Upgradient and Downgradient Potential Vorticity Fluxes Produced by Forced Rossby Waves. Part II: Parameter Sensitivity and Physical Interpretation

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ABSTRACT

We examine the potential vorticity (PV) flux produced by forced Rossby waves in a two-layer quasigeostrophic model, using a perturbation analysis. Rossby waves are excited by external forcing applied to the upper layer. The southward PV flux is produced in the lower layer by the higher-order Rossby waves that are excited by nonlinear wave–wave interactions, whereas the northward PV flux is produced in the upper layer. The direction of the PV flux is consistent with that obtained by an eddy-resolving model of the wind-driven circulation in previous studies. The southward PV flux is produced in a wide parameter range comparable to the eddy-resolving model. The basic features of the PV flux remain unchanged even in the limit of weak stratification. In this limit, stratification has nearly no effect on the flow, except that it isolates the lower layer from the direct effects of external forcing. The mechanism of the southward PV flux is explained using basic features of the barotropic Rossby waves and does not depend on details of the model. Furthermore, the resonant triad interaction of Rossby waves does not affect the PV flux. Stratification weakens or strengthens the PV flux depending on the horizontal scale of the external forcing.

1. Introduction

A western boundary current and its eastward extension, such as the Gulf Stream, Kuroshio, and Kuroshio Extension, are flanked by recirculation gyres (Kawai 1972; Worthington 1976; Hogg 1983; Hogg et al. 1986; Jayne et al. 2009). The volume transport of each of the recirculation gyres flanking the northern and southern sides of the Gulf Stream is 20–30 Sv ($1 \text{ Sv} \equiv 10^6 \text{ m}^3 \text{ s}^{-1}$; Hogg 1992). The transport of the Gulf Stream is locally increased from about 95 to 150 Sv by the recirculation gyres (Hogg 1992). Pioneering numerical experiments by Holland and Rhines (1980) showed that the eddy potential vorticity (PV) flux produced by mesoscale perturbations drives recirculation gyres. Two possible mechanisms maintaining recirculation gyres have been proposed. One is PV homogenization (Rhines and Young 1982, hereinafter RY82). RY82 proposed that stirring of the PV by mesoscale perturbations drives the deep recirculation gyres. However, they assumed without proof that the PV flux is oriented in the downgradient direction of the time-mean PV.

The other possible mechanism is the rectification of Rossby wave motion, which is proposed by observational,

numerical, and theoretical studies (Thompson 1977; Hogg 1988; Malanotte-Rizzoli et al. 1995; Mizuta 2009; Waterman and Jayne 2011). Planetary and topographic Rossby waves tend to dominate in the mesoscale perturbations near the western boundary current at periods from a few tens of days to a few months (Thompson and Luyten 1976; Thompson 1977; Hogg 1981; Imawaki 1985; Bower and Hogg 1992). Haidvogel and Rhines (1983, hereinafter HR83) represented the strong perturbation of the western boundary current by an external forcing that is oscillatory in time and localized in space. They showed that Rossby waves excited by forcing drives a mean flow that is qualitatively the same as recirculation gyres. The advantage of the Rossby wave rectification mechanism is that the PV flux is determined by linear dynamics. This contrasts with the study by RY82, which assumed the downgradient PV flux without proof.

However, neither RY82 nor HR83 can completely reproduce the PV flux obtained in the numerical experiment by Holland and Rhines (1980). That is, the PV flux in Holland and Rhines (1980) is northward in the surface layer and southward in the deep layer, implying that the PV flux in the surface layer is oriented in the upgradient direction of the mean PV. Hence, the assumption by RY82 does not hold in the surface layer. The PV flux obtained by HR83, who used a barotropic

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model, is northward and only partly consistent with that of [Holland and Rhines \(1980\)](#). Thus, the mechanism maintaining the recirculation gyres has not been sufficiently understood. To resolve this difficulty, it is important to examine more precisely the character of the PV flux by Rossby waves. [Waterman and Jayne \(2012\)](#) extended the study by [HR83](#) and examined effects of the mean flow and the horizontal divergence on Rossby waves. However, the character of the PV flux by Rossby waves has not been fully understood.

In [Mizuta \(2018, hereinafter Part I\)](#), the southward PV flux was produced in the deep layer when stratification and nonlinearity were included in the [HR83](#) experiment, implying that the direction of the PV flux was consistent with that in [Holland and Rhines \(1980\)](#) in both the surface and deep layers. The southward PV flux is produced in both the linear case, in which linear theory of the wave–wave and wave–mean flow interaction is applicable to a good approximation, and in the nonlinear case, in which the nonlinear effect is of the same order as the beta effect. However, the mechanism producing the southward PV flux is still not clear enough. In addition parameters such as the horizontal scale and frequency of the forcing were fixed in [Part I](#). The southward PV flux is produced by the harmonic wave, which is excited by nonlinearity of Rossby waves. [Longuet-Higgins and Gill \(1967\)](#) showed that Rossby waves can form a resonant triad, exchanging energy effectively between resonant waves through the nonlinear interaction. However, it is not clear if the resonant interaction affects the production of the PV flux. Hence, we further examine the meridional component of the PV flux produced by the Rossby wave in [Part II](#).

In [Part II](#), the PV flux produced by Rossby waves is examined for two purposes. The first purpose is to examine whether the southward PV flux in the deep layer is commonly produced in the wide parameter range. We focus on the perturbation analysis, which is shown in [Part I](#) to be able to qualitatively reproduce nonlinear effects even in a moderately nonlinear case. In [Part I](#), we showed theoretically that the northward PV flux has to dominate when Rossby waves are excited by external or nonlinear forcing. Hence, it may be counterintuitive that the southward PV flux is produced by nonlinearity. The second purpose of [Part II](#) is to obtain a physical explanation of why the southward PV flux is produced. [Part II](#) is organized as follows: In [section 2](#), we describe the formulation of the perturbation analysis. We describe the numerical method and results of the perturbation analysis in [section 3](#). We examine the mechanism producing the southward PV flux for a special case in which stratification is weak in [section 4](#) and a more general case in [section 5](#). We summarize the results from these sections in [section 6](#).

2. Formulation

a. Perturbation analysis

We consider motions excited by external forcing that is oscillatory in time and localized in space in a two-layer, flat-bottomed ocean. It was shown in [Part I](#) that the basic feature of the PV flux is determined by the barotropic and the first baroclinic modes. Hence, we employed a two-layer model here for simplicity, instead of the Regional Ocean Model System (ROMS; [Haidvogel et al. 2000](#)) employed in [Part I](#). By doing so, we can explore the distribution of PV flux in a wide range of parameters. We have confirmed that results of a two-layer model and ROMS are similar to each other if the same forcing and internal deformation radius are used.

When the quasigeostrophic approximation is assumed, the nondimensional equations of the PV are

$$q_{k,t} + \varepsilon J(\psi_k, q_k) + 2\psi_{k,x} = 2W_e(x, y)e^{-it}\delta_{k1}, \quad (2.1)$$

where

$$q_k = \nabla^2 \psi_k + \frac{F}{2}(-1)^k(\psi_1 - \psi_2)$$

is the PV, ψ_k is the streamfunction, x and y are the eastward and northward coordinates, respectively, t is time, and the subscript $k = 1, 2$ denotes the upper- and lower-layer variables, respectively. Because eddy motions, which are considered the source of the waves, are intense near the surface ([Richardson 1983](#)), the forcing on the right-hand side of (2.1) is confined to the upper layer. All variables have been scaled with a characteristic velocity U , the angular frequency ω of the forcing, and

$$k_\beta = \frac{\beta}{2\omega}, \quad (2.2)$$

which represents half of the maximum wavenumber of the barotropic Rossby wave at the frequency of ω . The coefficient β is the meridional gradient of the Coriolis parameter. The nondimensional parameter,

$$\varepsilon = \frac{Uk_\beta}{\omega},$$

is a measure of nonlinearity, and

$$F = \frac{1}{(r_d^{\text{dim}} k_\beta)^2}$$

corresponds to the reciprocal of the Burger number defined with k_β and the internal deformation radius r_d^{dim} in dimensional units. For simplicity, we assume the upper- and lower-layer thicknesses to be the same. We

discuss later that fundamental features of the PV flux do not change even if the ratio of the upper- and lower-layer thicknesses takes a general value.

Assuming that ε is small, we expand ψ_k in an asymptotic series in ε :

$$\psi_k = \psi_k^{(0)} + \varepsilon \psi_k^{(1)} + \varepsilon^2 \psi_k^{(2)} + \cdots \quad (2.3)$$

Substituting (2.3) into (2.1) yields the $O(1)$ equations

$$\psi_k^{(0)} = \psi_k^p e^{-it} \quad \text{and} \quad (2.4)$$

$$-iq_k^p + 2\psi_{k,x}^p = 2W_e \delta_{k1}, \quad (2.5)$$

and the $O(\varepsilon)$ equations

$$\psi_k^{(1)} = \psi_k^h e^{-2it} + \bar{\psi}_k^{(1)}, \quad (2.6)$$

$$-2iq_k^h + 2\psi_{k,x}^h = -\frac{1}{2}J(\psi_k^p, q_k^p) \equiv w_k^h, \quad \text{and} \quad (2.7)$$

$$2\bar{\psi}_{k,x}^{(1)} = -\frac{1}{2}J(\psi_k^{*p}, q_k^p) \equiv \bar{w}_k^{(1)}, \quad (2.8)$$

where

$$q_k^p = \nabla^2 \psi_k^p + \frac{F}{2}(-1)^k(\psi_1^p - \psi_2^p).$$

The q_k^h is determined from ψ_k^h in the same manner as q_k^p , and the asterisk represents the complex conjugate. Equation (2.5) implies that the wave, which has the same frequency as the external forcing, is excited to the lowest order. This wave is referred to as the primary wave. Equations (2.7) and (2.8) imply that nonlinearity of the primary wave drives the mean flow $\bar{\psi}_k^{(1)}$ and the wave ψ_k^h , whose frequency is twice as large as that of the primary wave. We refer to ψ_k^h as the harmonic wave. The harmonic wave drives the mean flow of $O(\varepsilon^3)$:

$$2\bar{\psi}_{k,x}^{(3)} = -\frac{1}{2}J(\psi_k^{h*}, q_k^h) \equiv \bar{w}_k^{(3)}. \quad (2.9)$$

Because $O(\varepsilon^2)$ motions do not include the mean flow, the mean flow is dominated by $\bar{\psi}^{(3)}$ in the region where $\bar{\psi}^{(1)}$ is negligible. For the rest of Part II, we drop superscript p for simplicity. Hence ψ_k and q_k without a superscript represent the streamfunction and PV, respectively, of the primary wave.

By using the barotropic and baroclinic modes of ψ , which are respectively defined as

$$\psi_b = \frac{\psi_1 + \psi_2}{2} \quad \text{and} \quad (2.10)$$

$$\psi_c = \frac{\psi_1 - \psi_2}{2}, \quad (2.11)$$

and ψ_b^h and ψ_c^h , which are defined with ψ_k^h in the same manner as ψ_b and ψ_c , we can express (2.5) and (2.7) as

$$-i\nabla^2 \psi_b + 2\psi_{b,x} = W_e, \quad (2.12)$$

$$-i(\nabla^2 \psi_c - F\psi_c) + 2\psi_{c,x} = W_e, \quad (2.13)$$

$$-2i\nabla^2 \psi_b^h + 2\psi_{b,x}^h = w_b^h, \quad \text{and} \quad (2.14)$$

$$-2i(\nabla^2 \psi_c^h - F\psi_c^h) + 2\psi_{c,x}^h = w_c^h. \quad (2.15)$$

The forcing to the harmonic wave is

$$w_b^h = -J(\psi_b, q_b) - J(\psi_c, q_c) \quad \text{and} \quad (2.16)$$

$$w_c^h = -J(\psi_b, q_c) - J(\psi_c, q_b), \quad (2.17)$$

where

$$q_b = \nabla^2 \psi_b \quad \text{and} \quad (2.18)$$

$$q_c = \nabla^2 \psi_c - F\psi_c \quad (2.19)$$

are the barotropic and baroclinic components of the PV, respectively. Equations (2.12) and (2.13) have homogeneous solutions, $\psi_b = A_b e^{i(k_b x + l_b y)}$ and $\psi_c = A_c e^{i(k_c x + l_c y)}$, which respectively satisfy the dispersion relation:

$$(k_b + 1)^2 + l_b^2 = 1, \quad \text{and} \quad (2.20)$$

$$(k_c + 1)^2 + l_c^2 = 1 - F. \quad (2.21)$$

Equation (2.21) indicates that wavenumbers k_c and l_c are complex; hence ψ_c is evanescent when

$$F > 1. \quad (2.22)$$

Equation (2.22) is rewritten with variables in dimensional units as

$$\omega > \omega_c = \frac{\beta r_d^{\text{dim}}}{2}. \quad (2.23)$$

The typical period of Rossby waves observed near the western boundary current is between a few tens of days and a few months (Thompson and Luyten 1976; Thompson 1977; Hogg 1981; Imawaki 1985; Bower and Hogg 1992), satisfying the condition in (2.23). Thus, we focus on the case $F > 1$ in the present study. Note that both the primary and harmonic waves are evanescent when $F > 1$.

b. Meridional PV flux

As we considered in Part I for the continuously stratified ocean, the direction of the PV flux is constrained by the enstrophy equation (Rhines and Holland 1979;

Plumb 1986). Because this constraint is important in the present study, we briefly review it, focusing on the two-layer ocean. Equations (2.5) and (2.7) can be expressed in a general form,

$$-imq_k^M + 2\psi_{k,x}^M = w_k^M, \quad (2.24)$$

where $(m, M) = (1, p)$ or $(2, h)$ with $w_k^p = 2W_e \delta_{k1}$. By multiplying (2.24) by $q_k^*/4$, taking the real part, and introducing multiple time scales, we obtain the equation of enstrophy [or, more precisely, a wave action, because the denominator of the first term on the left-hand side of (2.25) corresponds to 2β]:

$$\delta \frac{\partial}{\partial T} \frac{|q_k^M|^2}{4} + Y_k = \frac{1}{4} \text{Re } w_k^M q_k^{M*}, \quad (2.25)$$

where Re denotes the real part, Y_k is the meridional component of the mean PV flux,

$$(X_k, Y_k) = \text{Re}(-\psi_{k,y}^M q_k^{M*}, \psi_{k,x}^M q_k^{M*}), \quad (2.26)$$

and $\delta^{-1}T$ represents the time larger than $O(1)$. The second term on the left-hand side of (2.25) is rewritten as

$$Y_k = \nabla \cdot \mathbf{J}_{Hk} + (-1)^k J^z, \quad (2.27)$$

where

$$\mathbf{J}_{Hk} = \text{Re} \left(\frac{|\psi_{k,x}^M|^2 - |\psi_{k,y}^M|^2}{2} - \frac{F|\psi_1^M - \psi_2^M|^2}{4}, \psi_{k,x}^M \psi_{k,y}^{M*} \right) \text{ and} \quad (2.28)$$

$$J^z = \text{Re} \frac{F}{4} (\psi_{1,x}^M + \psi_{2,x}^M)(\psi_1^{M*} - \psi_2^{M*}) \quad (2.29)$$

are the horizontal and vertical components of the enstrophy flux, respectively. The right-hand side of (2.25) represents the production of enstrophy by w_k^M , whereas the first and second terms on the left-hand side of (2.25) represent the temporal change of enstrophy and the meridional PV flux, which is equivalent to the divergence of enstrophy flux. Equation (2.25) indicates that the northward and southward PV flux is produced when w_k^M tends to increase and decrease enstrophy in the stationary state, respectively. No meridional PV flux is produced when w_k^M vanishes corresponding to the conventional nonacceleration theorem.

Taking the summation over k and integrating over a large area S_L , (2.27) yields

$$\int_{S_L} (Y_1 + Y_2) dx dy = \int_{\Gamma_L} (\mathbf{J}_{1H} + \mathbf{J}_{2H}) \cdot \mathbf{n} dl, \quad (2.30)$$

where Γ_L is the boundary of S_L and $\mathbf{n}$ is the unit vector normal to Γ_L , pointing outward. If motions are locally approximated by a plane wave on Γ_L , the enstrophy flux across Γ_L is expressed in terms of the group velocity $\mathbf{c}_g$:

$$\mathbf{J}_{1H} + \mathbf{J}_{2H} = \mathbf{c}_g \frac{|q_1^M|^2 + |q_2^M|^2}{4}$$

(Plumb 1986). If there is no source of enstrophy outside of S_L , $\mathbf{c}_g \cdot \mathbf{n}$ is positive. Then (2.30) implies that the sum of the upper- and lower-layer PV fluxes integrated over S_L is northward. Therefore, the northward PV flux has to dominate in S_L , whereas the PV flux can locally be southward in the region where w_k tends to decrease the enstrophy.

For later use, we derive an alternative expression of the meridional PV flux. That is, eliminating q_k^M from (2.24) and (2.25) yields

$$Y_k = -\frac{1}{2} \text{Re } w_k^{M*} \eta_k^M, \quad (2.31)$$

where

$$\eta_k^M = \frac{i\psi_{k,x}^M}{m} \quad (2.32)$$

is the northward displacement of the fluid particles.

3. Numerical experiment

Equations (2.12)–(2.15) were solved numerically, using a Green's function:

$$\psi_b(x, y) = \iint_S G_{b1}(x - x', y - y') W_e(x', y') dx' dy', \quad (3.1)$$

$$\psi_c(x, y) = \iint_S G_{c1}(x - x', y - y') W_e(x', y') dx' dy', \quad (3.2)$$

$$\psi_b^h(x, y) = \iint_S G_{b2}(x - x', y - y') w_b^h(x', y') dx' dy', \quad \text{and} \quad (3.3)$$

$$\psi_c^h(x, y) = \iint_S G_{c2}(x - x', y - y') w_c^h(x', y') dx' dy', \quad (3.4)$$

where G_{bm} and G_{cm} ($m = 1, 2$) are Green's functions defined as

$$G_{bm} = \frac{1}{4m} H_0^{(2)} \left(\frac{r'}{m} \right) e^{i[(x-x')/m]},$$

$$G_{cm} = \frac{1}{4m} H_0^{(2)} (i\kappa_m r') e^{i[(x-x')/m]},$$

$$\kappa_m = \sqrt{F - \frac{1}{m^2}}, \quad \text{and}$$

$$r' = \sqrt{(x - x')^2 + (y - y')^2}.$$

The $H_0^{(2)}$ is the zeroth-order Hankel's function of the second kind and S is the model domain, which is large enough compared to the horizontal scale of the forcing. Nonlinear forcings in (3.3) and (3.4) were obtained from (2.16) and (2.17). The external forcing W_e in (3.1) and (3.2) was

$$W_e = Ae^{-[(|x|+|y|)/a]}. \quad (3.5)$$

Because W_e given by (3.5) has a single sign in space, the horizontal average of W_e does not vanish. If W_e changes its sign in space, the meridional PV flux also tends to change sign even in the linear barotropic model (Mizuta 2014). Because we are interested in the effects of nonlinearity and stratification on the direction of the PV flux, we chose the form of (3.5) to avoid complexity. In addition, x and y derivatives of W_e are discontinuous along $x = 0$ and $y = 0$. However, we discuss later that this does not cause any problems.

Equations (3.1)–(3.4) include two independent parameters, a and $F^{1/2}$, which represent the horizontal scale of the forcing scaled with k_β^{-1} and the reciprocal of the internal deformation radius scaled with k_β^{-1} , respectively. We performed 51 experiments, changing a from 0.25 to 3 for moderate $F^{1/2}$ values (first and second rows in Table 1) and from 0.5 to 1 for large $F^{1/2}$ values (the third row in Table 1). When $F^{1/2}$ is large, the internal deformation radius becomes small; hence experiments for large values of a were not performed, as the number of grid boxes necessary to resolve the deformation radius becomes quite large in such experiments. The amplitude A of the forcing does not affect the pattern of the primary and harmonic waves. Because the velocity scale U was determined by the amplitude A^{dim} of the forcing in dimensional units as $U = A^{\text{dim}}/12.5k_\beta\omega$ in all experiments, we obtain $A = 12.5$. Here the factor 12.5 is determined empirically so that U represents the typical velocity. [It is shown from (3.1) and (3.3) that ψ_b and ψ_b^h do not depend significantly on a for $a \gg 1$, as G_{bm} attenuates with the horizontal scale of $O(1)$. Hence the appropriate scale of velocity U can be obtained using a constant A for $a \geq O(1)$. Although this is not the case for $a \ll 1$, $A = 12.5$ was used in all experiments to avoid complexity.]

We performed numerical calculation with variables with dimensional units and nondimensionalized results for analysis. In all experiments, the internal deformation radius r_d^{dim} was 43 km, which is a typical value near the Kuroshio Extension. We changed F by changing ω , which affects k_β in (2.2). For a typical example, when the

TABLE 1. Experimental conditions. Experiments were performed for all possible pairs of a and $F^{1/2}$ listed in the first and second rows, respectively. Two experiments with $a = 0.5$ and 1 were performed for each $F^{1/2}$ listed in the third row.

a	0.25, 0.5, 1, 1.5, 2, 2.5, 3
$F^{1/2}$	1.06, 1.13, 1.70, 2.55, 3.40
	4.24, 5.09, 5.94, 6.79, 7.64, 8.49, 9.34, 10.19

period of forcing is 100 days, k_β^{-1} is 72.7 km and $F^{1/2} = 1.70$. The horizontal scale ak_β^{-1} of the forcing in dimensional units ranged between 11.4 and 436 km. The dimension of the model domain was 4000 km by 4000 km for most of the experiments and 8000 km by 8000 km when ak_β^{-1} was large. The number of grid boxes was 512 in both x and y . We confirmed that the numerical solution does not substantially depend on the number of grids or the domain size.

As a typical example of the primary wave, Figs. 1a, 1b, and 1d show the horizontal distribution of $\text{Re}\psi_1$, $\text{Re}\psi_2$, and W_e for $a = 1$ and $F^{1/2} = 1.70$, respectively. The region away from the forcing is not shown because the PV flux is weak there. Because ψ_c is evanescent, ψ_b dominates in ψ_1 and ψ_2 except for the region near the forcing. The distribution of ψ_1 and ψ_2 is qualitatively the same as in HR83. Figure 1c shows the distribution of the northward PV flux by the primary wave in the upper layer. As we considered in section 2b, the PV flux is northward in the upper layer. No meridional PV flux is produced in the lower layer because the external forcing vanishes there (not shown). As a typical example of the harmonic wave, the horizontal distribution of $\text{Re}\psi_1^h$ and $\text{Re}\psi_2^h$ for the same experiment as in Fig. 1 is shown in Figs. 2a and 2b, respectively. Although the amplitude of ψ_1^h is large near $(x, y) = (0, 0)$ owing to the contribution by ψ_c^h (Fig. 2a), radiation of the barotropic wave is apparent in ψ_2^h (Fig. 2b). Note that the contour intervals are different between Figs. 2a and 2b. Figures 2c and 2d show the PV flux by the harmonic wave in the upper and lower layers, respectively. The PV flux is northward and southward in the upper and lower layers, respectively. Because the meridional PV flux by the primary wave vanishes in the lower layer, the southward PV flux by the harmonic wave dominates there.

The meridional distributions of zonally integrated PV fluxes in the lower layer for $a = 0.25, 0.5, 1, 2$, and 3 are compared in Figs. 3a–e for various values of $F^{1/2}$. The southward PV flux dominates in the lower layer for all values of a and $F^{1/2}$. The weak northward PV flux is also produced near $y \sim 0$ for $F^{1/2} = 1.13$, especially for small values of a (purple lines in Figs. 3a,b). Although figures are not shown, ψ_c increases as $F^{1/2} \rightarrow 1$, producing the

FIG. 1. The primary wave for $a = 1$ and $F^{1/2} = 1.70$. Contours of (a) $\text{Re}\psi_1$, (b) $\text{Re}\psi_2$, (c) the meridional PV flux in the upper layer, and (d) W_e . Solid and dashed contours indicate positive and negative values, respectively. Contour intervals are (a) 0.5, (b) 0.5, (c) 5, and (d) 2.

northward PV flux to the next order. However, ψ_c increases only logarithmically as $F^{1/2} \rightarrow 1$.

As a measure of the amplitude of the southward PV flux in the lower layer, the minimum value of the zonally integrated PV flux $Y_{\min}$ is plotted against $F^{1/2}$ and a in Figs. 4a and 4b, respectively. As $F^{1/2}$ increases, $Y_{\min}$ asymptotically approaches a constant for both $a = 0.5$ and 1 (Fig. 4a). Thus, the substantial amount of the southward PV flux is produced even if stratification is very weak. The horizontal lines in Fig. 4a indicate $Y_{\min}$ in the limit as $F^{1/2} \rightarrow \infty$, which will be derived in the next section. As $F^{1/2}$ increases, $Y_{\min}$ is asymptotic to these lines. The cyan, red, and black lines in Fig. 4b indicate $Y_{\min}$ for $F^{1/2} = 1.70$, 3.40, and ∞ , respectively. The PV flux $Y_{\min}$ increases in amplitude with a in $a < 2.5$ and slightly decreases in

$a > 2.5$. In the limit $F \rightarrow \infty$, $Y_{\min}$ remains the same order as that for a finite value of $F^{1/2}$ even when a takes various values.

As a measure of the horizontal scale of the PV flux, the latitude $y_{\min}$ of the minimum of the zonally integrated PV flux is plotted against a in Fig. 5. Except for the case $F^{1/2} = 1.13$, $y_{\min}$ is nearly proportional to a (cyan, orange, and red lines in Fig. 5). The purple, cyan, orange, and red vertical lines in Fig. 5 indicate the lines $a = F^{-1/2}$ for $F^{1/2} = 1.13$, 1.70, 2.55, and 3.40, respectively. The change in $y_{\min}$ with a is less significant, when $a < F^{-1/2}$ for $F^{1/2} = 1.13$ (purple line in Fig. 5). The effects of a and F on the primary wave, which determines the PV flux in the upper layer to the lowest order, have already been examined by Waterman and Jayne (2012) and are not examined here.

FIG. 2. The harmonic wave for $a = 1$ and $F^{1/2} = 1.70$. Contours of (a) $\text{Re } \psi_1^h$, (b) $\text{Re } \psi_2^h$, and the meridional PV flux in the (c) upper and (d) lower layers. Contour intervals are (a) 0.25, (b) 0.05, (c) 1, and (d) 0.05.

4. Weak stratification limit

a. Formulation

Based on the results in the previous section, we consider the mechanism of the southward PV flux, focusing on the limit as $F \rightarrow \infty$. Expanding ψ and q in (2.12), (2.13), (2.18), and (2.19) in an asymptotic series of F^{-1} ,

$$\psi_k = \psi_k^0 + F^{-1}\psi_k' + F^{-2}\psi_k'' + \cdots \quad \text{and} \quad (4.1)$$

$$q_k = q_k^0 + F^{-1}q_k' + F^{-2}q_k'' + \cdots, \quad (4.2)$$

we obtain

$$\psi_c^0 = 0, \quad (4.3)$$

$$q_c^0 = -\psi_c' = iW_e, \quad (4.4)$$

$$\psi_b' = \psi_b'' = \cdots = 0, \quad \text{and} \quad (4.5)$$

$$q_b' = q_b'' = \cdots = 0. \quad (4.6)$$

Equations (4.3) and (4.4) indicate that the baroclinic mode contributes only to the change in q_k without changing ψ_k to the lowest order, and (4.5) and (4.6) indicate that $\psi_b = \psi_b^0$ and $q_b = q_b^0$.

Using (2.12), we can decompose q_b into two components:

$$q_b = q_w + q_\beta, \quad \text{and} \quad (4.7)$$

$$q_w = iW_e, \quad q_\beta = -2i\psi_{b,x}, \quad (4.8)$$

where q_w and q_β represent the change in q_b caused by external forcing and beta effect, respectively. Combining (4.4), (4.6), and (4.8), we obtain

$$q_1^0 = q_b + q_c^0 = q_b + q_w \quad \text{and} \quad (4.9)$$

FIG. 3. Meridional distribution of the zonally integrated PV flux in the lower layer for $a =$ (a) 0.25, (b) 0.5, (c) 1, (d) 2, and (e) 3. The purple, cyan, orange, and red lines indicate the PV flux for $F^{1/2} = 1.13, 1.70, 2.55,$ and 3.40 , respectively.

FIG. 4. (a) Plot of $Y_{\min}$ as a function of $F^{1/2}$ for $a = 1$ (solid line with crosses) and 0.5 (dashed line with circles). The solid and dashed horizontal lines indicate $Y_{\min}$ in the limit as $F \rightarrow \infty$ for $a = 1$ and 0.5 , respectively. (b) Plot of $Y_{\min}$ as a function of a for $F^{1/2} = 1.70$ (cyan line with triangles), 3.40 (red line with circles), and ∞ (black line with crosses).

$$q_2^0 = q_b - q_c^0 = q_\beta. \quad (4.10)$$

Equation (4.10) indicates that the PV q_2^0 in the lower layer is equivalent to q_b except that it does not include the term q_w , which represents the direct contribution of external forcing.

Then, the forcing term on the right-hand side of (2.7) is expressed as

$$w_1^{h0} = 2W - W_\beta \quad \text{and} \quad (4.11)$$

$$w_2^{h0} = W_\beta, \quad (4.12)$$

where

$$W = -J(\psi_b, q_b), \quad W_\beta = -J(\psi_b, q_\beta), \quad (4.13)$$

represent the nonlinear forcing due to the advection of q_b and q_β by ψ_b , respectively, and the superscript 0 denotes the $O(1)$ variables.

As in the primary wave, ψ_c^h is $O(F^{-1})$. Then, using (2.31), the PV flux in the lower layer and the barotropic component of the PV flux by the harmonic wave, respectively, are

$$Y_2^{h0} = -\frac{1}{2} \text{Re} \eta_b^{h0} W_\beta^* \quad \text{and} \quad (4.14)$$

$$Y_b^{h0} = \frac{1}{2} (Y_1^{h0} + Y_2^{h0}) = -\frac{1}{2} \text{Re} \eta_b^{h0} W^*, \quad (4.15)$$

where

$$\eta_b^{h0} = \frac{i\psi_{b,x}^{h0}}{2}$$

is the barotropic component of the northward displacement of fluid particles by the harmonic wave. As considered in section 2b, Y_b^{h0} has to be northward in the sense that it is integrated over the entire domain. Thus, comparing (4.14) and (4.15) indicates that Y_2^{h0} becomes southward when W_β is out of phase with W . It can be readily shown that the above argument remains qualitatively the same, even if the

FIG. 5. Plot of $y_{\min}$ as a function of a for $F^{1/2} = 1.13$ (purple line with squares), 1.70 (cyan line with triangles), 2.55 (orange line with circles), and 3.40 (red line with crosses). The vertical purple, cyan, orange, and red lines indicate the lines $a = F^{1/2}$ for $F^{1/2} = 1.13$, 1.70 , 2.55 , and 3.40 , respectively.

ratio of the upper- and lower-layer thicknesses takes a general value.

Although we have considered a nearly barotropic ocean with weak stratification in this section, the PV is not barotropic. This might not seem physically consistent. Hence, before proceeding, we check the equation of enstrophy considered in section 2b in the limit as $F \rightarrow \infty$. For the primary wave, (2.25) is rewritten as

$$\begin{aligned}\nabla \cdot \mathbf{J}_{Hk} &= (-1)^{k-1} J^z + \frac{1}{4} \text{Re} \delta_{k1} W_e q_k^* \\ &= (-1)^{k-1} J^z + \frac{1}{4} \text{Re} \delta_{k1} W_e [q_b^* + (-1)^{k-1} q_c^*]\end{aligned}\quad (4.16)$$

in the stationary state. From (2.28), $\mathbf{J}_{Hk}$, which is included on the right-hand side of (4.16), reduces to the barotropic enstrophy flux,

$$\mathbf{J}_b = \text{Re} \left(\frac{|\psi_{b,x}|^2 - |\psi_{b,y}|^2}{2}, \quad \psi_{b,x} \psi_{b,y}^* \right),$$

as $F \rightarrow \infty$ for both the upper and lower layers ($k = 1, 2$). From (2.29) and (4.4), the right-hand side of (4.16) for $k = 1, 2$ also yields the same form as $F \rightarrow \infty$, and (4.16) becomes

$$\nabla \cdot \mathbf{J}_b = \text{Re} W_e q_b^*. \quad (4.17)$$

Equation (4.17) merely corresponds to the enstrophy equation of the barotropic mode. It can be shown that this is also the case for the harmonic wave. Thus, there is no inconsistency in the formulation in this section.

b. Southward PV flux

In this subsection, we examine the phase differences between q_b and q_β and between W and W_β , based on numerical experiments. We drop the subscript b , which denotes the barotropic mode, in the rest of this section because the baroclinic mode does not appear explicitly as $F \rightarrow \infty$. As a typical example of q , the black contours in the left panels of Fig. 6 show $\text{Re } q$ and $\text{Im } q$ for $a = 1$. Here Im denotes the imaginary part, and $\text{Re } q$ and $\text{Im } q$ represent the snapshot of the lowest-order PV, $q^{(0)} = \text{Re } q e^{-it}$, at $t = 0$ and $\pi/2$, respectively. For comparison, the black contours in the right panels of Fig. 6 show the snapshot of $q_\beta^{(0)} = \text{Re } q_\beta e^{-it}$, at $t = 0$ and $\pi/2$. The region $y < 0$ is not shown because q and q_β are symmetric about $y = 0$. Because $q - q_\beta = q_w$ is imaginary from (4.8), $q^{(0)}$ is identical to $q_\beta^{(0)}$ at $t = 0$ (Figs. 6a, b). Although $q^{(0)}$ increases from $t = 0$ to $\pi/2$ near $(x, y) = (0, 0)$, $q_\beta^{(0)}$ becomes weakly negative at $t = \pi/2$ in the same region (Figs. 6c, d). As the forcing attains a

maximum at $t = 0$, q and q_β are delayed and slightly advanced relative to the forcing. Because q_β becomes negative when the planetary vorticity is advected northward [see the second equation in (4.8)], the change in q_β from $t = 0$ to $\pi/2$ simply indicates that the northward flow is driven by the positive forcing.

As shown in (4.13), W and W_β are determined from the advection of q and q_β by ψ , respectively. When flow due to ψ is in the downgradient direction of q , W is positive, and vice versa. The blue contours in Fig. 6 indicate snapshots of $\psi^{(0)} = \psi e^{-it}$. At $t = 0$, the northward or northwestward flow near $(x, y) = (0, 0)$ is oriented in the downgradient direction of both q and q_β (Figs. 6a, b). At $t = \pi/2$, although the flow near $(x, y) = (0, 0)$ continues to be oriented in the downgradient direction of q , it is oriented in the upgradient direction of q_β .

Figure 7 compares the snapshot of the nonlinear forcing,

$$\Omega = -J(\psi^{(0)}, q^{(0)}) = W e^{-2it} - \frac{1}{2} J(\psi, q^*) \quad \text{and}$$

$$\Omega_\beta = -J(\psi^{(0)}, q_\beta^{(0)}) = W_\beta e^{-2it} - \frac{1}{2} J(\psi, q_\beta^*),$$

for $a = 1$ at $t = 0$ (top panels), $\pi/4$ (upper-middle panels), $\pi/2$ (lower-middle panels), and $3\pi/4$ (bottom panels). Note that Ω and Ω_β include the forcing to the mean flow as well as W and W_β . At both $t = 0$ and $\pi/2$, Ω is positive near $x = 0$ (Figs. 7a, e), as expected from Figs. 6a and 6c. Similarly, Ω_β is positive at $t = 0$ and negative at $t = \pi/2$ near $x = 0$ (Figs. 7b, f), as expected from Figs. 6b and 6d. Because the frequency of the harmonic wave is twice as large as that of the primary wave, the interval between $t = 0$ and $\pi/2$ corresponds to half of the harmonic wave's period. Therefore, W and W_β are nearly out of phase with each other. Figure 8 compares the real and imaginary parts of W and W_β for $a = 1$. We can confirm that W and W_β are nearly out of phase with each other, except in the region $0 < x < 1, 1 < y < 2$. Figure 9 shows the horizontal distribution of the PV flux in the lower layer obtained from (4.14). The southward PV flux dominates, corresponding to the phase difference between W and W_β . As $q^{(0)}$ includes a component proportional to W_e (see (4.7) and (4.8)), the discontinuities in x and y derivatives of W_e given by (3.5) cause discontinuities in Ω and W along $x = 0$ and $y = 0$ (Figs. 7c, e). However, because ψ_b^h and ψ_c^h are expressed in terms of the forcing by the primary wave using (3.3) and (3.4), ψ_b^h and ψ_c^h are continuous in x and y , causing no serious problems to other variables.

c. Semianalytic solution

In this subsection, we confirm that the results obtained in the previous subsection are consistent with the basic

FIG. 6. Comparison of (left) $q^{(0)}$ and (right) $q_{\beta}^{(0)}$. The black contours indicate snapshots of $q^{(0)}$ and $q_{\beta}^{(0)}$ at $t =$ (top) 0 and (bottom) $\pi/2$. Snapshots of $\psi^{(0)}$ are superimposed by blue contours. Blue arrows indicate the direction of the flow. Contour intervals are 1.5 and 1.8 for the black and blue contours, respectively.

features of Rossby waves. We also examine the effects of the resonant triad interaction (Longuet-Higgins and Gill 1967) on the PV flux. Substituting (3.5) into (2.12) and taking the Fourier transform in x , we obtain

$$\tilde{\psi}_{yy} - \lambda^2 \tilde{\psi} = \frac{2iaA}{a^2 k^2 + 1} e^{-|y|/a}, \quad (4.18)$$

where

$$\lambda = \sqrt{k(k+2)}$$

and the tilde denotes the Fourier component,

$$\tilde{\psi} = \int_{-\infty}^{\infty} \psi e^{-ikx} dx.$$

Then we obtain

$$\tilde{\psi} = \sum_{m=1}^2 R_m e^{il_m|y|}, \quad (4.19)$$

$$\tilde{q} = - \sum_{m=1}^2 (k^2 + l_m^2) R_m e^{il_m|y|}, \quad \text{and} \quad (4.20)$$

$$\tilde{q}_{\beta} = \sum_{m=1}^2 2k R_m e^{il_m|y|}, \quad (4.21)$$

where

$$l_1 = i\lambda, \quad l_2 = \frac{i}{a}, \quad (4.22)$$

$$R_1(k) = \frac{2ia^2 A}{\lambda(a^2 \lambda^2 - 1)(a^2 k^2 + 1)}, \quad \text{and} \quad (4.23)$$

FIG. 7. Snapshots of (left) Ω and (right) Ω_β at $t =$ (first row) 0, (second row) $\pi/4$, (third row) $\pi/2$, and (fourth row) $3\pi/4$. Contour interval is 2.

$$R_2(k) = -\frac{2ia^3A}{(a^2\lambda^2 - 1)(a^2k^2 + 1)}. \quad (4.24)$$

As (k, l_1) is real and satisfies the dispersion relation in the interval $-2 < k < 0$, we can decompose $\tilde{q}$ and $\tilde{q}_\beta$ into the resonant (or free) and forced components:

$$\tilde{q} = \tilde{q}_R + \tilde{q}_F, \quad \text{and} \quad (4.25)$$

$$\tilde{q}_\beta = \tilde{q}_R + \tilde{q}_{\beta F}, \quad (4.26)$$

where the subscripts R and F denote the resonant and forced components, respectively, and

$$\tilde{q}_R = -\sum_{m=1}^2 s_m (k^2 + l_m^2) R_m e^{il_m|y|}, \quad (4.27)$$

$$\tilde{q}_F = -\sum_{m=1}^2 (1 - s_m) (k^2 + l_m^2) R_m e^{il_m|y|}, \quad (4.28)$$

$$\tilde{q}_{\beta F} = \sum_{m=1}^2 2(1 - s_m) k R_m e^{il_m|y|}, \quad \text{and} \quad (4.29)$$

FIG. 8. Contours of (a) $\text{Re } W$, (b) $\text{Re } W_\beta$, (c) $\text{Im } W$, and (d) $\text{Im } W_\beta$. Contour interval is 1.5.

$$s_m(k) = \begin{cases} 1 & \text{when } m = 1 \text{ and } -2 \leq k \leq 0 \\ 0 & \text{otherwise} \end{cases}. \quad (4.30)$$

The resonant component of $\tilde{q}_\beta$ is the same as that of $\tilde{q}$, because $k^2 + l_1^2 + 2k = 0$.

Using (4.13) and the identity,

$$\tilde{f}g = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k - k') \tilde{g}(k') dk',$$

we obtain

$$\begin{aligned} \tilde{W} &= \frac{1}{4\pi} \int_{k/2}^{\infty} \sum_{m=1}^2 \sum_{n=1}^2 (k' l_n'' - k'' l_m') \\ &\quad \times (k^2 + l_m'^2 - k'^2 - l_n'^2) R_m' R_n'' e^{i(l_m' + l_n'')y} dk' \quad \text{and} \end{aligned} \quad (4.31)$$

$$\begin{aligned} \tilde{W}_\beta &= -\frac{1}{2\pi} \int_{k/2}^{\infty} \sum_{m=1}^2 \sum_{n=1}^2 (k' l_n'' - k'' l_m') \\ &\quad \times (k' - k'') R_m' R_n'' e^{i(l_m' + l_n'')y} dk', \end{aligned} \quad (4.32)$$

in $y > 0$, where

$$\begin{aligned} k'' &= k - k', \\ l_m' &= l_m(k'), \quad l_n'' = l_n(k''), \quad \text{and} \\ R_m' &= R_m(k'), \quad R_n'' = R_n(k''). \end{aligned}$$

Because $\tilde{W}$ and $\tilde{W}_\beta$ are antisymmetric about $y = 0$, the form in $y < 0$ is not shown. As we did for the PV, we decompose W and W_β into resonant and forced components:

$$\begin{aligned} W &= -J(\psi, q_R) - J(\psi, q_F) \\ &= W_R + W_F, \quad \text{and} \end{aligned} \quad (4.33)$$

FIG. 9. Contours of the meridional PV flux in the lower layer in the limit as $F \rightarrow \infty$. Contour interval is 0.15.

$$\begin{aligned} W_\beta &= -J(\psi, q_R) - J(\psi, q_{\beta F}) \\ &= W_R + W_{\beta F}. \end{aligned} \quad (4.34)$$

Substituting (4.33) and (4.34) into (4.14) and (4.15), we can also decompose Y_2^{h0} and Y_b^{h0} into resonant and forced components:

$$\begin{aligned} Y_2^{h0} &= -\frac{1}{2} \text{Re } \eta^{h0} W_R^* - \frac{1}{2} \text{Re } \eta^{h0} W_{\beta F}^* \\ &= Y_R^{h0} + Y_{\beta F}^{h0}, \quad \text{and} \end{aligned} \quad (4.35)$$

$$\begin{aligned} Y^{h0} &= -\frac{1}{2} \text{Re } \eta^{h0} W_R^* - \frac{1}{2} \text{Re } \eta^{h0} W_F^* \\ &= Y_R^{h0} + Y_F^{h0}. \end{aligned} \quad (4.36)$$

Note that the subscript b in Y_b^{h0} is dropped here.

Equation (4.31) implies that two waves with wavenumbers of $\mathbf{k}'_m = (k', l'_m)$ and $\mathbf{k}''_n = (k'', l''_n)$ drive motions with the wavenumber of $\mathbf{k}'_m + \mathbf{k}''_n = (k, l'_m + l''_n)$. The form on the right-hand side of (4.31) is reminiscent of the triad interaction of Rossby waves (Longuet-Higgins and Gill 1967). [Note that W is antisymmetric in y . Then it can be shown that motions increase linearly in y if resonance occurs.] However, l'_1 and l''_1 can take only negative values (l'_2 and l''_2 are imaginary) in the region $y > 0$, as the group velocity should be positive there [see (4.22)]. Then the resonance condition is not satisfied because $\mathbf{k}'_1 + \mathbf{k}''_1$ is located outside the circle in Fig. 10, which represents the dispersion curve of the primary wave (vector $\mathbf{OC}$ in Fig. 10; see the caption of Fig. 10 for proof), while the dispersion curve of the harmonic wave is located inside this circle (small circle in Fig. 10). Thus,

FIG. 10. Schematic of a triad of barotropic Rossby waves. The large and small circles indicate the dispersion curves of the primary and harmonic waves, respectively. Point O is the origin, $\mathbf{OA} = \mathbf{k}'_1$ and $\mathbf{OB} = \mathbf{k}''_1$ are the wavenumbers of the primary wave, and $\mathbf{OC} = \mathbf{k}'_1 + \mathbf{k}''_1$. Point P is the center of the large circle, and A' is the crossing of the line AP and the large circle. When B is located at A', C is located at Q, which is the crossing of the line OP and the large circle. When B is located on the arc OA A', C is located outside the large circle. Thus, if A and B are located in the region $l < 0$, C is located outside the large circle.

we cannot focus on a specific triad but instead must consider all triads including both resonant and forced components.

Figure 11a shows the distribution of the real and imaginary parts of $\tilde{q}_F$, $\tilde{q}_{\beta F}$, and $\tilde{q}_R$ at $y = 0$. Because $\text{Re } \tilde{q}_F = \text{Re } \tilde{q}_{\beta F} = \text{Im } \tilde{q}_R = 0$, they are not shown. The results shown in Fig. 11a can be interpreted by analogy with an oscillator forced by a periodic forcing, as the equation of the PV can be expressed in the form of the forced oscillator:

$$\frac{d\hat{q}^{(0)}}{dt} = -i\omega_0 \hat{q}^{(0)} + \hat{W}_e e^{-it}, \quad (4.37)$$

where the hat denotes the two-dimensional Fourier component,

$$\hat{q}^{(0)} = \iint_{-\infty}^{\infty} q^{(0)} e^{-i(kx+ly)} dx dy,$$

and

$$\omega_0 = -\frac{2k}{k^2 + l^2} \quad (4.38)$$

corresponds to the natural frequency of the oscillator. The resonance condition, $\omega_0 = 1$, is satisfied on the dispersion curve (the circle in Fig. 12). Because of the character of the forced oscillator, $\hat{q}^{(0)}$ is in phase with $\tilde{W}_e$ when $\omega_0 = 1$, whereas the phase of $\hat{q}^{(0)}$ relative to $\tilde{W}_e$ is $\pi/2$ for $\omega_0 < 1$ (regions B and C in Fig. 12) and $-\pi/2$ for $\omega_0 > 1$ (region A). Hence, $\tilde{q}_R$, which corresponds to resonant $\hat{q}^{(0)}$, is in phase with $\tilde{W}_e$ (dotted line in Fig. 11). The phase of $\tilde{q}_F$, which is the sum of $\hat{q}^{(0)}$ at $\omega_0 \neq 1$, is mostly $\pi/2$ except for the interval $-2 < k < -1$ (solid line in Fig. 11). From (2.18), (4.7), and (4.38), we obtain

$$\hat{q}_\beta^{(0)} = \omega_0 \hat{q}^{(0)}.$$

Hence $\tilde{q}_{\beta F}$ is imaginary and out of phase with $\tilde{q}_F$ when $k > 0$, owing to the contribution of $\hat{q}^{(0)}$ in the region C where $\omega_0 < 0$ (dashed line in Fig. 11).

Figure 13 shows the horizontal distribution of q_F , $q_{\beta F}$, and q_R for $a = 1$, which are obtained numerically using the inverse Fourier transform of (4.27)–(4.29). [Note that singularities of $O(k^{-1/2})$ at $k = 0, -2$ in (4.23) vanish when the inverse Fourier integral is performed with a complex variable $\theta = \cos^{-1}(k + 1)$.] As expected from Fig. 11a, the phase relative to the forcing at $(x, y) = (0, 0)$ is $\pi/2$ for q_F (Figs. 13a,d) and 0 for q_R (Figs. 13c,f). Similarly, Fig. 11b shows the distribution of $\text{Im} \tilde{\psi}_F$ and $\text{Re} \tilde{\psi}_R$. Note that $\text{Re} \tilde{\psi}_F = \text{Im} \tilde{\psi}_R = 0$. Because of the difference by the factor $-(k^2 + l_m^2)$ between ψ and q , $\tilde{\psi}_F$ and $\tilde{\psi}_R$ are mostly out of phase with $\tilde{q}_F$ and $\tilde{q}_R$, respectively. Also, $\tilde{\psi}_R$ has a larger amplitude for long waves compared to $\tilde{q}_R$ (dotted line in Fig. 11b).

Figure 14 shows the horizontal distribution of ψ_R and ψ_F for $a = 1$. Because the wavenumber of the long wave is more meridional than the short wave (see vectors **OA** and **OB** in Fig. 10), contours of ψ_R are more zonal compared to q_R (see gray lines in Figs. 13c and 14a). The phase relative to the forcing at $(x, y) = (0, 0)$ is π for ψ_R and $-\pi/2$ for ψ_F . The distributions of q , q_β , and ψ in Fig. 6 are consistent with those in Figs. 13 and 14.

Figures 15a and 15b compare the zonal integrals of each term in (4.35) and (4.36), respectively. For the barotropic component of the PV flux, Y_F^{h0} is positive and contributes to the northward PV flux (dashed line in Fig. 15b). Although the phase difference between $\tilde{q}_R$ and $\tilde{q}_F$ is $\pi/2$ except for the factor $e^{il_m|y|}$, Y_R^{h0} is southward (dotted line in Fig. 15b). The integral of Y^{h0} over the entire model domain is 0.079 and positive, as we considered in section 2b. The reason why Y_R^{h0} has the opposite sign to Y_F^{h0} may be qualitatively explained as follows. As we have seen, the distribution of q_F , q_R , and ψ may be expressed in an idealized form:

$$q_F = iBe^{-(\kappa|x|+\lambda y)}, \quad (4.39)$$

$$q_R = Ce^{-i(k_1x+l_1y)}, \quad \text{and} \quad (4.40)$$

$$\psi = -De^{-i(k_2x+l_2y)}, \quad (4.41)$$

where all parameters are real and positive. We assume for simplicity that k_1 , l_1 , k_2 , and l_2 vary slowly in space and C and D decrease slowly from $(x, y) = (0, 0)$, although these assumptions are not exactly valid. Substituting (4.39)–(4.41) into (4.33) yields

$$W_F = \begin{cases} -\frac{1}{2}(k_2\lambda - l_2\kappa)BDe^{-(\kappa x + \lambda y) - i(k_2x + l_2y)} & \text{in } x > 0 \\ -\frac{1}{2}(k_2\lambda + l_2\kappa)BDe^{\kappa x - \lambda y - i(k_2x + l_2y)} & \text{in } x < 0 \end{cases}, \quad \text{and} \quad (4.42)$$

$$W_R = \frac{1}{2}(k_1l_2 - l_1k_2)CDe^{-i[(k_1+k_2)x + (l_1+l_2)y]}. \quad (4.43)$$

Because (k_1, l_1) (i.e., the wavenumber of q_R) is more zonal than (k_2, l_2) (i.e., the wavenumber of ψ) the term in brackets of (4.43) is positive. In the region $x > 0$, l_2 tends to be smaller than k_2 (see blue contours in Fig. 13), mostly due to the contribution from short waves and partly due to ψ_F . Hence, the term in brackets in (4.42) is mostly positive. Then W_R is out of phase with W_F near $(x, y) = (0, 0)$, producing Y_R^{h0} , which has the opposite sign to Y_F^{h0} .

In contrast, the PV flux Y_2^{h0} in the lower layer becomes southward because of Y_R^{h0} (dotted and solid lines in

Fig. 15a). The forced component $Y_{\beta F}^{h0}$ also contributes to the southward PV flux near $y = 0$, where $q_{\beta F}$ is out of phase with q_F (dashed line in Fig. 15a). Thus, the southward PV flux in the lower layer is produced by the difference between q_F , q_R , and $q_{\beta F}$, which can be interpreted only by the basic features of the Rossby waves. Qualitatively the same results can be obtained for the other values of a .

5. Effects of stratification

To examine the effects of stratification on the PV flux, we consider the second-lowest terms of the perturbation

FIG. 11. Plots of (a) $\text{Im } \tilde{q}_F$ (solid), $\text{Im } \tilde{q}_{\beta F}$ (dashed), and $\text{Re } \tilde{q}_R$ (dotted), and (b) $\text{Im } \tilde{\psi}_F$ (solid) and $\text{Re } \tilde{\psi}_R$ (dotted) at $y = 0$ as a function of k .

expansions in (4.1) and (4.2). Substituting (4.3)–(4.6) into (2.16) and (2.17), and taking $O(F^{-1})$ terms, we obtain

$$w_b^{h'} = -J(\psi_c', q_w) \quad \text{and} \quad (5.1)$$

$$w_c^{h'} = -J(\psi_b^0, q_c') - J(\psi_c', q_\beta) - J(\psi_c', q_w). \quad (5.2)$$

Because (4.4) and (4.8) indicate that both ψ_c' and q_w are proportional to W_e , the last terms on the right-hand side of (5.1) and (5.2) vanish, implying that

$$w_b^{h'} = \eta_b^{h'} = 0. \quad (5.3)$$

By expanding η_k^M in (2.31) into vertical modes, we can decompose the lower-layer PV flux into the contributions from η_b^h and η_c^h :

$$\begin{aligned} Y_2^h &= -\frac{1}{2} \text{Re } \eta_b^h w_2^{h*} + \frac{1}{2} \text{Re } \eta_c^h w_2^{h*} \\ &= Y_{2b}^h + Y_{2c}^h. \end{aligned} \quad (5.4)$$

Combining (5.1)–(5.4), and keeping terms of $O(F^{-1})$ or lower, yields

$$\begin{aligned} Y_{2b}^h &= -\frac{1}{2} \text{Re } \eta_b^{h0} W_\beta^* - \frac{1}{F} \text{Re } \eta_b^{h0} [J(\psi_b^0, W_{e,x}) \\ &\quad - J(\psi_{b,x}^0, W_e)] \quad \text{and} \end{aligned} \quad (5.5)$$

$$Y_{2c}^h = \frac{1}{2F} \text{Re } \eta_c^{h'} W_\beta^*. \quad (5.6)$$

We have used the relation $q_c' = -2i\psi_{c,x}'$, which can be derived from (2.13) and (2.19). The first and second terms on the right-hand side of (5.5) represent the PV flux of $O(1)$ discussed in previous sections and the correction of

$O(F^{-1})$ to w_2^h , respectively. Although the baroclinic wave is evanescent, (5.6) indicates that the interaction between η_c^h and the barotropic forcing produces the baroclinic PV flux. Taking the $O(1)$ terms in (2.15) and using (2.32), (4.11), and (4.12), we can express $\eta_c^{h'}$ as

$$\eta_c^{h'} = \frac{W_x - W_{\beta,x}}{4}.$$

The solid, dashed, and dotted lines in Fig. 16 show the area integrals of Y_2^h , Y_{2b}^h , and Y_{2c}^h , which are directly calculated from the numerical solution as a function of $F^{1/2}$,

FIG. 12. Definitions of regions A, B, and C. The solid circle indicates the dispersion curve of the primary wave.

FIG. 13. Comparison of (left) $q_F e^{-it}$, (center) $q_{\beta F} e^{-it}$, and (right) $q_R e^{-it}$ at $t =$ (top) 0 and (bottom) $\pi/2$ shown by black contours. Snapshots of $\psi^{(0)}$ are superimposed by blue contours. Contour intervals are 1.5 and 1.8 for the black and blue contours, respectively. In (c) the gray solid line indicates the typical orientation of the contours of $q_R e^{-it}$.

respectively. For comparison, Y_{2b}^h and Y_{2c}^h estimated from (5.5) and (5.6) are also shown by the thick dashed and thick dotted lines, respectively. The PV flux estimated from (5.5) and (5.6) qualitatively agrees with that obtained from the numerical solution except for the interval near $F = 1$. By comparing (5.6) and the first term of (5.5), we see that Y_{2c}^h represents the effects of the change in lower-layer motions by stratification. The southward PV flux in the lower layer is weakened by Y_{2c}^h , which is northward, for both $a = 0.5$ and 1 (dotted lines in Fig. 16). This corresponds to the weakening of the lower-layer motions by stratification. The second term on the right-hand side of (5.5) weakens the southward PV flux, which is produced by the first term of (5.5), for $a = 0.5$ and strengthens it for $a = 1$ (dashed lines in Fig. 16). The change in Y_{2b}^h with $F^{1/2}$ dominates over that in Y_{2c}^h for $a > 1.5$ (not shown), corresponding to the increase of the southward PV flux with decreasing $F^{1/2}$ shown in Fig. 4b.

6. Summary

To better understand the wave-mean interaction in the western boundary region, we examined the effects of

stratification and nonlinearity on the PV flux produced by the forced Rossby waves. The perturbation analysis shows that the primary wave is excited directly by the forcing, whereas the harmonic wave is excited by the nonlinear interaction of the primary wave. The harmonic wave produces the southward PV flux in the lower layer, whereas the primary wave produces the northward PV flux in the upper layer. Because of the conventional nonacceleration theorem, no meridional PV flux is produced by the primary wave in the lower layer, in which the external forcing vanishes.

The southward PV flux dominates in the lower layer in the wide parameter range typical for the western boundary region. This may be counterintuitive, as the volume integral of the PV flux should be northward according to the enstrophy equation. Even in the limit of infinitesimally weak stratification, the amplitude of the PV flux remains on the same order as that obtained for finite stratification. The horizontal scale of the PV flux is nearly proportional to a , except for the case where F is close to unity.

Equations of the perturbation PV are greatly simplified in the limit of the weak stratification. In this limit,

FIG. 14. Snapshots of (left) $\psi_R e^{-it}$ and (right) $\psi_F e^{-it}$ at $t =$ (top) 0 and (bottom) $\pi/2$. Contour interval is 0.8. In (a) the gray solid line indicates the typical orientation of the contours of $q_R e^{-it}$ in Fig. 13c.

stratification has almost no effect on the flow, except that it isolates the lower layer from the direct effects of the external forcing. That is, the PV in the lower layer is determined only by the advection of the planetary vorticity. It is shown that a resonant triad interaction of Rossby waves is absent. Various triads of resonant (or free) and forced waves are substantial contributors to the PV flux. The southward PV flux is produced because the nonlinear forcing to the lower layer tends to be out of phase with the barotropic component of the nonlinear forcing. Although the phase of the nonlinear forcing is not determined by a simple process, it is explained only by basic features of Rossby waves and does not depend on details of experimental conditions. The difference between the forced and resonant components of the PV is a key feature that determines the phase of the nonlinear forcing (Figs. 6 and 13). The difference of

dominant wavenumbers between ψ and q shown in Fig. 13 is also important. One might expect that the southward PV flux may be reproduced by the interaction between two or three waves that are periodic in x , making a simpler interpretation of the southward PV flux possible. However, such combinations of the waves could not be found. This is probably due to the fact that the above-noted features of ψ and q are essentially caused by the superposition of infinite number of resonant and forced waves.

The effects of finite stratification on the PV flux are qualitatively consistent with the perturbation analysis with respect to F^{-1} , except for the case where F is close to unity. Because ψ_c increases logarithmically as $F \rightarrow 1$ (i.e., $\omega \rightarrow \omega_c$), ψ_k cannot be approximated by the asymptotic series of F^{-1} like (4.1) as $F \rightarrow 1$. Except for this case, stratification weakens the response of lower-layer

FIG. 15. Meridional distributions of the zonally integrated meridional PV flux. (a) Solid, dotted, and dashed lines indicate Y_2^{h0} , Y_R^{h0} , and $Y_{\beta F}^{h0}$, respectively. (b) The solid, dotted, and dashed lines indicate Y^{h0} , Y_R^{h0} , and Y_F^{h0} , respectively.

motions to the nonlinear forcing, whereas it weakens or strengthens the nonlinear forcing, depending on the value of a . When a is smaller or larger than about 1.5, the net effect of stratification is to weaken or strengthen the PV flux in the lower layer, respectively.

The direction of the PV flux obtained in the present study is consistent with that obtained in an eddy-resolving model by [Holland and Rhines \(1980\)](#) in both the upper and lower layers. Results of the present study may contribute to the understanding of the driving

mechanism of the recirculation gyres. In the present study, the external forcing was of single sign. When the forcing changes sign in space, the meridional PV flux also changes sign in space as expected from the entropy constraint in [section 2b](#). However, the basic features of the PV flux remain unchanged. It has not been confirmed that the mechanism of the southward PV flux proposed in the present study is substantial in the real ocean or in eddy-resolving numerical models. This is left for future research.

FIG. 16. Plots of the meridional PV flux, which is integrated in the entire domain, as a function of $F^{1/2}$ for $a =$ (a) 0.5 and (b) 1. Solid lines with crosses, dashed lines with circles, and dotted lines with triangles indicate Y_2^h , Y_{2b}^h , and Y_{2c}^h , respectively. The thick dashed and dotted lines indicate Y_{2b}^h and Y_{2c}^h obtained by the perturbation analysis, respectively.

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