtue of the chain rule, as

$$\frac{\partial}{\partial t} = -\omega \frac{\partial}{\partial \phi} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2}, \quad (6a)$$

$$\frac{\partial}{\partial x} = k \frac{\partial}{\partial \phi} + \varepsilon \frac{\partial}{\partial X}. \quad (6b)$$

Expand the SSIWs in asymptotic series of $\varepsilon$:

$$u = \sum_{n=0}^{\infty} \varepsilon^n u_n, \quad (7a)$$

$$v = \sum_{n=0}^{\infty} \varepsilon^n v_n, \quad (7b)$$

$$\eta = \sum_{n=0}^{\infty} \varepsilon^n \eta_n, \quad \text{and} \quad (7c)$$

$$\omega = \sum_{n=0}^{\infty} \varepsilon^n \omega_n. \quad (7d)$$

Insert (6) and (7) into (3) and the $O(1)$ terms yield

$$-\omega \frac{\partial u_0}{\partial \phi} = -k \frac{\partial \eta_0}{\partial \phi}, \quad (8a)$$

$$-\omega \frac{\partial v_0}{\partial \phi} = 0, \quad \text{and} \quad (8b)$$

$$-\omega \frac{\partial \eta_0}{\partial \phi} + k \frac{\partial u_0}{\partial \phi} = 0. \quad (8c)$$

The solutions to (8) are
with the nondimensional dispersion relation \( \omega_0^2 = k^2 \) that corresponds to the dispersion relation of short internal gravity waves and is consistent with the numerical analysis of the SSIWs in IHS (Fig. 4). Without loss of any generality, we choose \( \omega_0 = k \).

The equations at \( O(\epsilon) \) are

\[
-\omega_0 \frac{\partial u_i}{\partial \phi} + k \frac{\partial \eta_i}{\partial \phi} = -\frac{\partial u_0}{\partial T_1} + \omega_1 \frac{\partial u_0}{\partial \phi} - \frac{\partial \eta_0}{\partial X} - \chi k U \frac{\partial u_0}{\partial \phi},
\]

\[\text{(10a)}\]

\[
-\omega_0 \frac{\partial v_i}{\partial \phi} = -u_0, \quad \text{and}
\]

\[\text{(10b)}\]

\[
-\omega_0 \frac{\partial \eta_i}{\partial \phi} + k \frac{\partial u_i}{\partial \phi} = -\frac{\partial \eta_0}{\partial T_1} + \omega_1 \frac{\partial \eta_0}{\partial \phi} - \frac{\partial u_0}{\partial X} - \chi k U \frac{\partial \eta_0}{\partial \phi} - \chi k Z \frac{\partial u_0}{\partial \phi},
\]

\[\text{(10c)}\]

Eliminating singularity requires that the following equations must be satisfied:

\[
\frac{\partial A_0}{\partial T_1} + \frac{\partial c_s^0 A_0}{\partial X} = 0, \quad \text{(11a)}
\]

\[
\omega_1 = \frac{\chi k}{U + \frac{1}{2} Z} \quad \text{(11b)}
\]

where \( c_s^0 = \frac{\partial \omega_0}{\partial k} = 1 \) is the group velocity of SSIWs at \( O(1) \).

Given (11), the solutions to (10) are

\[
u_1 = \text{Re}\left\{ -\frac{1}{k} A_0 e^{i \phi} \right\}, \quad \text{and} \quad \eta_1 = \text{Re}\left\{ A_0 e^{i \phi} \right\}.
\]

By incorporating \( \eta_i \) into \( \eta_0 \) and then substituting (9) and (12) into (4), we arrive at the equation for SSIW energy density. By averaging over one wavelength or wave period, the wave energy equation can be written as

\[
\left( \frac{\partial \langle \epsilon \rangle}{\partial T_1} + \epsilon \frac{\partial \langle \epsilon \rangle}{\partial T_2} + c_s \frac{\partial \langle \epsilon \rangle}{\partial X} \right) = \delta \sigma \langle \epsilon \rangle + O(\epsilon^2),
\]

\[\text{(13)}\]

where \( \langle \epsilon \rangle = |A_0|^2/2 + O(\epsilon^2) \) is the energy density of SSIWs averaged over one wavelength or wave period and

\[
c_s = \frac{\partial \omega_0}{\partial k} = \left( 1 + \frac{\delta}{2} Z \right) + \delta U, \quad \text{and} \quad \text{(14a)}
\]

\[
\sigma = -\frac{\delta}{2} \frac{\partial U}{\partial X} + \frac{1}{2} \frac{\partial Z}{\partial X}, \quad \text{and} \quad \text{(14b)}
\]

where \( c_s \) is the group velocity that determines the energy propagation of SSIWs. The term \( \delta U \) represents the Doppler shift by the horizontal velocity of NIWs, while the term \( \delta Z/2 \) represents the refraction effect due to the layer thickness fluctuations induced by NIWs. The coefficient \( \sigma \) is the growth rate determining the rate of change of \( \langle \epsilon \rangle \) in a reference frame following the SSIWs. The dynamics behind (14) can be better understood by introducing wave action density \( \langle \epsilon \rangle = \langle \epsilon \rangle/\omega_0 \), where \( \omega_0 = (1 + \delta Z/2)k \) is the intrinsic wave frequency (Bretherton and Garrett 1968). According to the ray tracing relation (Lighthill 1978), we have

\[
\frac{1}{\omega_0} \left( \frac{\partial \omega_0}{\partial T_1} + \epsilon \frac{\partial \omega_0}{\partial T_2} + c_s \frac{\partial \epsilon}{\partial X} \right) = -\frac{3}{2} \frac{\partial U}{\partial X} + O(\epsilon^2). \quad \text{(15)}
\]

Substituting (15) into (13) yields

\[
\frac{\partial \langle \epsilon \rangle}{\partial T_1} + \epsilon \frac{\partial \langle \epsilon \rangle}{\partial T_2} + c_s \frac{\partial \langle \epsilon \rangle}{\partial X} = O(\epsilon^2) \approx 0, \quad \text{(16)}
\]

which states that wave action of SSIWs is approximately conserved during their interactions with NIWs.

The validity of solutions (14)–(16) is demonstrated in appendix B by comparing them to numerical simulations of the reduced gravity model with all nonlinear terms included. With the aid of (16), the modulation dynamics of SSIWs by NIWs becomes quite clear. Following the SSIWs, \( \langle \epsilon \rangle \) is amplified in convergence (downwelling) zones of NIWs because of the converging wave action flux of SSIWs, as indicated by (14a) and (16). Furthermore, the converging currents of NIWs can cause an increase in \( k \) and thus \( \omega_0 \) by squeezing the phase contours of SSIWs. These two effects work in concert to result in a rapid amplification of \( \langle \epsilon \rangle \), and thus \( \langle \epsilon^2 \rangle = \omega_0^2 \langle \epsilon \rangle \). It should be noted that the energy of SSIWs is not conserved during their interactions with NIWs. In divergence zones of NIWs, the energy of SSIWs is increased because of the increased \( \omega_0 \). The opposite is true in divergence zones of NIWs. Dynamically, this energy change of SSIWs results from energy exchange between SSIWs and NIWs. As evidenced by (4) and (9), both the local potential and kinetic energy are transferred from NIWs to SSIWs in convergence zones of NIWs and vice versa.

The above analysis indicates that SSIWs are enhanced and damped in convergence and divergence regions of NIWs, respectively. However, this does not mean that NIWs have no net effects on the intensity of SSIWs. Imagine that there are two SSIWs with the same initial energy density \( \langle \epsilon_0 \rangle \): one occurs in a convergence region with a growth rate \( \sigma > 0 \) and the other arrives at a divergence region with a decay rate of \( -\sigma < 0 \). Then their total energy density, \( \langle \epsilon_0 \rangle \langle \epsilon^2 \rangle = \langle \epsilon_0 \rangle^2 \langle \epsilon^2 \rangle \), will increase with
time because $e^{\alpha t} + e^{-\alpha t}$ is an increasing function of $t$. This suggests that SSIWs in general tend to be more energetic in the presence of NIWs.

Wave action conservation of SSIWs in the presence of NIWs suggests that the modulation mechanism described above can be understood from the perspective of wave–mean flow interactions, except that here the “wave” and “mean flow” correspond to SSIWs and NIWs, respectively. This makes the modulation mechanism distinct from the previously proposed triad wave resonance mechanisms (Niwa and Hibiya 1997; Danioux and Klein 2008). Unlike the triad wave resonance where NIWs only interact with their harmonics, the modulation mechanism proposed here allows NIWs to interact with SSIWs in a broad range of frequencies.

b. SSIW–NIW interaction in a 3D primitive equation system

The above analysis based on the reduced gravity model suggests that the underlying dynamics for the modulation of SSIWs by NIWs can be understood in terms of wave action conservation of SSIWs in the presence of NIWs. In this section, we extend these solutions to a 3D primitive equation system.

The linearized equations for the SSIWs in the 3D primitive equation system are

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + W \frac{\partial u}{\partial z} = f v - \frac{\partial p}{\partial x} = \left( u \frac{\partial U}{\partial x} + w \frac{\partial U}{\partial z} \right),$$  \hspace{1cm} (17a)

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} + W \frac{\partial v}{\partial z} = -fu - \left( u \frac{\partial V}{\partial x} + w \frac{\partial V}{\partial z} \right),$$  \hspace{1cm} (17b)

$$0 = \frac{\partial p}{\partial z} + b,$$  \hspace{1cm} (17c)

$$\frac{\partial b}{\partial t} + U \frac{\partial b}{\partial x} + W \frac{\partial b}{\partial z} + wN^2 = -\left( u \frac{\partial B}{\partial x} + w \frac{\partial B}{\partial z} \right),$$  \hspace{1cm} (17d)

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$  \hspace{1cm} (17e)

where $u$ ($U$) is the 3D velocity, $b$ ($B$) is the buoyancy, and $p$ ($P$) is the density-normalized pressure associated with the SSIWs (NIWs). Similar to the derivations in section 3a, the coordinate has been rotated to make the $y$ axis parallel with the SSIW front so that all the $y$ derivatives can be dropped.

Approximating the SSIWs and NIWs by the first baroclinic mode yields

$$u = u_1(x,y,t) \frac{\sqrt{HF_1(z)}}{c_1}, \quad U = U_1(x,y,t) \frac{\sqrt{HF_1(z)}}{c_1},$$  \hspace{1cm} (18a)

$$v = v_1(x,y,t) \frac{\sqrt{HF_1(z)}}{c_1}, \quad V = V_1(x,y,t) \frac{\sqrt{HF_1(z)}}{c_1},$$  \hspace{1cm} (18b)

$$w = w_1(x,y,t) \sqrt{H}N_0 G_1(z), \quad W = W_1(x,y,t) \sqrt{H}N_0 G_1(z),$$  \hspace{1cm} (18c)

$$p = g' \eta_1(x,y,t) \frac{\sqrt{HF_1(z)}}{c_1}, \quad P = g' Z_1(x,y,t) \frac{\sqrt{HF_1(z)}}{c_1},$$  \hspace{1cm} (18d)

$$b = g' \eta_1(x,y,t) N^2 \frac{\sqrt{H}G_1(z)}{c_1}, \quad b = g' Z_1(x,y,t) N^2 \frac{\sqrt{H}G_1(z)}{c_1},$$  \hspace{1cm} (18e)

where $H$ is the water depth, $c_1$ is the gravity wave speed of the first baroclinic mode, $N_0 = c_1/H$ is the reference buoyancy frequency, $g' = c_1^2/H$ is the reduced gravity, and $G_1$ ($F_1$) is the eigenfunction for the first baroclinic mode of vertical (horizontal) velocity satisfying $\int_0^H G_1 G_1 N^2 \, dz = 1$ ($\int_0^H F_1 F_1 \, dz = c_1^2$). Here the scaling factors are chosen in such a way that

$$\int_{-H}^H \frac{1}{2} (u^2 + v^2) \, dz = \frac{H}{2} (u_1^2 + v_1^2),$$  \hspace{1cm} (19a)

$$\int_{-H}^H \frac{b^2}{2N^2} \, dz = \frac{1}{2} g' \eta_1^2.$$  \hspace{1cm} (19b)

Substituting (18) into (17), the governing equations for the first baroclinic mode of SSIWs in presence of the first baroclinic mode of NIWs become:

$$\frac{\partial u_1}{\partial t} + sU_1 \frac{\partial u_1}{\partial x} + \frac{s}{2} W_1 \frac{u_1}{H},$$  \hspace{1cm} (20a)

$$= fu_1 - g' \frac{\partial \eta_1}{\partial x} - s \left( u_1 \frac{\partial U_1}{\partial x} + \frac{1}{2} w_1 \frac{U_1}{H} \right),$$  \hspace{1cm} (20b)

$$\frac{\partial v_1}{\partial t} + sU_1 \frac{\partial v_1}{\partial x} + \frac{s}{2} W_1 \frac{v_1}{H},$$  \hspace{1cm} (20c)

$$= -fu_1 - s \left( u_1 \frac{\partial V_1}{\partial x} + \frac{1}{2} w_1 \frac{V_1}{H} \right),$$  \hspace{1cm} (20d)

$$\frac{\partial \eta_1}{\partial t} + sU_1 \frac{\partial \eta_1}{\partial x} + sW_1 \frac{\eta_1}{H} + w_1,$$  \hspace{1cm} (20e)

$$= -s \left( \frac{1}{2} w_1 \frac{\partial Z_1}{\partial x} + w_1 \frac{Z_1}{H} \right),$$  \hspace{1cm} (20f)

$$\frac{\partial w_1}{\partial x} - \frac{w_1}{H} = 0.$$  \hspace{1cm} (20g)
where $s$ is the nondimensionalized projection coefficient defined as $\sqrt{Hc^{-3}} [\mu F_1^2] dz$.

If stratification is uniform, that is, $N$ is a constant, $F_1$ will be a cosine function. In this case, $s$ will be exactly zero, so that SSIWs do not interact with NIWs. However, for a realistic oceanic setting where stratification is typically very strong in the upper several hundred meters and becomes much weaker in the deep ocean, the magnitude of $F_1$ is much stronger in the upper ocean according to the Wentzel–Kramers–Brillouin (WKB) theory, leading to a positive value of $s$. In IHS, $s$ is estimated to be 2.2 because of the sharp thermocline centered around 65 m (Fig. 1c).

Using the same nondimensionalized variables as in (2), the nondimensionalized form of (20) is

$$\frac{\partial u_1}{\partial t} + s\delta \left( U_1 \frac{\partial u_1}{\partial x} + \frac{1}{2} w_1 U_1 \right) = e v_1 - \frac{s}{\partial x} \left( e u_1 \frac{\partial U_1}{\partial x} + 1 \right) w_1 U_1, \quad (21a)$$

$$\frac{\partial v_1}{\partial t} + s\delta \left( U_1 \frac{\partial v_1}{\partial x} + \frac{1}{2} w_1 v_1 \right) = -e v_1 - \frac{s}{\partial x} \left( e u_1 \frac{\partial V_1}{\partial x} + 1 \right) w_1 V_1, \quad (21b)$$

$$\frac{\partial \eta_1}{\partial t} + s\delta \left( \frac{1}{2} U_1 \frac{\partial \eta_1}{\partial x} + e W_1 \eta_1 \right) + w_1 = -s\delta \left( e \frac{\partial Z_1}{\partial x} + 1 \right) w_1 Z_1, \quad (21c)$$

$$\frac{\partial u_1}{\partial x} - w_1 = 0. \quad (21d)$$

For the sake of neatness, we again drop primes for the nondimensional variables here and hereinafter.

The associated nondimensional energy equation for SSIWs is

$$\frac{\partial e_1}{\partial t} = P_{a,1} + P_{w,1} + P_{\eta,1} + P_{\nu,1} + P_{b,1}, \quad (22)$$

where

$$e_1 = \frac{1}{2} \left( u_1^2 + v_1^2 + \eta_1^2 \right),$$

$$P_{a,1} = -s\delta \left( \frac{\partial}{\partial x} U_1 \frac{u_1^2 + v_1^2}{2} + \frac{1}{2} U_1 \eta_1^2 \right),$$

$$P_{w,1} = -\frac{\partial}{\partial x} (\eta_1 u_1),$$

$$P_{\eta,1} = -s\delta e \left( u_1 \eta_1 \frac{U_1}{\partial x} + u_1 v_1 \frac{V_1}{\partial x} \right),$$

$$P_{\nu,1} = -s\delta e \left( u_1 \nu_1 U_1 + v_1 \nu_1 V_1 \right),$$

$$P_{b,1} = -\frac{s\delta e}{2} \left( \eta_1 U_1 \frac{\partial U_1}{\partial X} - s\delta w_1 \eta_1 \frac{Z_1}{2} \right).$$

Here $e_1$ is the energy density of SSIWs, $P_{a,1}$ is the advection term, $P_{w,1}$ is the convergence/divergence of SSIW energy flux due to pressure work, $P_{\eta,1}$ ($P_{\nu,1}$) is the kinetic energy exchange between NIWs and SSIWs through the horizontal (vertical) shear of NIWs, and $P_{b,1}$ is the potential energy exchange between NIWs and SSIWs. Note that $P_{\nu,1}$ has no counterpart in the energy equation [(4)] of the reduced gravity model since the horizontal current is vertically uniform in the reduced gravity model.

Following the procedures in section 3a (see appendix C for details), the SSIW energy equation [(22)] can be approximated as

$$\frac{\partial (e_1)}{\partial t} + e \frac{\partial}{\partial x} (e_1) + s\delta (e_1) = \delta \sigma (e_1) + O(e^2), \quad (23)$$

where

$$c_g = 1 + \frac{s}{2} \frac{Z_1}{2} + s\delta U_1, \quad \text{and} \quad (24a)$$

$$\sigma = -s \left( 5 \frac{\partial U_1}{\partial X} + 1 \delta Z_1 - \frac{1}{2} V_1 \right), \quad (24b)$$

where $c_g$ is the group velocity of SSIWs while $\sigma$ is the growth rate determining the rate of change of $\langle e \rangle$ in a reference frame following the SSIWs.

The SSIW energy equations (23) and (24) in the 3D primitive equation system are similar to their counterparts in the reduced gravity model, that is, (13) and (14). Aside from the projection coefficient $s$, there is only one difference between the growth rate $\sigma$ in the primitive equation system [(24b)] and reduced gravity system [(14b)]. Compared to (14b), an additional term $s V_1/z$ arises in (24b). This term corresponds to $P_{\nu,1}$ in (22) and originates from the energy exchange between SSIWs and NIWs through the vertical shear of NIWs (see appendix C for details). This energy exchange is absent in the reduced gravity model as there is no vertical shear in the reduced gravity model. However, as demonstrated in the following subsection (see Fig. 6), the energy exchange due to the vertical shear of NIWs plays a negligible role compared to other terms in (24b). Therefore, wave action of SSIWs in the 3D primitive equation system is still approximately conserved in practice (see appendix C for details).

The above analysis suggests that the fundamental conclusions derived from the reduced gravity model hold to a large extent in the 3D primitive equation system. The modulation of SSIWs by NIWs can be primarily...
understood in terms of wave action conservation of SSIWs in the presence of NIWs. SSIWs are enhanced in convergence (downwelling) regions of NIWs and damped in divergence (upwelling) regions.

c. Validation of analytical analysis using IHS

1) COMPARISON OF GROUP VELOCITY AND GROWTH RATE OF SSIWs

The validity of (23) and (24) can be assessed based on the comparisons with the numerical solutions in IHS. Here we track the SSIW front shown in Fig. 5 from its initial emergence until it propagates out of the model domain. To separate the SSIW from the NIW, a spatial low-pass filter is used to isolate the NIW with a cutoff wavelength of 30 km while the SSIW is attained by using a spatial high-pass filter with a cutoff wavelength of 15 km. As both the NIWs and SSIWs exhibit little variability in the alongfront direction, we take a vertical section perpendicular to the front to analyze their interactions. As in the theoretical analysis in section 3b, the axes are rotated so that the x axis is perpendicular to the front. Sensitivity tests suggest that choosing different vertical sections perpendicular to the front does not have any substantial impact on the following conclusions.

The current $u$ ($U$) and buoyancy $b$ ($B$) associated with the SSIWs (NIWs) are projected onto the first baroclinic mode to obtain the projection coefficients $u_1$ ($U_1$) and $\eta_1$ ($Z_1$) based on (18). The center of the SSIW front in IHS, $x_c$, is defined as $(x_d + x_u)/2$, where $x_d$ and $x_u$ correspond to the locations of the largest negative and positive values of $w_1$. The energy density of the SSIW front in IHS, $e_1$, is computed following (22), and its mean value within the front $\langle e_1 \rangle$ is measured as $\int_{x_{zd}}^{x_{zu}} e_1 dx/(x_{zu} - x_{zd})$, where $x_{zd}$ and $x_{zu}$ are the nearest zero crossings of $w_1$ outside the interval $[x_d, x_u]$. With the aid of the time series of $x_c$ and $\langle e_1 \rangle$, the group velocity $c_{g, IHS}$ and growth rate $\sigma_{IHS}$ of the SSIW front in IHS can be estimated as $dx_c/dt$ and $\langle e_1 \rangle^{-1}d\langle e_1 \rangle/dt$, respectively.

The theoretical solutions of the group velocity and growth rate of the SSIW front are derived by substituting into (24) the values of $U_1$, $V_1$, $Z_1$, $\partial U_1/\partial x$, and $\partial Z_1/\partial x$ averaged within $[x_{zd}, x_{zu}]$. The group velocity derived from (24a) is consistent with $c_{g, IHS}$ (Fig. 6a). Particularly, they agree well with each other during the initial growing stage of the SSIW front (before 5d10h) with a discrepancy less than 10%. The larger error afterward might be attributed to the nonlinear effects as the amplitude of the SSIW front has undergone a substantial increase (Fig. 6b). As $u_1$ within the front is of the same direction as the group velocity (not shown), the nonlinear advection would contribute to a faster propagation in the numerical experiment than the linear model solution predicts.

The growth rate derived from (24b) is also consistent with $\sigma_{IHS}$ (Fig. 6b). Here the evolution of the SSIW front is divided into three stages. The first stage is the growing stage (before 5d10h) when its magnitude increases rapidly. The second is the saturation stage (5d10h–6d6h) with little magnitude variation. The final one is the decay stage (after 6d6h) when the magnitude starts to attenuate. The SSIW front resides in a convergence zone of NIWs during the first two stages but goes into a divergence zone in the final stage. The theoretical model [(24b)] shows good skill at the growing stage when the nonlinearity of the SSIW front is weak. Furthermore, it qualitatively reproduces the attenuation of the SSIW front at the decay stage because of the reversed sign of

![Fig. 6.](image-url)
\[ \sigma_{\text{HIS}} \text{. Not surprisingly, } (24b) \text{ overestimates the growth rate during the saturation stage when the nonlinear effects become important.} \]

A notable feature is that the time for the SSIW front to reside in the convergence zone of NIWs (i.e., the growing and saturation stages) is considerably longer than the half-wave period of NIWs (Fig. 6b). Around the location \((-25^\circ N\) of the SSIW front, the NIWs are characterized by a wave frequency \(\Omega\) around 1.6f (Fig. 3a), corresponding to a half-wave period of 0.37 days. In contrast, the time for the SSIW front to reside in the convergence zone of NIWs reaches up to 1.8 days. This difference results mainly from the fact that the SSIW front does not stand still but propagates at a speed of \(c_{g,\text{HIS}} \approx 2 \text{ m s}^{-1}\) (Fig. 6a). The frequency of NIWs in a reference frame following the SSIW front becomes \(\Omega = \Omega - k_N c_g\), where \(k_N = \sqrt{\Omega^2 - f^2/c_g}\) is the horizontal wavenumber of NIWs. This predicts a time period of \(-1.6\) days for the SSIW front to reside in the convergence zone of NIWs, close to the value 1.8 days estimated from IHS (Fig. 6b).

Finally, it should be noted that the energy exchange due to the vertical shear of NIWs, that is, \(sV_i/2\) in (24b), plays a negligible role compared to the remaining terms in (24b). The growth rates computed with and without the term \(sV_i/2\) agree well with each other with a discrepancy less than 5\% (Fig. 6b). Therefore, SSIW wave action in the 3D primitive equation system can still be treated as a conserved quantity in practice.

2) \textbf{COMPARISON OF ENERGY TRANSFER RATE FROM NIWS TO SSIWS}

The theoretical solutions in sections 3a and 3b suggest that the increase of energy density of SSIWs in convergence zones of NIWS is due to both the converging energy flux of SSIWs and the energy transfer from NIWs to SSIWs. While the former is fully expected in a convergent background flow, the latter is not trivial and needs to be verified. In this subsection, we compare the energy transfer rate from NIWs to SSIWs in IHS to that derived from the theoretical solutions. Based on (17), the dimensional energy transfer rate in IHS can be computed as

\[
\text{TR}_{\text{HIS}} = -sH\tau_f \left( \frac{7}{4} \frac{\partial U_i}{\partial x} + \frac{1}{2} \frac{\partial Z}{\partial x} - \frac{1}{2} V_i \right). \tag{26}
\]

Figure 7 displays several snapshots of \(\text{TR}_{\text{HIS}}\) during the growing stage of SSIW front. Consistent with our theoretical solutions, there is a pronounced energy transfer from the NIWs to the SSIW front. In particular, both the spatial distribution and magnitude of \(\text{TR}_{\text{HIS}}\) agree well with those derived from the theoretical solutions. This provides further evidence for the validity of theoretical solutions.

4. Validation of analytical analysis using realistic numerical simulations

\textit{a. Model configurations}

In sections 2 and 3, we proposed a new mechanism for modulation of SSIWs through SSIW–NIW interaction based on the idealized numerical simulation and theoretical analyses. To test whether the theory has any relevance to reality, we perform two more realistic simulations using ROMS configured for the entire Gulf of Mexico and forced by reanalysis atmospheric forcing (Fig. 8). The horizontal resolution of ROMS is set at 3 km with 60 vertical layers. The nonlocal K-profile parameterization (Large et al. 1994) is again used to parameterize vertical mixing.

Both simulations start from 21 March 2010 using Hybrid Coordinate Ocean Model (HYCOM) data assimilation (Chassignet et al. 2007) as the initial and boundary conditions and last for 90 days. Only the data in the last 20 days are used for the analysis shown below. One simulation, which is referred to as control run (hereinafter GoM-C), is forced by the 6-hourly and 0.25°-resolution atmospheric surface variables (e.g., wind and surface air temperature) obtained from the ERA-Interim reanalysis dataset (Dee et al. 2011). The other simulation, which is referred to as the filtered run (hereinafter GoM-F), uses the same atmospheric variables as in GoM-C except that the winds are daily averaged. As the inertial period in the Gulf of Mexico ranges from 1.6 days at 18°N to 1 day at 30°N, the daily averaged winds contain little variance at inertial and higher frequencies, leading to much suppressed NIWs in GoM-F. Therefore, a comparison between these two experiments with identical model resolutions and physics parameterizations can provide a useful evaluation for the influence of NIWs on SSIWs. To minimize topographic effects, we applied the analysis to the central Gulf of Mexico (23.5°–27.5°N, 95°–85°W) where the water depth is greater than 1000 m.

\textit{b. SSIWs in GoM-C and GoM-F}

In GoM-C, energetic NIWs are excited by the 6-hourly wind forcing, as evidenced by a pronounced peak around
$f$ in the frequency spectrum of large-scale (>30 km) horizontal convergence/divergence $\nabla_H U$ at 100 m (Fig. 9a). More than 60% of $\nabla_H U$ variance comes from the near-inertial [(0.8–2)$f$] band, suggesting that NIWs make a dominant contribution to large-scale convergence/divergence. The energetic NIWs may be linked to the sea breeze whose period is close to the inertial period in the northern Gulf of Mexico (Zhang et al. 2009).

Figure 10 displays the area-mean (23.5°–27.5°N, 95°–85°W) frequency spectra of large-scale (>30 km) and small-scale (<15 km) vertical velocity at various depths in GoM-C. For the large-scale vertical velocity, most of its variance comes from the subinertial (<0.8$f$) and near-inertial [(0.8–2)$f$] frequency bands. For instance, the contribution from the superinertial frequency band (>2$f$) is less than 8%, 7%, and 7% at 100, 300, and 500 m, respectively. Consistent with the previous numerical studies (e.g., Danioux et al. 2008; Komori et al. 2008), the frequency spectrum of large-scale vertical velocity exhibits a pronounced peak at $f$ and a secondary peak at 2$f$ (Fig. 10a). The former is directly related to wind-generated NIWs while the latter may result from the triad wave resonance transferring energy from wind-generated NIWs to their harmonics (Danioux and Klein 2008). In contrast, the superinertial frequency band accounts for more than 50% of small-scale vertical velocity variance at various depths (Fig. 10b), revealing the dominant role of SSIWs in generating small-scale vertical velocity in the ocean interior. In particular, there are no notable peaks at $f$ and its harmonic frequencies in the frequency spectrum of small-scale vertical velocity.

To test the theory developed in sections 2 and 3 in a realistic setting, we compare the superinertial (>2$f$) small-scale (<15 km) vertical velocity $w_{ss}$ in convergence ($\nabla_H U < 0$) and divergence ($\nabla_H U > 0$) zones of large-scale flows in GoM-C. Here $\nabla_H U$ is computed from the large-scale horizontal velocity at 100 m. Using velocity at different depths between the surface and
200 m does not change the results presented below significantly. Consistent with the analytical solutions, $w_{ss}$ in the ocean interior (at 300 m) exhibits marked asymmetry between convergence ($\nabla_H \mathbf{U} < 0$) and divergence ($\nabla_H \mathbf{U} > 0$) zones of large-scale flows (Fig. 11a). The mean $w_{ss}^2$ in large-scale flow convergence zones is $0.8 \times 10^{-9} \text{m}^2 \text{s}^{-2}$, 24% larger than that in divergence zones (statistically significant at 95% significance level based on a Wilcoxon rank-sum test). This difference is more striking between strong convergence and divergence zones (Fig. 11b). For instance, mean $w_{ss}^2$ is 54% larger in strong convergence ($\nabla_H \mathbf{U} < -3.0 \times 10^{-6} \text{s}^{-1}$) regions than in strong divergence ($\nabla_H \mathbf{U} > 3.0 \times 10^{-6} \text{s}^{-1}$) regions (statistically significant at 95% significance level). Furthermore, the strong $w_{ss}$ values are found to mainly occur in the convergence regions with more than 75% of the strong $w_{ss}$ events of $|w_{ss}| > 25 \text{ m day}^{-1}$ identified in the convergence ($\nabla_H \mathbf{U} < 0$) zones (Fig. 11c). In contrast, only less than 1% of the strong $w_{ss}$ events appear in the strong divergence ($\nabla_H \mathbf{U} > 3.0 \times 10^{-6} \text{s}^{-1}$) regions, providing further supports for the theory.

In GoM-F, the NIWs are substantially suppressed as the daily mean winds do not contain any energy in the near-inertial frequency band for the Gulf of Mexico. Consequently, $\nabla_H \mathbf{U}$ becomes much weaker in GoM-F than in GoM-C (Fig. 9a). The $\nabla_H \mathbf{U}$ variance at 100 m decreases from $3.4 \times 10^{-12} \text{s}^{-2}$ in GoM-C to $1.5 \times 10^{-12} \text{s}^{-2}$ in GoM-F. To further quantify the contribution of different frequency bands to the difference of $\nabla_H \mathbf{U}$ variance between GoM-C and GoM-F, we introduce a cumulated contribution $C(\omega)$ in the frequency domain defined as

$$C(\omega) = \frac{\int_{0}^{\omega} P_{\text{con}}(s) - P_{\text{fir}}(s) \, ds}{\int_{0}^{\omega} P_{\text{con}}(s) - P_{\text{fir}}(s) \, ds},$$

(27)

where $P$ is the mean frequency spectrum of $\nabla_H \mathbf{U}$ averaged over the central Gulf of Mexico (23.5°–27.5°N, 95°–85°W), $\omega_N$ is the Nyquist frequency, and the subscripts con and fir represent GoM-C and GoM-F, respectively. The difference $C(2f) - C(0.8f)$ is about 1.05, suggesting that the decrease of $\nabla_H \mathbf{U}$ variance in GoM-F is mainly due to the much weaker NIWs (Fig. 9b). It should be noted that the subinertial $\nabla_H \mathbf{U}$ variance is slightly weaker in GoM-C than in GoM-F. Although the exact reasons remain unclear, possible candidates may include the dissipation of subinertial flow due to the horizontal eddy viscosity induced by NIWs (Müller 1976; Polzin 2010).

In spite of the fact that both GoM-C and GoM-F are conducted at the same resolution of 3 km, the small-scale vertical velocity becomes significantly weaker in
GoM-F than in GoM-C (Fig. 12a). For instance, the area-mean (23.5°–27.5°N, 95°–85°W) small-scale vertical velocity variance at 300 m is $1.5 \times 10^{-9}$ m$^2$ s$^{-2}$ in GoM-C, while it decreases to $0.8 \times 10^{-9}$ m$^2$ s$^{-2}$ in GoM-F (the difference is statistically significant at 95% significance level). We note that about 55% of small-scale vertical velocity variance difference at 300 m between GoM-C and GoM-F is due to the superinertial frequency band while the near-inertial and subinertial frequency bands only account for 31% and 14% of the difference, respectively (Fig. 12b). The situation is similar at other depths. This suggests that the stronger small-scale vertical velocity in GoM-C than in GoM-F is primarily due to the intensified SSIWs.

It should be noted that the 6-hourly winds used in GoM-C do not contain any energy in the superinertial frequency band, so the stronger SSIWs in GoM-C are unlikely to be directly caused by the winds. Furthermore, as the enhanced small-scale superinertial vertical velocity variance in GoM-C is not confined to the harmonic frequencies of $f$ but spreads over all the superinertial frequencies (Fig. 12b), it cannot be simply ascribed to the triad wave resonance mechanism (Danion and Klein 2008), either. However, this broadband enhancement is consistent with our modulation mechanism that allows NIWs to exert an influence on SSIWs with a broad range of frequencies. This lends support to the important role of the modulation mechanism in intensifying SSIWs in the reality.

5. Summary and discussion

In this study, the influence of NIWs on SSIWs and their associated vertical velocity is theoretically and numerically analyzed using high-resolution ROMS simulations. We present a dynamic mechanism for modulating SSIW strength by background NIWs. It shows that in
convergence (downwelling) regions of NIWs, energy flux of SSIWs converges and energy is transferred from NIWs to SSIWs, leading to enhanced SSIWs. The opposite is true in divergence (upwelling) zones of NIWs. The underlying dynamics can be understood in terms of wave action conservation of SSIWs in the presence of background NIWs.

The theoretical solution is validated by high-resolution ROMS simulations forced by realistic atmospheric forcing in the Gulf of Mexico. The simulations show that the strengths of SSIWs within convergence and divergence regions of NIWs exhibit a marked asymmetry with significantly stronger small-scale superinertial vertical velocity in convergence regions. By removing near-inertial wind forcing, the NIWs are substantially suppressed. This further leads to a significant decrease in the small-scale vertical velocity variance throughout the superinertial frequency band.

The modulation mechanism proposed in this study is analogous to the modulation of surface gravity wave strength by internal waves proposed by Alpers (1985). Its dynamics can be understood from the perspective of wave–mean flow interactions where the “wave” and “mean flow” correspond to SSIWs and NIWs, respectively. This makes the modulation mechanism proposed here distinct from the triad wave resonance mechanisms proposed in previous studies (Niwa and Hibiya 1997; Danioux and Klein 2008). While NIWs can interact only with their harmonics through the triad wave resonance, the modulation mechanism allows NIWs to exert an influence on SSIWs with a broad range of frequencies. Furthermore, the modulation mechanism proposed in this study should be distinguished from frontogenesis (Hoskins and Bretherton 1972) and filamentary intensification (McWilliams et al. 2009), so that small-scale vertical motions in these mechanisms are mainly confined to the surface mixed layer (Thomas et al. 2008). In contrast, the SSIWs have a much deeper vertical structure and can generate strong vertical velocity in the ocean interior.

Finally, the modulation mechanism proposed in this study provides a way of transferring energy from NIWs to SSIWs. The energy transfer is transient as the energy is transferred from NIWs to SSIWs in convergence regions of NIWs with reversed energy transfer when SSIWs propagate through divergence regions. However, this energy transfer could become permanent in the case that intensified SSIWs in the convergence region of NIWs become highly nonlinear and break. Such a scenario deserves further investigations in future studies, as it could potentially contribute to turbulent mixing in the oceans.

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APPENDIX A

Wave Action Conservation for the Two-Dimensional Case

In this appendix, we generalize the results in section 3a to the two-dimensional case in which the \( y \) derivatives are not zero. The nondimensional equations are

\[
\frac{\partial \mathbf{u}}{\partial t} + \delta \left( \frac{U \partial \mathbf{u}}{\partial x} + V \frac{\partial \mathbf{u}}{\partial y} \right) = e \nabla \cdot \mathbf{F} = \mathbf{P}_b + \mathbf{P}_s, \quad (A1a)
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + \delta \left( \frac{U \partial \mathbf{v}}{\partial x} + V \frac{\partial \mathbf{v}}{\partial y} \right) = -e \nabla \cdot \mathbf{u} - e \delta \left( \frac{u \partial U}{\partial x} + \frac{v \partial U}{\partial y} \right), \quad (A1b)
\]

\[
\frac{\partial \eta}{\partial t} + \delta \left( \frac{U \partial \eta}{\partial x} + V \frac{\partial \eta}{\partial y} \right) + (1 + \delta Z) \left( \frac{\partial \mathbf{u}}{\partial x} + \frac{\partial \mathbf{v}}{\partial y} \right) + e \delta \left( \frac{u \partial Z}{\partial x} + \frac{v \partial Z}{\partial y} \right) = -\delta \eta \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right), \quad (A1c)
\]

where \( X = ex \) and \( Y = ey \) are two slowly changing coordinates.

The nondimensional energy equation associated with (A1) is

\[
\frac{\partial e}{\partial t} + \delta \frac{\partial \mathbf{E}}{\partial x} + e \delta \frac{\partial \mathbf{E}}{\partial y} = -\nabla \cdot \mathbf{F} + \mathbf{P}_b + \mathbf{P}_s, \quad (A2)
\]

where \( e = (1 + \delta Z)(u^2 + v^2)/2 + \eta^2/2 \) is the energy density of SSIs, \( \mathbf{F} = (1 + \delta Z)\mathbf{u} \) is the energy flux of SSIs due to pressure work, and \( \mathbf{P}_b (\mathbf{P}_s) \) is the potential (kinetic) energy exchange between NIWs and SSIs:

\[
\mathbf{P}_b = \frac{1}{2} \delta \mathbf{e} \left( \frac{\partial \mathbf{u}}{\partial x} + \frac{\partial \mathbf{v}}{\partial y} \right), \quad \text{and} \quad (A3a)
\]

\[
\mathbf{P}_s = -\delta e(1 + \delta Z) \left( \frac{uv \partial \mathbf{u}}{\partial x} + \frac{uv \partial \mathbf{v}}{\partial y} + \frac{uv \partial \mathbf{u}}{\partial x} + \frac{uv \partial \mathbf{v}}{\partial y} \right). \quad (A3b)
\]

We now introduce the fast and slow variables:

\[
\frac{\partial \phi}{\partial t} = -\omega, \quad \frac{\partial \phi}{\partial x} = k, \quad \frac{\partial \phi}{\partial y} = l, \quad T_1 = et, \quad T_2 = e^2t, \quad X = ex, \quad Y = ey, \quad (A3c)
\]

where \( \omega \) and \( (k, l) \) can be treated as the local wave frequency and wavenumber of SSIs, respectively. The temporal and spatial derivatives can be expanded, in virtue of the chain rule, as

\[
\frac{\partial}{\partial t} = -\omega \frac{\partial}{\partial \phi} + e \frac{\partial}{\partial T_1} + e^2 \frac{\partial}{\partial T_2}, \quad (A4a)
\]

\[
\frac{\partial}{\partial x} = k \frac{\partial}{\partial \phi} + e \frac{\partial}{\partial X}, \quad \text{and} \quad (A4b)
\]

\[
\frac{\partial}{\partial y} = l \frac{\partial}{\partial \phi} + e \frac{\partial}{\partial \phi}, \quad (A4c)
\]

Expand the perturbed quantities in asymptotic series of \( e \). The \( O(1) \) terms of (A1) are

\[
-\omega_0 \frac{\partial \eta_0}{\partial \phi} = -k \frac{\partial \eta_0}{\partial \phi}, \quad (A5a)
\]

\[
\omega_0 \frac{\partial \eta_0}{\partial \phi} = -l \frac{\partial \eta_0}{\partial \phi} \quad \text{and} \quad (A5b)
\]

\[
-\omega_0 \frac{\partial \eta_0}{\partial \phi} + k \frac{\partial \eta_0}{\partial \phi} + l \frac{\partial \eta_0}{\partial \phi} = 0. \quad (A5c)
\]

The solutions to (A5) are

\[
(u_0, v_0, \eta_0) = \text{Re} \left\{ A_0 e^{ik} \left( \frac{k}{\omega_0}, \frac{l}{\omega_0} \right) \right\}, \quad (A6)
\]

with the dispersion relation \( \omega_0 = \sqrt{k^2 + l^2} \).

The \( O(e) \) terms of (A1) are

\[
-\omega_0 \frac{\partial \eta_1}{\partial \phi} + k \frac{\partial \eta_1}{\partial \phi} + l \frac{\partial \eta_1}{\partial \phi} = -\omega_0 \frac{\partial \eta_0}{\partial \phi} + \omega_1 \frac{\partial \eta_0}{\partial \phi} - \frac{\partial \eta_0}{\partial \phi} \quad \text{and} \quad (A7a)
\]

\[
-\omega_0 \frac{\partial \eta_1}{\partial \phi} + l \frac{\partial \eta_1}{\partial \phi} = -\omega_0 \frac{\partial \eta_0}{\partial \phi} + \omega_1 \frac{\partial \eta_0}{\partial \phi} - \frac{\partial \eta_0}{\partial \phi} \quad \text{and} \quad (A7b)
\]

\[
-\omega_0 \frac{\partial \eta_1}{\partial \phi} + k \frac{\partial \eta_1}{\partial \phi} + l \frac{\partial \eta_1}{\partial \phi} = -\omega_0 \frac{\partial \eta_0}{\partial \phi} + \omega_1 \frac{\partial \eta_0}{\partial \phi} + \frac{\partial \eta_0}{\partial \phi} \quad \text{and} \quad (A7c)
\]

To get rid of the singularity, the following equations must be satisfied:

\[
2 \left( \frac{\partial A_0}{\partial T_1} + \frac{\partial A_0}{\partial Y} + \frac{\partial A_0}{\partial X} \right) + A_0 \left( \frac{\partial \eta_0}{\partial X} + \frac{\partial \eta_0}{\partial Y} \right) = 0, \quad (A8a)
\]

and

\[
\omega_1 = \chi(kU + lV) + \frac{\chi}{2} \omega_0 Z, \quad (A8b)
\]

where \( \epsilon_{xy}^0 = \omega_{x0} \partial k \) and \( \epsilon_{y0}^0 = \omega_{00} \partial l \) are the \( O(1) \) group velocity.

With the aid of (A8), the solutions to (A7) are
u_1 = \text{Re}\left\{ -\frac{\chi}{2} \frac{Z}{\omega_0} k A_0 e^{i\phi} - \frac{i}{\omega_0} \left[ \frac{\partial}{\partial T_1} \left( A_0 \frac{k}{\omega_0} \right) + \frac{\partial A_0}{\partial X} - A_0 \frac{l}{\omega_0} \right] e^{i\phi} \right\}, \quad (A9a)

u_1 = \text{Re}\left\{ -\frac{\chi}{2} \frac{Z}{\omega_0} l A_0 e^{i\phi} - \frac{i}{\omega_0} \left[ \frac{\partial}{\partial T_1} \left( A_0 \frac{l}{\omega_0} \right) + \frac{\partial A_0}{\partial Y} + A_0 \frac{k}{\omega_0} \right] e^{i\phi} \right\}, \quad (A9b)

\eta_1 = 0. \quad (A9c)

Substituting (A6) and (A9) into (A2) and averaging over one wave period or wavelength yield

$$\frac{\partial \langle e \rangle}{\partial T_1} + e \frac{\partial \langle e \rangle}{\partial T_2} + \frac{\partial c_{sv} \langle e \rangle}{\partial X} + \frac{\partial c_{sv} \langle e \rangle}{\partial Y} = -\delta \langle e \rangle \left[ \frac{1}{2} \left( \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} \right) + \cos^2 \alpha \frac{\partial U}{\partial X} + \sin \alpha \cos \alpha \frac{\partial U}{\partial Y} + \sin^2 \alpha \frac{\partial V}{\partial Y} \right] + O(\varepsilon^2), \quad (A10)$$

where \(\langle e \rangle = |A_0|^2/2 + O(\varepsilon^2)\) is the energy density averaged over one wavelength or wave period, \(\alpha = \arctan(l/k)\) is the azimuth of the horizontal wave vector, and the group velocity is

$$c_{sv} = \frac{\partial a_0}{\partial k} = \left( 1 + \frac{\delta}{2} Z \right) \cos \alpha + \delta U, \quad (A11a)$$

$$\frac{1}{\omega_1} \left( \frac{\partial a_0}{\partial T_1} + e \frac{\partial a_0}{\partial T_2} + c_{sv} \frac{\partial a_0}{\partial X} + c_{sv} \frac{\partial a_0}{\partial Y} \right) = -\delta \left[ \frac{1}{2} \left( \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} \right) + \cos^2 \alpha \frac{\partial U}{\partial X} + \sin \alpha \cos \alpha \frac{\partial U}{\partial Y} + \sin^2 \alpha \frac{\partial V}{\partial Y} \right] + O(\varepsilon^2). \quad (A12)$$

Substituting (A12) into (A10) yields

$$\frac{\partial \langle n \rangle}{\partial T_1} + e \frac{\partial \langle n \rangle}{\partial T_2} + \frac{\partial c_{sv} \langle n \rangle}{\partial X} + \frac{\partial c_{sv} \langle n \rangle}{\partial Y} = O(\varepsilon^2) \approx 0. \quad (A13)$$

Equation (A13) states that the wave action of SSIWs is approximately conserved in the presence of NIWs.

**APPENDIX B**

**Validation of the Analytical Solutions (14)–(16)**

In this section, the validity of our analytical solutions (14)–(16) is tested against numerical simulations based on the reduced gravity model with full dynamics included:

$$\frac{\partial u_t}{\partial t} + u_t \frac{\partial u_t}{\partial x} + v_t \frac{\partial u_t}{\partial y} = f v_t - g \frac{\partial \eta_t}{\partial x}, \quad (B1a)$$

$$\frac{\partial v_t}{\partial t} + u_t \frac{\partial v_t}{\partial x} + v_t \frac{\partial v_t}{\partial y} = -f u_t - g \frac{\partial \eta_t}{\partial y}, \quad \text{and} \quad (B1b)$$

$$\frac{\partial \eta_t}{\partial t} + u_t \left( H + \eta_t \right) + v_t \left( H + \eta_t \right) = 0, \quad (B1c)$$

where \(u_t = u + U\) and \(\eta_t = \eta + Z\) represent the total velocity and interface elevation. A double periodic domain is constructed with a horizontal resolution of 200 m. We use \(g' = 0.01 \text{ m s}^{-2}\), \(H = 400 \text{ m}\), and \(f = 1 \times 10^{-4} \text{ rad s}^{-1}\).

The initial interface elevation is superposed by an NIW and an SSIW (Fig. B1). Both propagate toward the positive horizontal axis and have no variability along the horizontal axis, analogous to the situation in IHS. The wave frequency of NIW, \(\Omega\), is set to be 1.1 for a horizontal current speed of 0.2 m s\(^{-1}\). Then the horizontal wavenumber of NIW, \(k_N\), can be uniquely determined from the dispersion relation of internal wave \(\Omega = \sqrt{c^2 k_N^2 + f^2}\). The horizontal structure of SSIW is modeled as a Gaussian wave packet:

$$\eta = \eta_0 e^{-\left(x-k_N y\right)^2/L^2} \cos(k_N x), \quad (B2)$$

where \(\eta_0 = 1 \text{ m}, k_N = 2\pi l/(6 \text{ km})\), and \(L = 10 \text{ km}\). Sensitivity tests suggest using different initial values for the amplitude, wavenumber, and wave frequency of NIW, and SSIW does not make any substantial impact on the following conclusions as long as the spatial and temporal scales of NIW and SSIW are well separated from each other and the nonlinearity of SSIW is negligible.
We use a high-pass (low-pass) filter with a cutoff wavelength of 15 (30) km to attain signals associated with the SSIW (NIW). As $\eta(x, t)$ is modeled as a Gaussian wave packet, a Hilbert transform is used to estimate its envelope $\Lambda(x, t)$ and phase $\theta(x, t)$. Then the center of SSIW, $x_c(t)$, is defined as the location where $\Lambda(x, t)$ reaches its maximum. In this case, the group velocity can be computed as $dx_c/dt$. The horizontal wavenumber of SSIW, $k$, is estimated based on a least squares fit to $\theta(x, t)$. The energy density of SSIW, $\langle \varepsilon \rangle$, is computed following (4) as $\langle \varepsilon \rangle = \left\{ 1/[2(x_2 - x_1)] \right\} \int_0^{x_2} (Z + H)(u^2 + v^2) + g^2 \eta^2 dx$, where $x_1$ and $x_2$ are the locations at which $\Lambda(x, t)$ decreases to 10% of its maximum value. Finally, $\langle n \rangle$ is evaluated as $\langle n \rangle = \langle \varepsilon \rangle/\omega_i$, where $\omega_i = ck[1 + Z/(2H)]$. The analytical solutions to these variables are computed by substituting into (14)–(16) the values of $U, Z, \partial U/\partial x$, and $\partial Z/\partial y$ averaged between $x_1$ and $x_2$.

![FIG. B1. Initial condition for $Z + \eta$.](image)

![FIG. B2. The evolution of (a) energy density $\langle \varepsilon \rangle$, (b) action density $\langle n \rangle$, (c) horizontal wavenumber $k$, and (d) group velocity $c_g$ of SSIW derived from the reduced gravity model simulations (solid red) and theoretical solutions (14)–(16) (dashed blue). Here $\langle \varepsilon \rangle$, $\langle n \rangle$, and $k$ have been normalized by their initial values.](image)
Figure B2 shows the evolution of $\langle e \rangle$, $\langle n \rangle$, $c_g$, and $k$ computed from the numerical simulations and analytical solutions (14)–(16). The analytical solutions agree well with numerical simulations. The difference in $\langle e \rangle$, $\langle n \rangle$, $c_g$, and $k$ between the analytical solutions and numerical simulations is within 10% of their initial values. We conclude that the analytical solutions are valid.

APPENDIX C

Perturbation Analysis in a 3D Primitive Equation System

The $O(1)$ terms of (21) are

$$-\omega_0 \frac{\partial u_0}{\partial \phi} = -k \frac{\partial \eta_0}{\partial \phi}, \quad (C1a)$$

$$-\omega_0 \frac{\partial \eta_0}{\partial \phi} = 0, \quad (C1b)$$

$$-\omega_0 \frac{\partial \eta_0}{\partial \phi} + w_0 = 0, \quad \text{and}$$

$$k \frac{\partial u_0}{\partial \phi} - w_0 = 0. \quad (C1c)$$

The solutions to (C1) are

$$(u_1^0, v_1^0, w_1^0, \eta_1^0) = \text{Re} \{A_0 e^{i \phi} (1, 0, i k, 1) \} \quad (C2)$$

with the associated dispersion relation $\omega_0 = k$.

The $O(\varepsilon)$ terms of (21) are

$$-k \left( \frac{\partial u_1}{\partial \phi} - \frac{\partial \eta_1}{\partial \phi} \right) = i (\omega_1 - s \chi U_1 k) u_1,$$

$$\frac{\partial u_1}{\partial T_1} - \frac{\partial \eta_1}{\partial X} = -\frac{sk}{2} u_1, \quad (C3a)$$

$$-k \frac{\partial u_1}{\partial \phi} = -u_1 - \frac{sk}{2} u_1 V_1, \quad (C3b)$$

$$-k \frac{\partial \eta_1}{\partial \phi} + w_1 = i \left( \omega_1 - \frac{sk}{2} U_1 k \right) \eta_1$$

$$-\frac{\partial \eta_1}{\partial T_1} = -s \chi \omega_1 Z_1, \quad \text{and}$$

$$k \frac{\partial u_1}{\partial \phi} - w_1 = -\frac{sk}{2} \omega_1. \quad (C3c)$$

To get rid of the singularity, the following equation must be satisfied:

$$\frac{\partial A_k}{\partial T_1} + \frac{\partial \omega_0}{\partial \phi} = 0, \quad \text{and} \quad (C4a)$$

$$\frac{\partial c_g}{\partial T_1} + \frac{\partial \omega_0}{\partial \phi} = 0, \quad \text{and} \quad (C4b)$$

where $c_g = \partial \omega_0 / \partial k = 1$ is the $O(1)$ group velocity of SIWVs.

With the aid of (C4), the solutions to (C3) are

$$u_1^0 = \text{Re} \left\{ \left( -\frac{sX}{2} Z_1 + \frac{sX}{2} U_1 \right) A_0 e^{i \phi} \right\}. \quad (C5a)$$

$$v_1^0 = \text{Re} \left\{ \left( -i + \frac{sX}{2} V_1 \right) A_0 e^{i \phi} \right\}, \quad (C5b)$$

$$w_1^0 = \text{Re} \left\{ i k \left( -\frac{sX}{2} Z_1 + \frac{sX}{2} U_1 \right) A_0 e^{i \phi} + \frac{\partial A_0}{\partial X} e^{i \phi} \right\}, \quad \text{and} \quad (C5c)$$

$$\eta_1^0 = 0. \quad (C5d)$$

Substituting (C2) and (C5) into (22) and averaging over one wave period or wavelength yield:

$$\langle e_1 \rangle = \frac{A_0^2}{2} \left( 1 + \frac{s \delta}{2} U_1 - \frac{s \delta}{2} Z_1 \right) + O(\varepsilon^2), \quad (C6a)$$

$$\langle P_{u,1} \rangle = -s \delta \varepsilon \left( \frac{3}{4} \frac{\partial U_1}{\partial X} + O(\varepsilon^3) \right), \quad (C6b)$$

$$\langle P_{w,1} \rangle = -e \frac{\partial}{\partial X} (e_1^1) + O(\varepsilon^3), \quad (C6c)$$

$$\langle P_{u,1}^V \rangle = -s \delta \varepsilon \left( \frac{1}{4} \frac{\partial U_1}{\partial X} (e_1^1) - \frac{1}{2} (e_1^1) V_1 \right) + O(\varepsilon^3), \quad \text{and} \quad (C6d)$$

$$\langle P_{k,1} \rangle = -s \delta \varepsilon \left[ \left( \frac{1}{2} \frac{\partial Z_1}{\partial X} + \frac{3}{4} \frac{\partial U_1}{\partial X} \right) (e_1^1) + \frac{3}{2} Z_1 \frac{\partial (e_1^1)}{\partial X} \right] + O(\varepsilon^3). \quad (C6e)$$

Substituting (C6) into (22) yields (23) and (24).

Introduce the intrinsic wave frequency $\omega_i = k(1 + s \delta Z_1 / 2)$. According to the ray tracing relation (Lighthill 1978), we have

$$\frac{1}{\omega_i} \left( \frac{\partial \omega_i}{\partial T_1} + \epsilon \frac{\partial \omega_i}{\partial T_2} + c_g \frac{\partial \omega_i}{\partial \phi} \right) = -\delta \frac{3}{2} \frac{\partial U_1}{\partial X} + O(\varepsilon^3). \quad (C7)$$

The equation for wave action density $n_1 = e_1 / \omega_i$ is

$$\frac{\partial \langle n_1 \rangle}{\partial T_1} + \epsilon \frac{\partial \langle n_1 \rangle}{\partial T_2} + c_g \frac{\partial \langle n_1 \rangle}{\partial \phi} = \delta \frac{1}{2} V_1 + O(\varepsilon^3). \quad \text{(C8)}$$

REFERENCES

