An Exact, Steady, Purely Azimuthal Flow as a Model for the Antarctic Circumpolar Current

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Abstract

The problem of flow moving purely in the azimuthal direction on a sphere is considered. An exact solution for an incompressible (constant density), inviscid fluid, which admits a velocity profile below the surface and along the surface, is constructed; this can be regarded as a model for the Antarctic Circumpolar Current (ACC). The new approach adopted here is to model the processes that produce the observed structure of the ACC by the introduction of a nonconservative body force. It is shown that if the body force is conservative, then the governing equations necessarily lead to profiles that are quite unrealistic. However, with a suitable choice of body force, which reverts to conservative outside the ACC, any velocity profile of any width can be constructed as an exact solution of the system. A fairly simple choice is made in this note in order to present some specific results: a profile on the surface that is zero outside the arc of the ACC, with a maximum at its center and decaying with depth. It is shown that the methods developed here can be used to produce ever more complicated profiles to correspond to different data. Indeed, the basic example that this study introduces can be regarded as one of the jets that compose the ACC, and the results allow for any number of such jets. Although only one velocity profile is described, it is emphasized that many different choices, motivated by direct velocity observations in specific regions, are possible within the model. In conclusion, a few comments are made outlining the way in which this exact solution can be embedded within more general and complete discussions of the ACC and its properties.

1. Introduction

Exact solutions in fluid mechanics are rare and must be treated with some caution and some respect. On the one hand, these solutions are likely to be very limited in their application to a particular physical phenomenon with all its inherent complexity. On the other hand, they are precise and clear in their validity, detail, and structure. Of course, one of the skills of the theorician interested in using a mathematical framework to describe and understand physical processes is to learn how to combine the exact with the approximate in order to make progress. Here, we present one aspect of this process: the construction of an exact solution that relates to a flow observed in the oceans on Earth. Ultimately, we would wish to describe the motions, for example, that include surface waves and their interaction with variable currents: a complex flow. In addition, the topography of the ocean bottom and the existing landmasses around the globe add further complications to the flows observed on Earth. All these difficulties are set aside for the purposes of this discussion. Here, in particular, we will focus on the problem of providing an exact description of the flow that moves completely around the globe on a circular path. This, we propose, can be regarded as a model for the Antarctic Circumpolar Current (ACC). We note that nonlinear exact solutions for oceanic east–west wave propagation are available in the research literature, but these have been derived using the $f$-plane...
approximation close to a fixed latitude (as in Pollard 1970; Mollo-Christensen 1979) or, for equatorial flows, the β-plane approximation (see Constantin 2012, 2014; Henry 2013; Hsu 2014; Ionescu-Kruse 2015). Recently, however, exact nonlinear solutions in spherical coordinates (so with a more precise geometry) were presented in Constantin and Johnson (2016) for equatorial flows. The present paper builds on this work by showing not only that on a rotating, spherical Earth we can accommodate azimuthal flows that follow a small circle but also that a meridionally localized jet structure, which decays monotonically with depth, can be described.

The ACC is arguably the most significant current in our oceans and the only current that completely encircles the polar axis; almost unobstructed by landmasses, being constricted only in the region of the Drake Passage (about 800 km wide), it flows around Antarctica. This primarily wind-driven flow moves eastward through the southern regions of the Atlantic, Indian, and Pacific Oceans (see Fig. 1). Although the mean current speed is relatively low (4–25 cm s\(^{-1}\), about 3% of the ambient wind speed; the southern latitudes have some of the strongest westerly winds on Earth), the ACC transports vast volumes of water; indeed, its effects are felt from the surface down to depths as much as 4–5 km, it is about 23,000 km long, and in places its width extends over 2000 km. Unlike other major oceanic currents, the ACC is not a single flow, being composed of a number of high-speed, vertically coherent, seafloor-reaching jets (with speeds commonly exceeding 1 m s\(^{-1}\) and typically 40–50 km wide) that run largely parallel to the ocean ridge system that surrounds Antarctica, separated by zones of low-speed flow. [Further information can be found, e.g., in Ivchenko and Richards (1996), Rintoul et al. (2001), Firing et al. (2011), and Olbers et al. (2004).] Here, the essential, simplified, geometric model for this configuration is to consider flow in the neighborhood and following the path of a small circle at some given latitude around the polar axis of a spherical Earth. We will formulate the problem in spherical coordinates [as outlined in Constantin and Johnson (2016)] and combine this with a suitable flow structure for the ACC.

As with almost all exact solutions of fluid systems, we must expect that some elements of the physics cannot be accommodated in any form; the current exercise is no exception. Although we can incorporate an arbitrary flow that follows a small circle on a rotating, spherical Earth, it is altogether unrealistic to expect that a lot of the observed detail can be incorporated. But we do demonstrate that an exact solution is available that allows for any chosen flow profile (both at the surface and varying with depth), although unchanging as it moves in the azimuthal direction on a circular path. The modeling of fairly realistic profiles—that is, profiles that are observed in some average sense—is impossible using the conventional Euler equation but accessible if a non-conservative force field is introduced. [One of the essential difficulties of working with a conservative system comes about as a consequence of the Taylor–Proudman theorem; see Taylor (1917) and Proudman (1916), as we briefly discuss in section 2.] There can be no doubt that many of the properties, both observed and assumed, in these oceanic flows, come about by the action of non-conservative processes. Because the governing Euler equation (which we take as providing our overarching theoretical principle) can admit a general body force, we take advantage of this in our development. The upshot is that we can describe, via an exact solution, a broad family of flows, which may extend over a wide latitudinal arc and behave in any desired manner below the surface, allowing for a decrease in speed with depth but not necessarily dropping to zero on the bottom (even though this is a natural choice). So we can, for example, model data for the ACC, which indicates that there is a current even at considerable depths [e.g., 5 cm s\(^{-1}\) at about 2 km, with 15 cm s\(^{-1}\) at the surface in some regions; see Ivchenko and Richards (1996) and Firing et al. (2011)]. Certainly we can accommodate profiles that enable the appropriate volume of water to be transported by the ACC (see Johnson and Bryden 1989), and for this the details of the vertical structure are less important.

![Fig. 1. Sketch of the ACC. It resides between two circumpolar fronts: the inner Polar Front and the outer Subantarctic Front (red curves). Landmasses are drawn in black and oceanic regions in white, with depths shallower than 3.5 km shaded blue.](image-url)
Therefore, the model that we shall work with is that of an essential core of fluid, of arbitrary velocity profile, that constitutes the ACC moving eastward around Earth with unchanging form in that direction.

The plan in this paper is to outline the problem in spherical coordinates [and more details of this and associated geometries can be found in Constantin and Johnson (2016)] and the special choices that underpin the construction of the exact solution that is appropriate for the ACC. We will explain that this involves the need to use a nonconservative force field in order to produce a realistic flow but that this automatically caters for many of the ad hoc mechanisms invoked in other models. Indeed, we advocate the use of exact solutions, the more complete the better, before there is any attempt to model other physical processes. The way in which we can impose the shape of the profile, and the pressure condition at the free surface, will be described and an example presented. We conclude with a brief indication of the possible ways in which this exact solution can be used as part of a wider theoretical study of the ACC and its properties.

2. Formulation of the problem and of the equations

The fundamental assumption that predicates our analysis is that we work with an incompressible, inviscid fluid. [The density of water changes by about 0.025% when there is a change of 500 kPa in pressure, and it is generally accepted that the Reynolds number is extremely large for these oceanic flows; see Maslowe (1986).] We introduce a set of (right handed) spherical coordinates \((r, \theta, \phi)\); \(r\) is the distance (radius) from the center of the sphere, \(\theta\) (with \(0 \leq \theta \leq \pi\)) is the polar angle and \(\phi\) (with \(0 \leq \phi \leq 2\pi\)) is the azimuthal angle, that is, the angle of longitude. The North and South Poles are at \(\theta = 0, \pi\), respectively, and the equator is on \(\phi = \pi/2\); the Antarctic Circumpolar Current sits at about \(\theta = 3\pi/4\). The unit vectors in this \((r, \theta, \phi)\) system are \((e_r, e_\theta, e_\phi)\), respectively, and the corresponding velocity components are \((u, v, w)\); \(e_\phi\) points from west to east, and \(e_\theta\) from north to south (see Fig. 2).

The Euler equation and the equation of mass conservation are, respectively,

\[
\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + v \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial \phi}\right)(u,v,w) \\
+ \frac{1}{r} \left(-v^2 - w^2, uv - w^2 \cot \theta, uw + vw \cot \theta\right) \\
= -\frac{1}{\rho} \left(\frac{\partial p}{\partial r} - \frac{p}{r \sin \theta} \frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi}\right) + (F_r, F_\theta, F_\phi),
\]

and

\[
\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi} = 0,
\]

where \(p(r, \theta, \phi)\) is the pressure in the fluid, and \((F_r, F_\theta, F_\phi)\) is the body force vector. These equations describe the problem in a coordinate system with its origin at the center of the sphere; we now associate \((e_r, e_\theta, e_\phi)\) with a point fixed on the sphere, which is rotating about its polar axis. Thus, on the left of Eq. (2), we must introduce the additional terms

\[
2\Omega \times \mathbf{u} \quad \text{(Coriolis)} \quad \text{and} \quad \Omega \times (\Omega \times \mathbf{r}) \quad \text{(centripetal acceleration),}
\]

where

\[
\Omega = \Omega(e_r \cos \theta - e_\theta \sin \theta), \quad \mathbf{u} = u e_r + v e_\theta + w e_\phi, \quad \mathbf{r} = r e_r,
\]

with \(\Omega \approx 7.29 \times 10^{-5} \text{ rad s}^{-1}\) the constant rate of rotation of Earth; these two contributions can be written as

\[
2\Omega (-w \sin \theta, -w \cos \theta, u \sin \theta + v \cos \theta) \\
- r\Omega^2 (\sin^2 \theta, \sin \theta \cos \theta, 0).
\]

In this work, an important ingredient is the use, and suggested form, of the body force vector \((F_r, F_\theta, F_\phi)\). We take the component in the \(r\) direction to be the familiar gravitational term (with \(g = \text{constant} \approx 9.81 \text{ m s}^{-2}\), and constancy is reasonable for the depths of the oceans on Earth); in the \(\theta\) direction, we introduce a general body
force $G(r, \theta)$ and zero in the $\phi$ direction. Thus, the body force vector is written as

$$[-g, G(r, \theta), 0].$$

(4)

At the free surface, $r = R + h(\theta, \phi)$, where $R \approx 6378$ km is the (mean) radius of Earth, we impose a surface pressure and the kinematic boundary condition:

$$p = P(\theta, \phi) \quad \text{on} \quad r = R + h(\theta, \phi),$$

(5)

and

$$u = \frac{v}{r} \frac{\partial h}{\partial \theta} + \frac{w}{r \sin \theta} \frac{\partial h}{\partial \phi} \quad \text{on} \quad r = R + h(\theta, \phi),$$

(6)

respectively. At the bottom of the ocean, $r = d(\theta, \phi)$, which we take to be an impermeable, solid boundary, we have the corresponding kinematic boundary condition:

$$u = \frac{v}{r} \frac{\partial d}{\partial \theta} + \frac{w}{r \sin \theta} \frac{\partial d}{\partial \phi} \quad \text{on} \quad r = d(\theta, \phi).$$

(7)

With reference to our formulation above, we note that there is now a general consensus that the dynamical balance of the ACC comes about by means of a forcing mechanism, but the exact nature of this forcing is still debated. There is no doubt that while turbulent fluxes play an important role in the momentum balance and on the vertical structure of the flow—see the discussions in Gallego et al. (2004), Smith and Marshall (2009), and Stewart et al. (2014)—the other relevant factors are the stress of the wind on the sea surface and the drag due to topography ("form drag"); the delicate issue is the balance between these mechanisms. Therefore, our inclusion of an arbitrary forcing function will be useful for testing flow models, the more so since our considerations also apply to a generic channel, geophysical fluid dynamic flow and so are not limited to the ACC (which we chose to discuss because it is the most widely studied flow of this type in our oceans).

The solution that we seek represents a steady flow, which is purely in the azimuthal direction, so $u = v = 0$ everywhere. Further, this flow does not vary in this direction, so $w = w(r, \theta)$, and then we have $p = p(r, \theta)$, $h = h(\theta)$, and $d = d(\theta)$. All the equations and boundary conditions are now satisfied, with

$$-\frac{w^2}{r} - 2\Omega w \sin \theta - r \Omega^2 \sin^2 \theta = -\frac{1}{\rho} p_r - g, \quad \text{and}$$

$$-\frac{w^2}{r} \cot \theta - 2\Omega w \cos \theta - r \Omega^2 \sin \theta \cos \theta = -\frac{1}{\rho_r} p_\theta + G(r, \theta);$$

(8)

(9)

the subscripts here denote partial derivatives. These equations, at least when linearized, exhibit the familiar geostrophic balance between the pressure gradient and Coriolis term (in addition to the contribution from the centripetal acceleration and the body force). Eliminating the pressure between Eqs. (8) and (9) yields the equation for $w$:

$$\left(\frac{w}{r \sin \theta} + \Omega\right)(r w_r \cos \theta - w_\theta \sin \theta) = -\frac{1}{2}(r G)_r = -F,$$

(10)

where it is convenient, when working with the azimuthal velocity component in the flow, we use $F$ rather than $G$. On the other hand, when we consider the pressure distribution associated with this flow field, as we describe later, we must revert to $G$, which then slightly complicates the analysis. Equation (10) can be solved by the method of characteristics by introducing

$$s = r \sin \theta$$

and then treating $w = \hat{w}(s, r)$; because of this transformation, it is convenient to express $F = \hat{F}(s, r)$, and then we obtain

$$\left(\frac{\hat{w}}{s} + \Omega\right)\hat{w}_r = \hat{F}(s, r).$$

(11)

We have written $r \cos \theta = -\sqrt{r^2 - r^2 \sin^2 \theta} = -\sqrt{r^2 - s^2}$ and chosen the negative sign as appropriate for the ACC (which sits at about $\theta = 3\pi/4$). The general solution of Eq. (11), for flows with $\hat{w} > 0$, is

$$\hat{w}(s, r) = -\Omega s + \sqrt{A(s) + 2s \int_0^s \hat{F}(s, r') \sqrt{r'} dr'},$$

(12)

where $A(s)$ is the arbitrary function of integration and $d(\theta) = d(s/r)$. We now make an important observation: if the force system is conservative, then $F = 0$ and $\hat{w}$ becomes an arbitrary function of $s$, that is, $\hat{w}$ is constant on lines $r \sin \theta = constant$, which are parallel to the axis of rotation; this is equivalent to the Taylor–Proudman theorem (usually stated as: a slow, steady, frictionless flow of a barotropic, incompressible fluid is two-dimensional and does not vary in the direction of the rotation vector). This, as we will mention shortly, has severe consequences for the possible flow structure in the azimuthal direction; indeed, it is overcoming this difficulty that is one of the drivers for our approach that makes use of a nonconservative system.

Our general formulation allows for $d = d(\theta)$, but we opt here—to simplify the presentation of these ideas—for the simple choice $d = R - D = constant$, where
$R$ is the average radius of Earth, and $D$ is the constant mean depth of the ocean. Further, we will develop the results for $\hat{w} = 0$ on the bottom (but see the comment below), and so the relevant solution from Eq. (12) is

$$\hat{w}(s, r) = -\Omega s + \sqrt{\Omega^2 s^2 + 2s \int_{R-D}^{R} \frac{\hat{F}(s, r')}{\sqrt{(r')^2 - s^2}} \, dr'},$$  \hspace{1cm} (13)$$

and for realistic (real) solutions we must aim to have $\hat{F} > 0$ (because the contributions from $\Omega$ are very small in this context). It is clear that a generalization of Eq. (13) will admit $\hat{w} = W(\theta)$ on $r = d(\theta)$, although the resulting calculations are rather less transparent.

The pressure can be determined directly from Eqs. (8) and (9); this is most conveniently accomplished by writing

$$\frac{p}{\rho} = -gr + \frac{1}{2} r^2 \Omega^2 \sin^2 \theta + \Pi(r, \theta)$$  \hspace{1cm} (14)$$

and then

$$\frac{\partial \Pi}{\partial r} = \frac{1}{r} K(r \sin \theta, r),$$

$$\frac{\partial \Pi}{\partial \theta} = K(r \sin \theta, r \cot \theta + r G(r, \theta),$$  \hspace{1cm} (15)$$

where

$$K(s, r) = \hat{w}^2(s, r) + 2s \Omega \hat{w}(s, r) = 2s \int_{R-D}^{R} \frac{\hat{F}(s, r')}{\sqrt{(r')^2 - s^2}} \, dr',$$

the last equality arising from Eq. (13). The existence of a unique solution for $\Pi$, which satisfies the general condition at the surface

$$\Pi = \Pi_0(\theta) \quad \text{on} \quad r = R,$$  \hspace{1cm} (17)$$

is guaranteed by virtue of the form of $\hat{w}$ given in Eq. (13) (which relates this function to $G$ via $F$). Indeed, it is this last observation that hides some tiresome complications but only in the presentation of the results: $K$, and therefore $\Pi$, in terms of $G$ (the original body force component) involves a number of additional integrals. Nevertheless, the complete description of the pressure at and below the surface (and on a free surface whose shape can be prescribed) is available.

### 3. Modeling the ACC

On the basis of the exact solution for the general profile given by Eq. (13), and with the corresponding choices (here) of $h(\theta) = 0$ and $d = R - D$, we now investigate how these results can be applied to flows around a midlatitude in the Southern Hemisphere, namely, the ACC (which typically sits about 40° to 60°S). Let the ACC be centered at $\theta = \Theta$ (so $\Theta = 3\pi/4$ or a little larger); the function $F(s, r)$ in Eq. (13), which is associated with the representation of the nonconservative system appropriate for this flow, is determined by making a suitable choice of velocity profile. In section 3a below we determine the most significant elements, via a balance of terms, for the governing equations (8)–(9), and we show that the solution to the linearized problem overestimates the nonlinear exact solution, albeit by a small amount. In section 3b, we discuss how the prescription of the velocity profile at the surface, as a suitable current there, determines the velocity profile beneath if we specify the forcing $F$. An alternative point of view is presented in section 3c, where we start from descriptions of the velocity profile throughout the flow, obtained from direct observations of the ACC in the Drake Passage (see Firing et al. 2011) and determine the corresponding forcing $\hat{F}$.

#### a. Analysis of the balances and linearization

In the governing equations [Eqs. (8)–(9)], the centripetal and gravitational acceleration may be absorbed into a modified pressure

$$P^i = p + \rho gr - \frac{1}{2} \rho \Omega^2 r^2 \sin^2 \theta$$

to transform these equations to

$$\frac{w^2}{r} - 2\Omega w \sin \theta = -\frac{1}{\rho} P^i,$$  \hspace{1cm} (18)$$

$$\frac{w^2}{r} \cot \theta - 2\Omega w \cos \theta = -\frac{1}{\rho r} P^i + G(r, \theta).$$  \hspace{1cm} (19)$$

Given that $2\Omega r \sin \theta \approx 300 \text{ m s}^{-1}$, and that the maximal speed of the ACC does not exceed $3 \text{ m s}^{-1}$, we see that the nonlinear advection terms in Eqs. (18)–(19) are small perturbations of the linear flow $w_0$, which is governed by the system

$$2\Omega w_0 \sin \theta = -\frac{1}{\rho} P^i_0,$$  \hspace{1cm} (20)$$

$$-2\Omega w_0 \cos \theta = -\frac{1}{\rho r} P^i_0 + G(r, \theta).$$  \hspace{1cm} (21)$$

Hence, the radial (modified) pressure gradient must balance the correspondent component of the Coriolis force, and since $\sin \theta = \cos \theta$ in the region of the ACC and...
the meridional pressure gradient is typically relatively small compared to the radial gradient, the leading-order balance is

$$G(r, \theta) \approx -2\Omega w_0 \cos \theta.$$  

(22)

The outcome of this analysis is that the forcing plays a key role in the dynamics of the ACC, being of the order of the meridional component of the Coriolis force. In this context, note that classical models of the ACC do not account for a sufficiently large forcing and this has led to excessively large values of the mass transport—almost 8 times larger than the observed values [see the discussion in Wolff (1999) and Marshall et al. (2016)].

We now show that the nonlinear advection terms, while small, nevertheless have an overall effect on the flow. Indeed, implementing the approach we used for Eqs. (8)–(9) to the linearized version of the system [Eqs. (20)–(21)] leads to

$$\Omega(\tilde{w}_0)_r = \frac{\tilde{F}(s, r)}{\sqrt{r^2 - s^2}}$$  

(23)

instead of Eq. (11). Therefore, the linear solution $\tilde{w}_0 \approx 0$ (for eastward flow) that vanishes on the bed $r = d(\theta)$ is

$$\tilde{w}_0(s, r) = \frac{1}{\Omega s} \int_{\theta=0}^{\theta=\theta} \frac{\tilde{F}(s, r')}{\sqrt{(r')^2 - s^2}} dr'.$$

Comparing this with Eq. (12), for $A(s) = \Omega^2 s^2$, yields

$$\dot{w}(s, r) = -\Omega s + \sqrt{\Omega^2 s^2 + 2\Omega s \tilde{w}_0(s, r)}$$

$$= \frac{2\Omega s \tilde{w}_0(s, r)}{\Omega s + \sqrt{\Omega^2 s^2 + 2\Omega s \tilde{w}_0(s, r)}}$$

$$= \frac{\tilde{w}_0(s, r)}{1 + \frac{2\tilde{w}_0(s, r)}{\Omega s}},$$

whenever $\tilde{w}_0(s, r) > 0$. This shows that, necessarily, the nonlinear terms trigger an overall slowing down of the flow. Certainly, the local effect is very small since $\Omega s \approx 300 \text{ m s}^{-1}$ means that even in the fast flow regions where the speed of the ACC reaches 1.5 m s$^{-1}$ (see Apel 1987), the nonlinear correction amounts to only about 0.4 cm s$^{-1}$. But, more significantly, our approach produces a detailed structure and estimates, without recourse to any approximations. Furthermore, observations provided by the Mercator Ocean database for the month of August (see http://www.mercator-ocean.fr/en/) show that in the Indian section of the southern sea off the coast of Africa, located between 39° and 45°S and 30° and 50°E, the ACC reaches speeds of the order of 1.5 m s$^{-1}$ in the near-surface region. Interpreting this average of measured current velocities as indicative for the value $w_0$ of the mean flow, we observe that for a meridional width of about 666 km in a near-surface layer of depth 375 m, the nonlinear correction accounts for a mass transport of 1 Sv = $10^6$ m$^3$ s$^{-1}$. Despite representing only a small proportion of the mean mass transport of the ACC, estimated at about 130 Sv (see Wolff 1999), this nonlinear correction is nonetheless significant as it exceeds the mass transport of all the world’s rivers combined.

b. Prescription of the velocity profile at the free surface

Throughout this subsection we suppose that the flow at the surface has a maximum (in the middle, we will take for simplicity) and drops to zero at two angles either side, so the ACC can be as wide as we wish. Furthermore, these zeros hold throughout the depth of the flow [by virtue of $\dot{w}(r \sin \theta) = \text{constant}$], and outside these boundaries the flow is taken to be stationary, so $\ddot{w} = 0$ here. Thus, we have a well-defined meridional arc where the ACC exists, of arbitrary length, outside which there is no motion. Now if we were to assume that the body force system is conservative, then $F = 0$ and the solution is described by $\dot{w}(r \sin \theta) = \text{constant}$ everywhere; this means, for example, that the maximum speed at the surface penetrates to the bed [and although there is some evidence that the ACC flows with some speed to great depths, see Ivchenko and Richards (1996), a conservative force model takes this to extremes]. A schematic depicting this flow configuration is shown in Fig. 3, which represents a section through Earth in the Southern Hemisphere, with the ocean surface and ocean bed (appearing as arcs of circles here) included. The velocity profile takes constant values on lines parallel to the polar axis and, plainly, this is unacceptable as a description of the ACC. However, our model that invokes a nonconservative system allows the profile to be both prescribed at the surface and have a suitable structure with depth for an appropriate choice of $F$ (i.e., $G$). To proceed, we will assume that the surface profile is imposed at a fixed radius $r = R$ in accordance with our earlier simplifying assumptions, and then to maintain this will require some appropriate pressure distribution, which can be determined from Eqs. (14) to (16). (If the free surface in the region of the current is not $r = \text{constant}$, its shape can be specified and the corresponding pressure distribution can again be determined, although the calculations in this case are rather more tedious.) Let

$$w = f(\theta) \text{ on } r = R,$$
then Eq. (13) gives  

\[ f(\theta) = -\Omega R \sin \theta + \sqrt{\Omega^2 R^2 \sin^2 \theta + 2R \sin \theta M(R \sin \theta)} \int_{R-D}^{R} \frac{F(R \sin \theta, r')}{\sqrt{(r')^2 - R^2 \sin^2 \theta}} \, dr'. \]  

(24)

At first sight, expression (24) might appear rather daunting; however, some simple choices are available that make this significantly more transparent (and, although not a necessary requirement within our exact solution, it is certainly advantageous). The two important observations are 1) we may simply choose the form of \( F \) (and although it might be tempting to interpret this as being generated by a particular physical, non-conservative system, that is beyond our remit within our proposed approach to modeling at this early stage of the investigation); and 2) \( F \) can accommodate any suitable behavior in depth, that is, its dependence on \( r \). Thus, a convenient choice for \( F \) is the separable form  

\[ \tilde{F}(s, r) = M(s)N'(r)\sqrt{r^2 - s^2}, \quad R - D \leq s < r \leq R, \]  

(25)

where the derivative on \( N \) (denoted by the prime) is to ease the following calculation; otherwise, \( M \) and \( N \) are arbitrary functions. When Eq. (25) is used in Eq. (24), we obtain  

\[
\begin{align*}
   f(\theta) &= -\Omega R \sin \theta \\
   &= -\Omega R \sin \theta + \sqrt{\Omega^2 R^2 \sin^2 \theta + 2R \sin \theta M(R \sin \theta)} \int_{R-D}^{R} \frac{F(R \sin \theta, r')}{\sqrt{(r')^2 - R^2 \sin^2 \theta}} \, dr'. 
\end{align*}
\]

(26)

and the ACC current is described by  

\[
\begin{align*}
   \dot{w}(s, r) &= -\Omega s \\
   &= -\Omega s + \sqrt{\Omega^2 R^2 \sin^2 \theta + 2s M(s)[N(r) - N(R - D)]} \\
   \text{for given } M \text{ and } N. \quad \text{Finally, to complete this version of the solution, we will need to select a suitable surface profile and the behavior with the depth [which here is controlled by the function } N(r) \text{].}
\end{align*}
\]

The profile that we choose must satisfy the general requirements (of a maximum speed on \( \theta = \Theta \), dropping to zero at any two given angles away from \( \theta = \Theta \)) and not pose too many analytical difficulties (generated by the dependence on \( s = r \sin \theta \)). With these points in mind, we introduce the surface azimuthal velocity profile  

\[
\begin{align*}
   f(\theta) &= \begin{cases} 
   \dot{w}_0 \cos(\theta - \Theta) - \dot{\theta}_0 \sin \theta, & \Theta - \dot{\theta}_0 \leq \theta \leq \Theta + \dot{\theta}_0, \\
   0, & |\theta - \Theta| > \theta_0,
\end{cases} \quad \text{(28)}
\end{align*}
\]

where \( \dot{w}_0 = w_0/(1 - \cos \theta_0) \), so we have a maximum speed of \( w_0 \) at \( \theta = \Theta \), with zero at \( \theta = \Theta \pm \dot{\theta}_0 \) for some \( \dot{\theta}_0 > 0 \), and we certainly expect \( \theta_0 < \pi/18 \). We use Eq. (28) in Eq. (26), find \( M(R \sin \theta), \) that is, \( M(s), \) to give  

\[
2s M(s)[N(r) - N(R - D)] = \dot{w}_0 \left[ \frac{s}{R} \sin \theta - (\cos \theta) \sqrt{1 - \frac{s^2}{R^2} - \cos \theta_0} \right]^2 \\
+ 2\Omega s \dot{w}_0 \left[ \frac{s}{R} \sin \theta - (\cos \theta) \sqrt{1 - \frac{s^2}{R^2} - \cos \theta_0} \right], \quad \text{(29)}
\]

Knowing \( N(r) \), Eq. (29) defines the function \( M(s) \). To fix \( N \), let us use the simplest realistic depth profile: exponential decay from the given surface speed at any point to zero on the bottom. (We have already commented that we could relax this and allow a nonzero speed on the bed of the ocean if that would provide a better model in some circumstances). We set  

\[
N(r) = N_0 e^{\alpha(r - R)}, \quad \text{(30)}
\]

where \( N_0 \) and \( \alpha > 0 \) are constants. All these choices are inserted into the general expression for \( \dot{w} \), Eq. (27), to
produce a profile that varies across a meridional arc and also with depth; an example is shown in Fig. 4. This is presented as a surface that depicts the speed in the flow, with the surface current (varying from zero to a maximum) clearly evident in the right panel. (The orientation is identical to that used in Fig. 3, so the ocean surface, as shown in this figure, is below the ocean bed; the zero on the bottom and along the edges of the meridional arc are therefore on the upper edge and the left and right edges, respectively, in the figure.).

An important consequence of our approach to this modeling problem, that is, by introducing a non-conservative body force, which is used to mimic the complicated processes that produce (it is believed) the observed type of ACC profiles, is that we need it only where the ACC exists. Outside the ACC, we take the flow to be stationary—the only motion in our model is that of the ACC—and the stationary state requires simply the conventional conservative body force to maintain the existing pressure distribution. Thus, we define $G$ [see Eqs. (4) and (9)] so that it is zero in the regions where $w = 0$, which, in our example, is for $|\theta - \Theta| > \theta_0$; we then see that the pressure determined from Eqs. (14) to (16), with $K = 0$, is that consistent with only gravity, centripetal and Coriolis contributing. Furthermore, it is instructive to observe that the body force is also conservative if there is no dependence on $r$, that is, $N(r) = \text{constant in our chosen form for } \bar{F}$, for then $\bar{F} = 0$ [see Eq. (25)]; it is the requirement to produce a suitable behavior of the velocity with depth that forces us to consider a nonconservative system in our model.

c. Matching to descriptions from field data

In our general discussion above, we have presented a theory that does not select a specific form of non-conservative forcing that is driven by any particular physical principles that might underpin the flow. We now indicate how this can be done by referring to the vertical structure descriptions of the ACC in the Drake Passage from direct velocity observations, presented in Firing et al. (2011).

The first type of mean velocity profile, advocated by Ferrari and Nikurashin (2010), is linear, with $\bar{w}_0(s, r) = a(s)(r - d)$ for some function $a(s) > 0$, where the speed vanishes on the bed $r = d$. The corresponding forcing, consistent with our formulation in Eq. (23), is then $F(r, \theta) = -\Omega r \cos \theta a(r \sin \theta)$, with the function $a(s)$ determined from observational data. An alternative mean velocity profile, fitted to the data, which takes the form $\bar{w}_0(s, r) = a(s)[e^{r/L(s)} - e^{r/L(s)}]$, with the length scale $L(s)$ between 1100 and 1700 m, increasing slightly from south to north, was found to be reasonably accurate in this region (see Firing et al. 2011). Since south to north in our coordinate system means that $\theta$ decreases around $3\pi/4$, a reasonable choice is $L(s) = bs$, and then, from Eq. (23), we find that the corresponding forcing is $F(r, \theta) = -\Omega b a(r \sin \theta)(\tan \theta)^{1/(b \sin \theta)}$ for suitable choices of the function $a(s) > 0$ and the constant $b > 0$.

4. Discussion and conclusions

This analysis has demonstrated how it is possible to produce an exact model ("exact" in the sense of an exact...
solution of the full set of governing inviscid equations written in spherical coordinates) for the ACC. We have shown that the conventional conservative system has some significant shortcomings but that the essential difficulties are overcome if we invoke a nonconservative body force to maintain the ACC. This, we have argued, is a suitable mathematical maneuver that successfully replaces the ad hoc modeling of the many non-conservative processes that certainly contribute to this flow. Indeed, our approach enables considerable freedom in the choice of velocity profile, surface distortion, and surface pressure distribution.

The ocean–atmosphere interactions in the circumpolar belt where the ACC resides are significant and varied. Satellite data and observation by vessels show consistently large eastward wind speeds (of average 8–12 m s\(^{-1}\)) with little variation in wind direction. The wind regime is dominated by frequent storms and has strong meridional variations, with the maximal wind speed near 50\(^\circ\)S, and gale force winds occur frequently (as much as 20\% of the time; see Tomczak and Godfrey 1994). The winds blowing over the ocean surface drive a near-surface flow, and the latitudinal variations in the wind velocities create areas of convergence and divergence at the surface that are balanced by upwelling and downwelling flows. Importantly, from our point of view, the wind regime generates a predominantly zonal flow in a fragmented system of more or less intense circumpolar jet streams; observations indicate that about 75\% of the total flow in the oceanic region of the ACC occurs in zonal jets, which occupy about 19\% of the cross-sectional area (see Tomczak and Godfrey 1994). Indeed, it has been observed that these zonal jets are not restricted to a near-surface layer but extend to great depths (in most places to the ocean floor). Drag due to topography (form drag, i.e., drag that arises from the pressure distribution generated by the shape of an object) slows down the current, preventing the zonal flow from accelerating indefinitely, despite being continuously fed with momentum by the westerly wind stress along an unobstructed band of latitude: the eastward flow establishes a pressure difference across the four major north–south submarine ridges that are encountered (and crossed at gaps, a process that enhances the filamentary structure of the ACC; see Klinck and Nowlin 2001). The associated jets that contribute to the ACC can be included within our exact solution simply by combining a number of solutions of the type that we have described, that is, a maximum at the center, dropping to zero on either side, and this repeated a number of times, the junction between each pair of jets being along the zero-speed line at a suitable latitude. All this is possible, we believe, because the Reynolds stresses and viscous terms are very small in these flows (see Olbers et al. 2004), and this permits us to model the flow dynamics by the inviscid Euler equations but with meridional nonconservative forcing. Furthermore, the wind-driven forcing, by which energy is transferred to the system, can be captured in our model by allowing pressure changes at the surface; the classical theory of surface gravity waves, on the other hand, takes the pressure on the surface to be constant, that is, atmospheric pressure. [The mechanism by which the wind imparts momentum on the water, using various models, is discussed in Johnson (2012) and Walsh et al. (2013).]

We regard the current as the eventual by-product of wind blowing over the water, but we remain agnostic as to how exactly that generation takes place, and, importantly, such details are not required in our model.

The fact that we have described the construction of an exact solution should be regarded as no more than the starting point for a careful and systematic theoretical investigation of these flows; a number of possibilities spring to mind. Our results presented here allow for an underlying flow to be modeled as accurately as we wish, at least in principle, by choosing an appropriate body force; this is clearly an avenue that needs further investigation, presumably based on available data and physical models for the flow. With a suitable background flow in place, it is natural, we believe, to examine the effects on wave propagation. Wave–current interactions are of considerable interest, and our exact solution makes this type of investigation readily accessible. The procedure is to perturb the solution described here, regarded as a prescribed background state, by adding surface waves and examining their general properties, evolution, and so on. The wave–current interaction then becomes a problem that can be studied as an irrotational wave perturbation of a background pure current flow; see Constantin and Johnson (2015), where this type of approach has been successfully applied to equatorial wave–current interactions. Another avenue would be to allow the flows of the type developed here to evolve slowly in the azimuthal direction; this will enable slow depth changes, for example, to be incorporated. Of course, both these give rise to approximate—no longer exact—solutions, for which suitable asymptotic techniques are likely to be relevant.

Finally, we make a brief general observation about the interpretation of our modeling assumptions in the light of available oceanographic data. We take the view that the proficiency of comprehensive and complicated numerical models that simulate the complex dynamics of the Southern Ocean actually spur the need for an understanding, at a basic level, of the analytical structure of these problems, helping to elucidate what is possible and
what is allowed. The practical benefit of an analytically tractable model is that it offers a global perspective, albeit of a simplified version of the system as a whole, but with the inherent ability to grasp the dominant features and the fundamental processes of the overall dynamics. The interactive exploration of analytical ideas and numerical simulations is a very powerful tool.

In conclusion, we submit that this new, exact solution, which provides a model for the Antarctic Circumpolar Current, can be the starting point for further, and more directly relevant, theoretical and numerical investigations of this important oceanic flow.

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REFERENCES


