Small-Scale Potential Vorticity in the Upper-Ocean Thermocline

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ABSTRACT

Twenty Electromagnetic Autonomous Profiling Explorer (EM-APEX) floats in the upper-ocean thermocline of the summer Sargasso Sea observed the temporal and vertical variations of Ertel potential vorticity (PV) at 7–70-m vertical scale, averaged over $O(4\text{--}8)$ -km horizontal scale. PV is dominated by its linear components—vertical vorticity and vortex stretching, each with an rms value of $\sim 0.15f$. In the internal wave frequency band, they are coherent and in phase, as expected for linear internal waves. Packets of strong, $>0.2f$, vertical vorticity and vortex stretching balance closely with a small net rms PV. The PV spectrum peaks at the highest resolvable vertical wavenumber, ~ 0.1 cpm. The PV frequency spectrum has a red spectral shape, a -1 spectral slope in the internal wave frequency band, and a small peak at the inertial frequency. PV measured at near-inertial frequencies is partially attributed to the non-Lagrangian nature of float measurements. Measurement errors and the vortical mode also contribute to PV in the internal wave frequency band. The vortical mode Burger number, computed using time rates of change of vertical vorticity and vortex stretching, is 0.2–0.4, implying a horizontal kinetic energy to available potential energy ratio of ~ 0.1 . The vortical mode energy frequency spectrum is 1–2 decades less than the observed energy spectrum. Vortical mode energy is likely underestimated because its energy at vertical scales > 70 m was not measured. The vortical mode to total energy ratio increases with vertical wavenumber, implying its importance at small vertical scales.

1. Introduction

Oceanic variations at horizontal scales smaller than $O(100)$ km, vertical scales less than the order of tens of meters, and frequencies between inertial and buoyancy frequencies are generally thought to be internal waves. The most prominent features of internal waves are that they propagate and do not possess Ertel potential vorticity (PV). Müller (1984) proposes that a PV-carrying finestructure, termed vortical motion, coexists with internal waves at the same spatial scales, and perhaps the same temporal scales. In the present analysis, *vortical motion* is defined as a flow component carrying PV at any spatial and temporal scale. The *vortical mode* is the eigenmode of the normal mode decomposition of the linear equations of motion on an f plane; it is the linear limit of vortical motion, equivalent to large-scale geostrophic flow. Vortical motion also includes quasigeostrophic flow at the mesoscale, and two-dimensional stratified turbulence at submesoscale and small scales (Müller 1984). The role of vortical

motion on basin scales and mesoscales, and evidence for potential-vorticity-carrying finestructure in the ocean interior are discussed in Kunze and Lien (2019). Vortical motion does not propagate and has kinematic and dynamic properties distinct from internal waves, as demonstrated by recent numerical model simulations of a collapsing wake (Watanabe et al. 2016). While internal waves propagate away from the collapsing wake, total potential vorticity, including a significant nonlinear component, remains in the wake and does not propagate, analogous to results of linear Rossby adjustment in which linear potential vorticity is left behind.

Ertel potential vorticity is defined as

$$\pi = (2\mathbf{\Omega} + \nabla \times \mathbf{u}) \cdot \nabla \psi, \quad (1)$$

where $2\mathbf{\Omega} = [0 \quad 2|\mathbf{\Omega}| \cos \phi \quad 2|\mathbf{\Omega}| \sin \phi] = [0 \quad f_y \quad f_z]$, ϕ is latitude, $\mathbf{\Omega}$ is Earth's rotation rate vector, $\mathbf{u}$ is oceanic velocity, and ψ is any conserved quantity. The parameter $\nabla \times \mathbf{u} = [\zeta_x \quad \zeta_y \quad \zeta_z]$ is the relative vorticity vector. In this analysis, we chose the conserved quantity $\psi = z - \eta$, where η is vertical displacement. This definition differs from the conventional choice of using buoyancy for ψ .

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The potential vorticity defined here has a dimension of vorticity. It is a constant when the fluid is at rest, similar to the available potential vorticity defined by [Wagner and Young \(2015\)](#). The vertical displacement of isopycnal surface η is defined as $\rho[x, y, z_0 + \eta(x, y, z_0, t), t] = \rho_b(z_0)$, where ρ is the local density and $\rho_b(z_0)$ is the background density at the equilibrium depth $z_0 = z - \eta(x, y, z_0, t)$. In the absence of turbulence mixing and intrusions, $z - \eta$ is conserved. Perturbation potential vorticity can be expressed as

$$\pi' = \zeta_z - f_z \partial_z \eta - f_y \partial_y \eta - \zeta_x \partial_x \eta - \zeta_y \partial_y \eta - \zeta_z \partial_z \eta. \quad (2)$$

Hereinafter, f_z is denoted as f for simplicity. At small aspect ratios, the third term on the right-hand side is smaller than the second term and can be ignored. The linear component of PV, $\zeta_z - f \partial_z \eta$, is the difference between the vertical component of relative vorticity ζ_z and vortex stretching $f \partial_z \eta$. The nonlinear component, $-\zeta_x \partial_x \eta - \zeta_y \partial_y \eta - \zeta_z \partial_z \eta$, is the sum of twisting and stretching of isopycnals by the relative vorticity vector. PV in the ocean can only be modified by external forcing or turbulent dissipation. In the absence of nonadiabatic and forcing processes, PV is conserved following the flow, that is, $(D/Dt)\pi' = 0$.

[Briscoe \(1977\)](#) and [Müller et al. \(1988\)](#) report the presence of current finestructure at vertical scales smaller than 10 m in the upper Sargasso Sea observed during the Internal Waves Experiment (IWEX). [Müller \(1984\)](#) hypothesizes that the current finestructure represents the potential-vorticity-carrying motion (i.e., vortical motion) at small scales. During the last three decades there have been extensive observational, theoretical, and numerical studies of vortical motions to investigate their properties, interactions with internal waves, and importance to lateral dispersion.

Because vortical motions carry PV and internal waves do not, the temporal and spatial structures of PV are due to those of vortical motion. [Müller et al. \(1988\)](#), using IWEX measurements in the Sargasso Sea, estimate linear potential vorticity averaged over horizontal scales of 25–900 m and describe its horizontal and temporal structures. Their PV has a $-3/2$ frequency spectral slope in the internal wave frequency band and a $2/3$ blue horizontal wavenumber spectral slope. The linear vortical motion, referred to as the vortical mode, has an energy of $2 \times 10^{-4} \text{ m}^2 \text{ s}^{-2}$, about one-tenth that of internal waves. The vertical vorticity ζ_z is nearly $10f$ averaged over horizontal scales of $O(10)$ m, and $0.2f$ averaged over horizontal scales of $O(1)$ km. Unfortunately, their estimates of ζ_z based on three measurements on each depth were contaminated by horizontal divergence, making their estimates of linear potential

vorticity questionable. Consequently, [Müller et al. \(1988\)](#) conclude that further studies are needed to measure PV accurately. In the present analysis, vertical vorticity and vertical divergence are computed using 20 velocity measurements at each isopycnal surface so that the contamination error is minimized.

[Kunze and Sanford \(1993\)](#) performed a spatial survey near Ampere Seamount and estimated both the linear and nonlinear component of PV. They report linear PV fluctuations at horizontal wavelengths of 6–15 km and vertical wavelengths of 50–380 m with vertical vorticity about $0.2f$. The observed vortical motion near Ampere Seamount is likely generated by flow past the seamount ([D'Asaro 1988](#)). [Kunze \(1993\)](#), using the same dataset, discusses the available potential energy (APE) to horizontal kinetic energy (HKE) ratio as a function of Burger number ($B = N\alpha/f\beta$, where N is buoyancy frequency, α horizontal wavenumber, and β vertical wavenumber). At Burger number ~ 1 , observed energy ratios APE/HKE are $O(1)$. [Kunze \(1993\)](#) cannot distinguish internal waves from the vortical mode based on observed APE/HKE because both motions have $\text{APE}/\text{HKE} = O(1)$ at $B \sim 1$.

[Pinkel \(2008\)](#) analyzed shear data taken by Doppler sonars operated during the Surface Heat Budget of the Arctic Ocean (SHEBA) experiment and concludes that the subinertial vortical motion has a white vertical wavenumber shear spectrum. [Pinkel \(2014\)](#), using multiyear fast CTD profile data, reports a clear spectral gap in vertical strain between vortical motions and internal waves at vertical scales greater than 20 m. The Burger number is estimated to be $O(0.1)$ based on the rate of internal wave vertical strain spreading into subinertial frequencies. [Polzin et al. \(2003\)](#) explain that the observed decrease of shear to strain ratio with increasing vertical wavenumber at small vertical scales is due to the vortical motion's high shear to strain ratio. They separate internal waves and vortical motions using correlation analysis and construct a vertical strain spectrum for vortical motion. Based on an APE/HKE energy ratio observed at subinertial frequencies, they estimate a Burger number of 1.3.

[Lelong and Riley \(1991\)](#) study the interaction among internal waves and vortical motions with a multiscale mathematical model and report that the nonlinear interaction between internal waves cannot generate vortical motions. The wave–vortical motion interaction provides mechanisms for wave–wave energy exchange. The interaction between vortical motions may generate internal waves.

[Polzin and Ferrari \(2004\)](#) propose that vortical motion is responsible for the isopycnal dispersion and the evolved structure observed during NATRE (North Atlantic

Tracer Release Experiment). They construct the vortical motion velocity spectrum following the strain spectrum proposed by Polzin et al. (2003). The vortical motion produces an isopycnal diffusivity of $1\text{ m}^2\text{ s}^{-1}$, consistent with observations. Sundermeyer et al. (2005) demonstrate that isopycnal dispersion results from small-scale geostrophic processes associated with turbulence mixing patches in a stratified flow during the Coastal Mixing and Optics (CMO) dye release experiment. The observed 1–10-km horizontal diffusivity was $1\text{--}10\text{ m}^2\text{ s}^{-1}$, 10 times greater than that predicted by internal wave dispersion (Young et al. 1982), implying strong lateral dispersion by vortical motion.

Lien and Müller (1992a) discuss the procedures to separate horizontal divergence $\Gamma = \nabla_h \cdot \mathbf{u}_h$, vertical vorticity ζ_z , and vortex stretching $f\partial_z\eta$ into internal wave and linear vortical mode components. The vortical mode is horizontally nondivergent, $\Gamma = 0$, whereas internal waves have no linear PV, $\zeta_z - f\partial_z\eta = 0$. They also describe a method of separating internal wave energy from the vortical mode (= geostrophic flow) using PV and Burger number. Kinematic properties of internal waves and the vortical mode, and the procedures for energy separation are reviewed in appendix A.

Alternative methods to separate internal waves from vortical motions have been proposed using shipboard or aircraft velocity and density measurements (Bühler et al. 2014, 2017; Lindborg 2015). Callies et al. (2016) decomposes mesoscale winds into internal waves and vortical motions in the tropopause and lower stratosphere using aircraft measurements. Their separation scheme is performed in the spectral domain and therefore the phase information is lost. The method proposed by Lien and Müller (1992a) retains the phase information.

Here, we present a unique set of measurements taken by a swarm of 20 Electromagnetic Autonomous Profiling Explorer (EM-APEX) floats in the upper thermocline of the Sargasso Sea (section 2). These measurements allow estimates of linear and nonlinear components of PV, ζ_z , $f\partial_z\eta$, and Γ at vertical length scales of 7–70 m averaged over horizontal scales of $O(4\text{--}8)$ km. Computation methods are discussed in section 3. Spectral properties of PV, ζ_z , $f\partial_z\eta$, and Γ in the internal wave frequency band are discussed in section 4. A special focus is to confirm the balance of vertical vorticity and vortex stretching for linear internal waves. Kinematic properties in the internal wave frequency band are consistent with linear internal waves, such that packets of strong ζ_z and $f\partial_z\eta$ balance, resulting in small PV (section 4). Spectral characteristics of PV are discussed in section 5. The total energy of the linear vortical mode is computed using the observed two-dimensional

frequency vertical wavenumber spectrum of PV and Burger number (section 6). Burger number is computed using time rates of change of vertical vorticity and vortex stretching of the vortical mode. In section 7 we discuss the method proposed by Bühler et al. (2014, 2017) and Lindborg (2015) to separate internal wave energy from vortical mode energy in horizontal wavenumber Fourier space using shipboard measurements. In summary, in the upper thermocline of the Sargasso Sea, the amplitude of internal waves is nearly one decade stronger than that of the vortical mode observed at a vertical length scale of 7–70 m and averaged over a horizontal scale $O(4\text{--}8)$ km. However, the linear vortical mode energy is likely underestimated because our observations do not capture linear vortical mode energy at vertical scales greater than 70 m.

2. LatMix experiment and EM-APEX float measurements

a. LatMix experiment

An array of 20 EM-APEX floats was deployed in the upper ocean of the Sargasso Sea southeast of Cape Hatteras in summer 2011 as part of the Scalable Lateral Mixing and Coherent Turbulence (short title: LatMix) Departmental Research Initiative funded by the Office of Naval Research (Fig. 1) (Shcherbina et al. 2015). During the experiment, EM-APEX floats measured temperature T , salinity S , pressure P , and horizontal velocity components (U , V) between the surface and $\sim 150\text{-m}$ depth, with some deeper profiles to $\sim 250\text{-m}$. Extensive effort was made to program all floats to rise in near synchrony. Simultaneous profiles minimize the oceanic space–time aliasing. Ten EM-APEX floats were equipped with dual FP-07 fast thermistors to measure turbulent temperature fluctuations. An average turbulent thermal vertical diffusivity of $5 \times 10^{-6}\text{ m}^2\text{ s}^{-1}$ at $\sim 30\text{-m}$ depth (potential density of 25.42 kg m^{-3}) derived from the floats' thermistor measurements was consistent with the average diapycnal diffusivity estimated by collocated tracer (rhodamine) observations (Lien et al. 2016). Turbulence measurements are not discussed further here.

Floats were deployed three times during the LatMix experiment (Fig. 1a). This paper focuses on the first deployment (D1) of 20 floats in a weakly confluent region during 4–10 June 2011. Floats were intended to be deployed in three concentric circles of 0.5-, 1-, and 2-km radii, with roughly six floats on each circle but, because of the oceanic horizontal shear, float arrays were not perfectly concentric. During its 7-day mission, this swarm advected 20 km southeastward (Fig. 1b) and

FIG. 1. (a) Float trajectories of three deployments (D1–D3) during the LatMix experiment. Black curves represent float trajectories on 4–10 Jun 2011 (D1), magenta curves on 13–16 Jun (D2), and blue curves on 17–20 Jun (D3). Background color shows the AVHRR image of SST on 4 Jun. (b) Trajectories (gray curves) of 20 floats (red dots) during D1 and the trajectory (thick black curve) of the float array center (black crosses). Floats were deployed in three concentric circles of radii of 0.5, 1, and 2 km. Blue contours mark the perimeter of the float array. (c) Photo of EM-APEX float. Sensors on the float are labeled.

remained coherent, expanding by about a factor of 2 and stretching slightly in a northwest–southeast direction. The float array collected a total of 2546 profiles.

b. EM-APEX floats

The EM-APEX float combines the standard Teledyne Webb Research Corp. APEX profiling float with an electromagnetic subsystem that measures the motion-induced electric fields generated by the ocean currents moving through the vertical component of Earth's magnetic field (Fig. 1c) (Sanford et al. 2005). The APEX float uses buoyancy changes to profile the ocean to a maximum depth of 2000 m. When on the sea surface, the

float's GPS position and profile data (except for the raw FP-07 data) are transmitted over the Iridium global satellite communications system. The time for GPS position acquisitions and data transmission by Iridium often causes floats ~20 min of sea surface advection.

Temperature and salinity measurements are taken every 10–25 s by a Sea-Bird Electronics SBE41CP CTD at 2–3-m vertical resolution. The velocity sensor operates on the same principles of motional induction applied on the Absolute Velocity Profiler (Sanford et al. 1985) and the Expendable Current Profiler (Sanford et al. 1982). The electric field sensing electrodes are located on the top end of the floats (Fig. 1c). Other

necessary measurements are magnetic compass heading and instrument tilt. The float descends and ascends at a typical speed of $\sim 0.14 \text{ m s}^{-1}$ and rotates at a period of $\sim 12 \text{ s}$. The vertical profiling speed and the rotational rate of the float decrease when the float profiles across a strong pycnocline or when it turns around at the sea surface or targeted depth. The motional-induced electric field is determined by a sinusoidal fit to the measured voltages using the basis functions from the horizontal components of the magnetometer measurements. The fit is made over a 50-s-long segment of data, and the averaging data window moves 25 s between successive fits, yielding $\sim 7\text{-m}$ vertically averaged velocity measurements every 3.5 m. These sinusoidal fits and the root-mean-square (rms) residuals are transferred to the APEX float controller for storage and later transmission over Iridium when the float surfaces. The 7-m vertical averaging velocity measurement limits our analysis at scales greater than finescale. The float's horizontal current uncertainty is $\sim 0.015 \text{ m s}^{-1}$ (Sanford et al. 1985). The velocity uncertainty leads to vertical vorticity noise of $O(0.01f)$, where $f = 0.044 \text{ cph} = 7.71 \times 10^{-5} \text{ s}^{-1}$, averaged over the $O(4\text{--}8)\text{-km}$ float array. The vertical vorticity noise is one decade smaller than the observed vertical vorticity signal in this analysis. Horizontal velocity measurements taken by EM-APEX floats are relative to depth-independent constants (Sanford et al. 1978). The depth-independent constant is typically determined as the residual between the depth-averaged velocity measured by the float and by GPS fixes.

Atmospheric forcing was measured from nearby shipboard sensors. During the first deployment of the LatMix experiment (D1), the surface wind stress was generally less than 0.05 N m^{-2} , except during a couple bursts of 0.15 N m^{-2} on 4 and 6 June. The net surface heat flux revealed a clear diurnal cycle of -200 W m^{-2} at night and $\sim 600 \text{ W m}^{-2}$ during the day. Due to direct sunlight striking the Ag/AgCl electrodes, velocity errors were introduced in the upper 60 m, sometimes extending deeper than 70 m, during the day, were stronger closer to the surface, and appeared as large velocity variations (Figs. 2c,d). These unusual velocity errors in daylight prevented us from computing and correcting for the depth-independent velocity in our velocity measurements. In the latter two float missions, experiments D2 and D3, a sunshade cover was added to the electrodes, eliminating velocity noise due to the sunlight effect. Depth-independent constants computed from 20 floats have a standard deviation of $\sim 0.015 \text{ m s}^{-1}$, likely associated with the float's velocity measurement uncertainty. In this analysis, positions of float profiles are interpolated linearly using GPS positions. Each

vertical profiling cycle takes about one hour. Therefore, float positions at time scales greater than a few hours are reasonably accurate.

Fluctuations in the meridional horizontal current were mostly semidiurnal and more apparent after 6 June. Density variations were associated mostly with temperature variations. The base of the surface mixed layer, defined as the shallowest depth where the density exceeded the near surface value by more than 0.1 kg m^{-3} , varied between 10 and 20 m, typical of the summer Sargasso Sea. The buoyancy frequency peaked at the base of the mixed layer and decayed nearly exponentially with depth. Below 80 m the mean vertical shear was 0.002 s^{-1} and the mean buoyancy frequency was 0.0063 s^{-1} (3.6 cph). To avoid the velocity errors introduced by sunlight on sensors, velocity measurements above 80 m are excluded from the following analysis.

c. Horizontal kinetic energy and potential energy frequency spectra, GM comparison

Frequency spectra of horizontal velocity, rotary velocity, horizontal kinetic energy, and potential energy are computed and compared with those predicted by the GM76 model (Cairns and Williams 1976). In the present analysis, spectra are computed using a multitaper spectral method with two tapers (Percival and Walden 1993). Horizontal velocity observed between 80- and 150-m depths are Wentzel-Kramers-Brillouin (WKB)-scaled, that is, $u_{\text{WKB}} = u(\bar{N}/N)^{1/2}$, where $\bar{N} = 3.6 \text{ cph}$ is the vertically and temporally averaged buoyancy frequency (Leaman and Sanford 1975). Frequency spectra of WKB-scaled velocity measured by each float are computed at each depth and averaged in the vertical. Depth-averaged frequency spectra of all floats are further averaged to yield the float-array-averaged frequency spectrum (Fig. 3). At frequencies above 0.06 cph, spectral shapes of horizontal velocity components and horizontal kinetic energy are in agreement with GM76 predictions, but spectral levels are weaker than the GM76 spectrum by a factor of ~ 2 . Clockwise (in time) velocity spectra are stronger than counterclockwise velocity spectra near the inertial frequency, consistent with linear internal wave theory. Both meridional velocity and counterclockwise (in time) spectra exhibit a clear semidiurnal spectral peak.

We compute vertical displacement η as $\rho[x, y, z_0 + \eta(x, y, z_0, t), t] = \rho_b(z_0)$. The background density ρ_b is computed by averaging density over time and all floats at fixed depths at 2-m vertical intervals. WKB-scaled vertical displacement is computed as $\eta_{\text{WKB}} = \eta(\bar{N}/N)^{1/2}$. Frequency spectra of vertical displacement $\Phi_{\eta}(\omega)$ are computed at each depth between 80 and 150 m, then

FIG. 2. Summary of surface forcing and EM-APEX float measurements averaged over 20 floats during the first deployment. (a) Surface wind stress τ and (b) net surface heat flux H_f estimated from shipboard meteorological sensors. (c)–(i) Mean profiles (left columns) and contours (right columns) of zonal velocity U , meridional velocity V , temperature T , salinity S , density σ_θ , vertical shear squared S^2 , and buoyancy frequency squared N^2 , respectively. Thick black curves in contour plots represent the base of the surface mixed layer. Shadings in the left columns represent 95% confidence intervals of mean profiles. Except for zonal and meridional velocity mean profiles, 95% confidence intervals of mean profiles are too small to be visible.

averaged vertically and over the float array. The averaged potential energy spectrum is computed as $(1/2)\overline{N^2\Phi_{\eta_{\text{WKB}}}(\omega)}$. The vertical displacement frequency spectral shape agrees with the GM76 prediction above

the semidiurnal frequency but differs at near-inertial frequencies. The observed total energy spectrum is about 60% of the GM76 level between the semidiurnal frequency and 0.3 cph, but is much weaker near the

FIG. 3. Frequency spectra of (a) zonal velocity, (b) meridional velocity, (c) clockwise velocity, (d) counterclockwise velocity, (e) horizontal kinetic energy, (f) available potential energy, and (g) total energy, and (h) the ratio of observed total energy to the GM76 prediction. These spectra are WKB scaled, averaged between 80- and 150-m depth, and averaged over the float array. Black curves and shadings are the mean and 95% confidence interval. Red curves are GM76 predictions. Horizontal dashed lines in (h) mark the ratios of 0.6 and 1. Inertial frequency f and semidiurnal tidal frequency D_2 are labeled.

inertial frequency, and closer to the GM76 level at the Nyquist frequency, 0.5 cph. Overall, internal wave energy during D1 was weaker, by a factor of ~ 2 , than the GM76 internal wave field.

3. Potential vorticity estimation methods

Linear and nonlinear components of PV are computed using simultaneous float measurements. Velocity and CTD measurements from all floats are interpolated linearly onto 1-h time intervals and 2-m vertical bins

between 80- and 150-m depth. Horizontal velocity and vertical displacement are projected onto isopycnal surfaces. However, because typical vertical displacements of ~ 3 m are nearly the same as the vertical resolution of the measurements, the isopycnal-following analysis does not produce significant differences from the isobaric analysis.

a. Estimates of vorticity vector (ζ_x , ζ_y , ζ_z)

The horizontal components of relative vorticity are estimated as $\zeta_x = -\partial_z v$ and $\zeta_y = \partial_z u$, assuming

$O(\partial_y w / \partial_z v, \partial_x w / \partial_z u) = O(\delta^2) \ll 1$, where δ is the aspect ratio. Vertical shears computed from all floats are averaged to form the mean vertical shear averaged over the spatial scale of the float array, $O(4\text{--}8)$ km.

The vertical component of relative vorticity, $\zeta_z = \partial_x v - \partial_y u$, averaged over the float array, is computed using

$$u(x_i, y_i) = \bar{u}(x_0, y_0) + \widehat{\partial_x u} \delta x_i + \widehat{\partial_y u} \delta y_i + \frac{1}{2} \widehat{\partial_x^2 u} (\delta x_i)^2 + \frac{1}{2} \widehat{\partial_y^2 u} (\delta y_i)^2 + \widehat{\partial_x \partial_y u} \delta x_i \delta y_i + \epsilon_{u_i}, \quad \text{and} \quad (3)$$

$$v(x_i, y_i) = \bar{v}(x_0, y_0) + \widehat{\partial_x v} \delta x_i + \widehat{\partial_y v} \delta y_i + \frac{1}{2} \widehat{\partial_x^2 v} (\delta x_i)^2 + \frac{1}{2} \widehat{\partial_y^2 v} (\delta y_i)^2 + \widehat{\partial_x \partial_y v} \delta x_i \delta y_i + \epsilon_{v_i}, \quad (4)$$

where $u(x_i, y_i)$ and $v(x_i, y_i)$ are velocity components at float position (x_i, y_i) . The overbar represents the mean over the float array and the overhat represents estimates of spatial derivatives. Parameters ϵ_u and ϵ_v are residuals of the least squares fit. Estimates of horizontal divergence ($\bar{\Gamma} = \widehat{\partial_x u} + \widehat{\partial_y v}$), deformation, shearing, and lateral strain rate are also computed. Estimates of vertical vorticity and horizontal divergence are the focus of this study. In the following analysis, we omit the accent $\widehat{}$ for convenience. The horizontal gradient of ζ_z is computed using fitted second derivatives, which will be used in [section 7](#). In this analysis, the standard deviation of residuals ϵ_u and ϵ_v is $\sim 0.013 \text{ m s}^{-1}$, close to the uncertainty of velocity measurements. This may contribute to the vertical vorticity and divergence uncertainty of $O(0.01f)$.

The quality of velocity gradient estimates and their derived variables depends on the shape of the float array and the array's number of velocity measurements. During the first LatMix deployment, the float array shape changed only slightly over the course of 7 days ([Fig. 1b](#)). [Kunze et al. \(1990\)](#) and [Lien and Müller \(1992a\)](#) discuss attenuation and contamination of estimates of Γ and ζ_z using a finite number of velocity sensors uniformly distributed around a circle in a horizontally homogeneous flow field. Transfer functions for estimates of vertical vorticity, horizontal divergence, and vertical strain due to horizontal area averaging, vertical averaging of velocity, and the finite vertical differencing effect for estimating vertical strain are discussed in [appendix B](#). The attenuation effect reflects the finite size of the circle such that fluctuations at spatial scales smaller than the size of the circle are not captured. The contamination effect arises from a finite separation of velocity measurements such that fluctuations of Γ at scales smaller than the horizontal separation of measurements contaminate estimates of ζ_z and vice versa. This contamination error can be reduced by increasing

simultaneous velocity measurements taken by the float array. At each time and isopycnal grid, velocity measurements taken by 20 floats are used to compute the means, first and second horizontal derivatives of horizontal velocity components using a least squares method, that is,

the number of velocity sensors. For example, the contamination error at the horizontal scale of the circle is 9% using three velocity sensors and decreases to 4% and 1.5% for four and six velocity sensors, respectively. According to the GM76 internal wave spectral model, estimates of vertical vorticity are significantly contaminated by horizontal divergence at frequencies exceeding ~ 0.1 cph (see [appendix C](#)). The quality of vertical vorticity estimates below 0.1 cph is confirmed by its close balance with the vortex stretching in the internal wave frequency band. These are discussed further in [section 4](#).

Vorticity vector $(\zeta_x, \zeta_y, \zeta_z)$ and horizontal divergence estimates are shown in [Fig. 4](#). Horizontal vorticities (vertical shears) are about two orders of magnitude greater than the vertical component of vorticity, and exhibit upward phase propagation of near-inertial waves at vertical wavelength ~ 25 m. Frequency spectra of horizontal vorticities show a prominent inertial wave peak ([Fig. 5](#)). Note that although energy is dominated by semidiurnal tides ([Fig. 3](#)), vertical shears are dominated by high-mode near-inertial waves. Vertical vorticity and horizontal divergence also have inertial peaks, though weaker than those of horizontal vorticities.

b. Estimates of isopycnal slope $\partial_x \eta$, $\partial_y \eta$, and vertical strain $\partial_z \eta$

For each float the vertical strain $\partial_z \eta$ is computed as the first vertical derivative of $\eta(x, y, z_0, t)$. Estimates of mean vertical strain and its horizontal gradients are computed by a linear least squares method so as to minimize the sum of the residual squared expressed as

$$\partial_z \eta(x_i, y_i) = \overline{\partial_z \eta}(x_0, y_0) + \widehat{\partial_x \partial_z \eta} \delta x_i + \widehat{\partial_y \partial_z \eta} \delta y_i + \epsilon_{\partial_z \eta_i}, \quad (5)$$

where $\epsilon_{\partial_z \eta}$ is the residual of the least squares fit. Isopycnal slopes $\partial_x \eta$ and $\partial_y \eta$ are computed following the

FIG. 4. Depth–temporal variations of normalized (a) zonal, (b) meridional, and (c) vertical components of relative vorticity, and (d) horizontal divergence. The normalized horizontal vorticity components are scaled down by a factor of 1/100 because of their much greater magnitudes than the vertical component of vorticity and horizontal divergence.

same procedure as horizontal gradients of horizontal velocity. Vertical displacements estimated from all

floats are used to fit to means and first and second horizontal derivatives of vertical displacements, that is,

$$\eta(x_i, y_i) = \bar{\eta}(x_0, y_0) + \widehat{\partial_x \eta} \delta x_i + \widehat{\partial_y \eta} \delta y_i + \frac{1}{2} \widehat{\partial_x^2 \eta} (\delta x_i)^2 + \frac{1}{2} \widehat{\partial_y^2 \eta} (\delta y_i)^2 + \widehat{\partial_x \partial_y \eta} \delta x_i \delta y_i + \epsilon_{\eta_i}, \quad (6)$$

where $\eta(x_i, y_i)$ is the vertical displacement measured at float position (x_i, y_i) , and ϵ_{η} is residual of the least squares fit.

Isopycnal slopes are about two orders of magnitude smaller than vertical strain, and are vertically coherent (Fig. 6). The meridional isopycnal slope is slightly stronger than the zonal slope, and shows a peak at the semidiurnal tidal frequency (Fig. 5). Vertical strain shows a spectral peak at the near-inertial frequency.

c. Estimates of potential vorticity

Estimates of vorticity vector, vertical strain, and isopycnal slopes averaged over the float array are used to compute the linear and nonlinear components of PV [(2)]. The isopycnal slope is $O(10^{-2})$ of the vertical strain. The third term in (2) is $O(10^{-2})$ smaller than the second term and will not be discussed in the following analysis. Hereinafter, relative vorticity and vortex stretching are normalized by planetary vorticity f . All normalized components of PV, scaled by f , are shown in

Fig. 7. Vertical strain and normalized vertical vorticity have similar magnitude. The vertical strain exhibits layer structure in the first half of the observational period, whereas the vertical vorticity shows low vertical mode structure. The meridional vorticity twisting term dominates nonlinearity due to the larger meridional isopycnal slope (Figs. 7c–e). But in general, linear terms are at least 5 times greater than nonlinear terms. Consequently, the total PV is well represented by the linear PV (Figs. 7f–g). In the following analysis, estimates of vertical strain are low-pass filtered to 7-m vertical scale to be consistent with the vertical averaging scale of velocity, vertical vorticity, and horizontal divergence.

4. Are there only internal waves in the internal wave frequency band?

Here, we discuss whether observed vertical vorticity, horizontal divergence, vertical strain, and PV in the

FIG. 5. Vertically averaged frequency spectra of (a) two horizontal vorticity (only vertical shear) components, (b) vertical vorticity and horizontal divergence, (c) zonal and meridional isopycnal slopes, and (d) vertical strain. The spectrum of vertical vorticity is also shown in (d) for comparison.

internal wave frequency band follow consistency relations for linear internal waves. Consistency relations for linear internal waves were first reported by Fofonoff (1969). Further consistency relations were revealed by Müller and Siedler (1976). Lien and Müller (1992b) extended previous analysis and report consistency

relations for both internal waves and vortical motions. These consistency relations have been used to test whether observations of velocity and vertical displacements follow kinematic characteristics of linear internal waves. Here, we discuss the two most fundamental consistency relations for linear internal waves.

FIG. 6. Depth–temporal variations of (a) zonal isopycnal slope, (b) meridional isopycnal slope, and (c) vertical strain. Note that isopycnal slopes are scaled by a factor of 100 because of their much smaller magnitudes than the vertical strain.

To our knowledge they have not been confirmed by observations.

a. Internal wave test 1: $PV = 0$?

For linear internal waves, the vertical component of relative vorticity balances vortex stretching so that the PV vanishes, that is, $\zeta_z - f\partial_z\eta = 0$. Numerical model simulations also confirm the absence of nonlinear PV for nonlinear internal waves. The absence of PV is one of the most prominent features for internal waves, but oceanic measurements have not been made to confirm it, primarily due to the lack of accurate vertical relative vorticity measurements at small scales.

Estimates of vertical vorticity and vertical strain are bandpass (0.025–0.1 cph; i.e., $0.6\text{--}2.3f$) filtered using a Butterworth two-poles nonlinear filter (Figs. 8, 9). The lower bound 0.025 cph, slightly lower than f , is chosen to include potential Doppler shifted internal waves. Vertical vorticity and vertical strain in the internal wave frequency band < 0.1 cph exhibit similar vertical and temporal patterns (Fig. 8). Simultaneous and collocated packets of strong vertical vorticity and vertical strain on 4–7 June at 100-m depth and on 8–9 June at 120-m depth illustrate their dynamic relevance. Time series of vertical vorticity and vertical strain (Fig. 9) show the close balance of strong vertical vorticity and vertical strain on 8–9 June with rms values of vertical vorticity and vertical strain about $0.16f$. The overall correlation between vertical vorticity and vertical strain is 0.7, with a 95% significance level of 0.06. Linear PV in the internal wave frequency band has an rms value of $0.08f$, about 50% of vertical vorticity or vertical strain. Most notable is the absence of elevated PV during periods of strong vertical vorticity and vertical strain. The linear

component of PV exhibits a hint of upward phase propagation at the near-inertial period.

Because of the observed small Rossby number, that is, $Ro = \zeta_z/f \ll O(1)$, vortical motion in our measurements should be weak at frequencies greater than f . Therefore, we expect PV to be weak in the internal wave frequency band. Because EM-APEX floats are not Lagrangian, but follow the horizontal current averaged between the surface and 150-m depth, observed PV may be, partially, explained as the advection of background PV by near-inertial waves (section 7b).

Vertical averages of normalized vertical vorticity and vertical strain frequency spectra agree with each other between f and 0.1 cph. In this frequency band they are significantly coherent and mostly in phase (Fig. 10). These are characteristics of linear internal waves, confirming that internal waves dominate in this frequency band. The GM76 vertical vorticity spectrum is computed to include the effects of 6-km float array averaging (assuming 3-km radius) and the contamination by horizontal divergence (appendixes B and C). It agrees well with the observed spectrum. Following Fig. 3h, we reduce the GM76 energy level by half to compute GM76 spectra of horizontal divergence, vertical vorticity, and vertical strain. Above 0.1 cph, the contamination effect by horizontal divergence becomes significant. The vertical strain spectrum is nearly white at frequencies above 0.1 cph, corresponding to vertical displacement noise of $O(0.1)$ m, likely due to our measurement uncertainty.

b. Internal wave test 2: $\partial_t(\zeta_z/f) = -\nabla_h u_h$

For linear internal waves, the time rate of change of normalized vertical vorticity or vertical strain

FIG. 7. Depth-temporal variations of normalized PV components: (a) vertical strain (normalized vortex stretching), (b) normalized vertical vorticity, (c)–(e) nonlinear components, (f) linear PV, and (g) the sum of linear and nonlinear components of PV. Note that the range of the color scale in (c)–(e) is different from other panels because nonlinear PV components are much smaller than the linear components.

should balance with the horizontal divergence, that is, $\partial_t(\zeta_z/f) = \partial_t(\partial_z \eta) = -\nabla_h u_h$. Observations of horizontal divergence were bandpass filtered between 0.025 and 0.1 cph. Horizontal divergence has vertical structure and temporal variation similar to time rates of change of vertical vorticity and vertical strain (Fig. 11). Packets of strong horizontal divergence appear in the layer at 80–100-m depth during 4–7 June and in a 130–110-m layer and above 100 m during 7–9 June. Because the linear vortical mode is horizontally nondivergent, the horizontal divergence at small scales should be contributed mostly by internal waves.

The time rate of change for bandpass filtered vertical vorticity and vertical strain agree well with the bandpass

filtered horizontal divergence, confirming the dynamic balance of linear internal waves (Figs. 11, 12). Correlation coefficients of horizontal divergence with time rates of change of the vertical vorticity and vertical strain are 0.75 and 0.65, respectively, with a 95% significance level of 0.06.

The observed horizontal divergence spectrum peaks at the inertial frequency, similar to the vertical vorticity spectrum (Fig. 13). At subinertial frequencies, horizontal divergence drops by nearly one decade, as expected for subinertial motions. The spectral level of the vertical vorticity at the lowest frequency is nearly the same as that at the inertial frequency. The vertical vorticity and horizontal divergence is significantly coherent except

FIG. 8. Depth–temporal variations in the internal wave frequency band of (a) normalized vertical vorticity, (b) vertical strain, and (c) normalized linear PV. All variables have been bandpass filtered in the internal wave low-frequency band between 0.025 and 0.1 cph.

at the highest frequencies. The most striking feature is that the horizontal divergence and vertical vorticity are nearly 90° out of phase below 0.1 cph, again confirming the dominance of internal waves.

The observed horizontal divergence spectrum is 0.5 the GM76 energy level including the area-averaging effect. For linear internal waves, frequency spectra of horizontal divergence, vertical vorticity, and vertical

FIG. 9. Time series of normalized vertical vorticity (black curve), vertical strain (red curve), and the normalized linear PV (shading) bandpass filtered between 0.025 and 0.1 cph between 84- and 142-m depth at 8-m intervals. The scale is illustrated at the top left.

FIG. 10. Depth averages of (a) frequency spectra of normalized vertical vorticity (black curve) and vertical strain (red curve), (b) coherence (Coh), and (c) phase spectra between vertical vorticity and vertical strain. In (a), the blue solid curve represents the area-averaged GM76 prediction of vertical vorticity, and the blue dashed curve includes contamination from horizontal divergence. Shadings represent 95% confidence intervals in (a) and (c) and the 95% significance level in (b). Solid dots and open circles in (b) and (c) represent significant and insignificant coherences, respectively.

strain should be related as $\Phi_{\nabla_h u_h}(\omega) = (\omega^2/f^2)\Phi_{\xi_z}(\omega) = (\omega^2/f^2)\Phi_{f\partial_z\eta}(\omega)$ (Lien and Müller 1992b). Including the area-averaging effect, the above relation between spectra of horizontal divergence and vertical vorticity is unchanged. With additional contamination effects, however, above 0.1 cph the vertical vorticity spectrum is contaminated by horizontal divergence and becomes nearly identical to the horizontal divergence spectrum at the buoyancy frequency 3.6 cph (Fig. C1c in appendix C).

5. Spectral characteristics of small-scale potential vorticity

The two-dimensional frequency–vertical wavenumber spectra of small-scale PV is computed (Fig. 14). Integrating the two-dimensional spectrum over the resolved vertical wavenumbers yields a red frequency spectrum with a nearly -1 spectral slope in the internal wave frequency band and a small peak at the inertial frequency. The total enstrophy, that is, the variance of PV, integrated in the observed frequency band is $0.02f^2$, 92% contributed by the linear component. In comparison, Müller et al. (1988) report a $-3/2$ frequency spectral slope for PV calculated from IWEX mooring observations. However, their estimates of vertical vorticity were contaminated by horizontal divergence. Their estimates

of enstrophy vary from $0.027f^2$ (averaging over a ~ 1 -km radius) to $225f^2$ (averaging over a 5-m radius). Our estimate of enstrophy, averaged over a horizontal scale $O(4-8)$ km, is smaller than that reported by Müller et al. (1988). The variance-preserving frequency spectrum indicates that strong enstrophy resides at subinertial frequencies, $\sim 60\%$ of total variance.

The vertical wavenumber spectrum of PV, computed by integrating the two-dimensional spectrum over the resolved frequencies, is nearly white at wavenumbers below 0.1 cpm. Because horizontal velocity measured by EM sensors represents a 7-m vertical average and the vertical strain is low-pass filtered at 7 m, the variance of potential vorticity above 0.1 cpm is not captured, reflected by the steep spectral drop at vertical wavenumbers > 0.1 cpm. The variance-preserving vertical wavenumber spectrum indicates that variance peaks at 0.1 cpm, suggesting that enstrophy resides at small vertical scales.

6. Linear vortical motion (vortical mode) estimates

a. Estimate of vortical mode Burger number

Lien and Müller (1992a) propose using normal mode decomposition to separate vertical vorticity, vortex

FIG. 11. Depth–temporal variations in the internal wave frequency band of (a) normalized horizontal divergence, (b) time rate of change of normalized vertical vorticity, (c) time rate of change of vertical strain, and (d) time rate of change of normalized linear PV. All variables have been bandpass filtered in the internal wave low-frequency band between 0.025 and 0.1 cph.

stretching, and horizontal divergence into the linear vortical mode and internal wave components [(A5)–(A9) in appendix A]. For the linear vortical mode, the vertical vorticity and vortex stretching are negatively correlated with a ratio of Burger number squared (i.e., dynamic aspect ratio) ($B^2 = N^2 \alpha^2 / f^2 \beta^2$) [(A2) in appendix A], that is, $\zeta_z^{\text{VM}} / f \partial_z \eta^{\text{VM}} = -B^2$. The linear potential vorticity can be expressed as $\text{PV} = \zeta_z^{\text{VM}} - f \partial_z \eta^{\text{VM}} = -(1 + B^2) f \partial_z \eta^{\text{VM}}$. The linear vortical mode Burger number can be computed using (A2). Theoretically, following (A5)–(A9), observed vertical vorticity and vertical strain can be separated into vortical mode and internal wave components. Unfortunately, computing ζ_z^{IW} using (A6) requires initial values of ζ_z^{IW} at all depths, which are unknown. To overcome this limitation, we estimate Burger number using time rates of change of normalized vertical vorticity and vertical strain, that is,

$$\frac{\partial_t \zeta_z^{\text{VM}} / f}{\partial_t \partial_z \eta^{\text{VM}}} = -B^2. \quad (7)$$

Note that Burger number is a function of horizontal and vertical wavenumbers. Here, we estimate Burger number using $\partial_t \zeta_z^{\text{VM}} / f$ and $\partial_t \partial_z \eta^{\text{VM}}$ in Fourier space (ω , β) and in real space, separately. Time rates of change of normalized vertical vorticity and vertical strain of the vortical mode are computed following (A5)–(A9) as

$$\partial_t \frac{\zeta_z^{\text{VM}}}{f} = \partial_t \frac{\zeta_z^{\text{obs}}}{f} - \partial_t \frac{\zeta_z^{\text{IW}}}{f} = \partial_t \frac{\zeta_z^{\text{obs}}}{f} + \Gamma, \quad \text{and} \quad (8)$$

$$\partial_t \partial_z \eta^{\text{VM}} = \partial_t \partial_z \eta^{\text{obs}} - \partial_t \partial_z \eta^{\text{IW}} = \partial_t \partial_z \eta^{\text{obs}} + \Gamma. \quad (9)$$

The $\partial_t \zeta_z^{\text{VM}} / f$ spectrum is weaker than that of $\partial_t \partial_z \eta^{\text{VM}}$ for all observed wavenumbers and frequencies (Fig. 15). Burger number is computed using spectral magnitudes, following (7), as

$$\hat{B}(\omega, k_z) = \left[\frac{\Phi_{\partial_t \zeta_z^{\text{VM}} / f}^{\text{VM}}(\omega, k_z)}{\Phi_{\partial_t \partial_z \eta^{\text{VM}}}^{\text{VM}}(\omega, k_z)} \right]^{1/4}.$$

It varies within 0.1–0.3 in the internal wave frequency band and 0.3–0.5 at subinertial frequencies. Phase spectra between $\partial_t \zeta_z^{\text{VM}} / f$ and $\partial_t \partial_z \eta^{\text{VM}}$ vary greatly, although they are mostly in the range $\pm(90^\circ\text{--}180^\circ)$.

Alternatively, we estimate Burger number $\bar{B}(T_{\text{lp}}, H_{\text{lp}})$ using linear regression analysis between $\partial_t \zeta_z^{\text{VM}} / f$ and $\partial_t \partial_z \eta^{\text{VM}}$, low-pass filtered with half-power at temporal scale T_{lp} and vertical scale H_{lp} . An example is shown for $T_{\text{lp}} = 10$ h and $H_{\text{lp}} = 10$ m (Fig. 16a). Low-pass filtered $\partial_t \zeta_z^{\text{VM}} / f$ and $\partial_t \partial_z \eta^{\text{VM}}$ are negatively correlated, as expected from (7). Burger number $\bar{B} = 0.34$ is found using linear regression analysis. Ordinary and orthogonal regression analyses yield nearly identical estimates of $\bar{B}$. Parameter $\bar{B}$ computed using different values of T_{lp} and H_{lp} yields

FIG. 12. Time series of time rate of change of normalized vertical vorticity (black curve) and time rate change of vertical strain (red curve) compared with the normalized horizontal divergence (shading) bandpass filtered between 0.025 and 0.1 cph, within the internal wave low-frequency band, between 84- and 142-m depth at 8-m intervals. The scale is illustrated at the top left.

estimates between 0.2 and 0.4, with a mean value of 0.3. At $T_{lp} > 30$ h and $H_{lp} < 30$ m, low-pass filtered $\partial_t \xi_z^{VM}/f$ and $\partial_t \partial_z \eta^{VM}$ are uncorrelated statistically (Fig. 16b). Overall, two different methods yield similar estimates of Burger number, ranging between 0.2 and 0.4. Our estimate of Burger number represents the value for the observed linear vortical mode averaged over a horizontal scale of $O(4-8)$ km. Our estimate of B is smaller than 1.26 and $O(1)$ reported by Polzin et al. (2003) and Kunze (1993), respectively, but slightly greater than 0.1 reported by Pinkel (2014).

b. Linear vortical mode energy

Müller et al. (1988) devise a method to compute the energy of the linear vortical mode. Following this method, spectra of total energy $\Phi_E^{VM}(\omega, \beta)$, kinetic energy $\Phi_{KE}^{VM}(\omega, \beta)$, and potential energy $\Phi_{PE}^{VM}(\omega, \beta)$ of the linear vortical mode can be expressed by the potential vorticity spectrum $\Phi_{PV}(\omega, \beta)$ [see (A10)–(A12)]. For the linear vortical mode, the ratio between the kinetic

energy and potential energy is $\Phi_{KE}^{VM}(\omega, \beta)/\Phi_{PE}^{VM}(\omega, \beta) = \bar{B}^2 = 0.04-0.16$ for $\bar{B} = 0.2-0.4$. Kunze (1993) derives the energy ratio for cyclostrophic motion. For the observed $Ro \sim 0.2$, the energy ratio reduces to 0.03–0.14.

A two-dimensional energy spectrum of the linear vortical mode is computed following (A10) using the observed potential vorticity spectrum and $\bar{B}(\omega, k_z)$ (see section 5) (Fig. 17). Assuming a constant Burger number of 0.3 or varying in the range between 0 and 1 yields a similar spectrum (not shown). Linear vortical mode energy resides mostly at subinertial frequencies and low vertical wavenumbers. In the frequency range of $1-3f$ and vertical wavenumber < 0.05 cpm, the vortical mode energy is at least two decades less than the observed total energy. The strongest contribution of the linear vortical mode energy is at 0.1 cpm, near our Nyquist vertical wavenumber. The kinetic energy and potential energy spectra for the linear vortical mode have the same frequency and vertical wavenumber distribution as its total energy [not shown; see (A11) and (A12)].

FIG. 13. Depth averages of (a) frequency spectra of normalized vertical vorticity (black curve) and horizontal divergence (red curve), (b) coherence, and (c) phase spectra between vertical vorticity and horizontal divergence. Blue dashed and solid curves in (a) are GM76 predictions of vertical vorticity as shown in Fig. 10. The magenta solid curve represents horizontal divergence spectrum of GM76 model including both the area-averaging and contamination effects. Shadings represent 95% confidence intervals in (a) and (c) and the 95% significance level in (b). Solid dots and open circles in (b) and (c) represent significant and insignificant coherences, respectively. The dashed line in (c) represents the referenced 90° phase.

The total energy frequency spectrum of the linear vortical mode is defined as $\Phi_E^{\text{VM}}(\omega) = \int_0^\infty \Phi_E^{\text{VM}}(\omega, \beta) d\beta$. However, we only measure a narrow vertical wavenumber range and, therefore, we estimate $\Phi_E^{\text{VM}}(\omega) = \int_{\beta_{\text{low}}}^{\beta_{\text{up}}} \Phi_E^{\text{VM}}(\omega, \beta) d\beta$, where $\beta_{\text{low}} = 1/70 \text{ cpm}$ and $\beta_{\text{up}} = 1/7 \text{ cpm}$. Our estimate $\Phi_E^{\text{VM}}(\omega)$ is 1–2 decades less than the total energy spectrum in the entire frequency band (Fig. 18). Parameter $\Phi_E^{\text{VM}}(\omega)$ computed using $\hat{B}(\omega, k_z)$ or a constant B in the range 0–1 yields a similar result. In the internal wave frequency band the linear vortical mode is expected to be weaker than internal waves. However, at subinertial frequencies most of the energy should be contributed by the vortical mode. We propose that the discrepancy is because we have underestimated the linear vortical mode energy at subinertial frequencies by missing the energy contributed by the vortical mode at vertical scales $> 70 \text{ m}$.

The rms amplitude of the total observed field is 0.03 m s^{-1} , and the linear vortical mode is 0.004 m s^{-1} . Note that the estimated linear vortical mode energy (Figs. 17, 18) represents only the linear geostrophic flow at vertical scales of 7–70 m averaged horizontally over 4–8 km, that is, the scale resolved by LatMix measurements. The difference between the total field and our estimate of linear vortical mode represents not only

internal waves, but also vortical motion at scales not captured by our measurements. Müller et al. (1988) report a linear vortical mode amplitude of $\sim 0.01 \text{ m s}^{-1}$ from IWEX observations, about twice our estimate from LatMix. However, their estimate of PV is contaminated by horizontal divergence.

Similarly, the vertical wavenumber spectrum of the linear vortical mode is computed by integrating the two-dimensional energy spectrum of the vortical mode over the frequency domain. The linear vortical mode has a -2 spectral slope below 0.1 cpm and is 1–2 decades smaller than the total vertical wavenumber spectrum. This implies that the vertical shear spectrum of the linear vortical mode is white, similar to internal waves. The contribution of vortical mode energy increases with vertical wavenumber. At 0.1 cpm , the vortical mode energy is about 25% of the total energy.

7. Discussion

a. Wave–vortex energy separation using shipboard measurements

Bühler et al. (2014) and Lindborg (2015) propose a technique to separate energy from internal wave and

FIG. 14. (a) Two-dimensional spectrum, (b) frequency spectrum, (c) variance preserving frequency spectrum, (d) vertical wavenumber spectrum, and (e) variance preserving vertical wavenumber spectrum of normalized PV. The black curve represents the total PV and the red curve linear PV. The blue line in (b) marks the reference -1 spectral slope. In (a) black contour lines represent constant horizontal wavenumbers for linear internal waves. Shadings in (b)–(e) represent the 95% confidence interval. The vertical dashed line in (a) marks the inertial frequency. The positive frequency corresponds to upward phase propagation.

FIG. 15. Two-dimensional frequency–vertical wavenumber spectra of (a) time rate of change of normalized vertical vorticity, and (b) time rate of change of vertical strain of the linear vortical mode. (c) Estimates of Burger number $\hat{B} \sim (\Phi_{\partial\xi_z/f}/\Phi_{\partial_t\partial_z\eta})^{1/4}$. (d) The phase between the time rate of change of normalized vertical vorticity and time rate of change of vertical strain. Black curves in (a), (b), and (d) represent contours of constant horizontal wavenumbers for internal waves. Vertical dashed lines mark the inertial frequency. The positive frequency corresponds to upward phase propagation.

vortical mode components using ship track velocity and buoyancy measurements in a horizontally isotropic flow. Bühler et al. (2017) extend the technique to a horizontally anisotropic flow assuming a uniform anisotropy across all horizontal wavenumbers. By this method, HKE is decomposed into rotational and divergent components. Using a hydrostatic assumption, the total energy spectrum of internal waves

$\Phi_E^{\text{IW}}(k)$ equals twice the divergent component of HKE, where k is along-track wavenumber. The total energy spectrum of the vortical mode is the difference between the observed total energy spectrum $\Phi_E^{\text{Obs}}(k)$ and the internal wave component, that is, $\Phi_E^{\text{VM}}(k) = \Phi_E^{\text{Obs}}(k) - \Phi_E^{\text{IW}}(k)$. Following Bühler et al. (2017), $\Phi_E^{\text{IW}}(k)$ can be computed using the longitudinal velocity ($u_{\parallel}$) spectrum $\Phi_{u_{\parallel}}(k)$, transverse velocity

FIG. 16. Estimates of linear vortical mode Burger number. (a) Scatterplot (dots) and the linear regression fit (red) of time-rate changes of normalized vertical vorticity and vertical strain low-pass filtered at 10 m in the vertical and 10 h in time. Burger number is estimated as the squared root of the fitted negative slope. (b) Estimates of Burger number following the linear regression analysis shown in (a) using different low-pass filtered time and vertical scales. Dashed red curves mark boundary of significant correlation. The big black dot marks the result shown in (a).

($u_{\perp}$) spectrum $\Phi_{u_{\perp}}(k)$, and cross-spectrum $C_{u_{\parallel}u_{\perp}}(k)$, that is,

$$\Phi_E^{\text{IW}}(k) = \Phi_{u_{\parallel}}(k) - \frac{1}{k} \int_k^{\infty} [\Phi_{u_{\perp}}(s) - \Phi_{u_{\parallel}}(s) + \Delta(s)] ds, \quad (10)$$

and

$$\Delta(k) = \frac{\sigma_{u_{\parallel}}^2 - \sigma_{u_{\perp}}^2}{\sigma_{u_{\parallel}u_{\perp}}} C_{u_{\parallel}u_{\perp}}(k). \quad (11)$$

Here, $\sigma_{u_{\parallel}}^2$ and $\sigma_{u_{\perp}}^2$ are variances of $u_{\parallel}$ and $u_{\perp}$, respectively, and $\sigma_{u_{\parallel}u_{\perp}}$ their covariance. For horizontally isotropic flow, $\Delta = 0$. For horizontally anisotropic flow, the wave–vortex decomposition scheme proposed by Bühler et al. (2014, 2017) and Lindborg (2015) requires the streamfunction and velocity potential of the velocity field to be uncorrelated, implying that the imaginary component of $C_{u_{\parallel}u_{\perp}}$ is negligible (Bühler et al. 2017). The latter represents an admissible check of the wave–vortex decomposition method in horizontally anisotropic flow. In horizontally isotropic flow, the wave–vortex decomposition method does not require the assumption of uncorrelated streamfunction and velocity potential of the velocity field (Callies et al. 2016).

During D1 of LatMix, R/V *Oceanus* made 77 repeated 30–40-km meridional sections at ~ 8 kt (1 kt ≈ 0.51 m s^{-1} ; Fig. 19). The shipboard survey covered most of the float measurement area. The shipboard 75-kHz ADCP took 2-min averaged horizontal velocity measurements in 8-m vertical bins. The CTD platform Triaxus was towed behind the ship, taking CTD

measurements between the sea surface and ~ 100 -m depth with ~ 2 -m vertical resolution. Along-ship wave-number spectra of HKE and APE between 60- and 100-m depths were computed using shipboard ADCP velocity and Triaxus CTD measurements of all 77 north–south segments. The depth range is chosen to be comparable to that of float measurements used in this analysis and is limited by the maximum 100-m depth of Triaxus measurements. The total energy spectra $\Phi_E^{\text{Obs}}(k)$ is computed and averaged vertically (Fig. 20).

The observed velocity field is horizontally anisotropic because $\sigma_{u_{\parallel}}^2 \neq \sigma_{u_{\perp}}^2$ (based on an F test) and $\sigma_{u_{\parallel}u_{\perp}} \neq 0$ (based on a t test). The imaginary component of $C_{u_{\parallel}u_{\perp}}$ has similar magnitude as its real component, suggesting that velocity potential and the velocity field streamfunction are correlated, violating the underlying assumption of the decomposition scheme. Regardless, the decomposition is performed following (10) and (11) considering both isotropic and anisotropic models. Because $|C_{u_{\parallel}u_{\perp}}|$ is 1–2 decades smaller than $\Phi_{u_{\parallel}}$ and $\Phi_{u_{\perp}}$ (not shown), the anisotropic correction $\Delta(k) \sim 0.03 C_{u_{\parallel}u_{\perp}}(k)$ is small so that isotropic and anisotropic models yield similar internal wave and vortical mode energy spectra (Fig. 20). Internal waves and the vortical mode have similar spectral shapes. The internal wave component represents $\sim 70\%$ of the total energy, and the vortical mode $\sim 30\%$. This result differs significantly from our analysis using EM-APEX float measurements. The discrepancy may arise because the two estimates are derived in different Fourier spaces. In section 6b, only the linear vortical mode on a vertical scale of 7–70 m, and horizontally averaged over $O(4\text{--}8)$ km, is

FIG. 17. Frequency–vertical wavenumber spectrum of (a) observed total energy and (b) total energy of the linear vortical mode computed using $\hat{B} = (\Phi_{\partial_t \xi_z / f} / \Phi_{\partial_t \partial_z \eta})^{1/4}$. (c) The ratio of the total energy of the vortical mode component to the observed total energy. Thin black contour lines represent constant horizontal wavenumbers for linear internal waves. Vertical dashed lines mark the inertial frequency. Positive frequency corresponds to upward phase propagation.

captured. The linear vortical mode is expressed in frequency and vertical wavenumber spaces, with only a very small subinertial frequency range, and narrow vertical wavenumber range, where the linear vortical mode exists. Here, the energy is separated in horizontal wavenumber Fourier space. Because the velocity potential and streamfunction are correlated, results from the wave–vortex decomposition presented in this section require further justification.

b. Shear effect of inertial wave advection on background PV

The observed PV varies at the inertial frequency (Figs. 8, 11, 14), which may arise partially because float measurements are non-Lagrangian. Because EM-APEX floats are advected by the depth-averaged horizontal current, the observed temporal variation of area-averaged PV following the float array, assuming no external source or sink, can be expressed as

$$\begin{aligned} \frac{dPV}{dt} &= \partial_t PV + \bar{U}_h \cdot \nabla_h PV + U'_h \cdot \nabla_h PV \\ &= \left(\frac{dPV}{dt} \right)_{\text{float}} + U'_h \cdot \nabla_h PV = 0, \end{aligned} \quad (12)$$

where U_h is the horizontal current averaged over the float array, $\bar{U}_h(t) = \int_{-H}^0 U_h(z, t) dz$ is the depth-average current, and $U'_h = U_h(z, t) - \bar{U}_h(t)$ the perturbation velocity. The effect of float's advection by surface current during data transmission and GPS acquisitions is not considered here.

Observed $(d_t PV)_{\text{float}}$ has a standard deviation of $0.062 f^2$ (Fig. 11). Here, we estimate the magnitude of $U'_h \cdot \nabla_h PV$ and compare with the magnitude of $(dPV/dt)_{\text{float}}$. We compute $\bar{U}_h(t)$ and $U'_h(z, t)$ using float measurements directly. Unfortunately, float velocity measurements in the upper 80 m suffer the sunlight effect during daylight and early evening hours. The $\bar{U}_h(t)$ and $U'_h(z, t)$ are approximated using velocity

FIG. 18. (a) Frequency spectra of total observed energy (thick black curve) and the linear vortical mode component (red and blue curves). (b) Vertical wavenumber spectra of total observed energy (thick black curve) and the linear vortical mode component (red and blue curves). Red curves represent linear vortical mode energy computed using Burger number as $\bar{B} = (\Phi_{\partial \xi_z / f} / \Phi_{\partial \eta / \partial z})^{1/4}$, blue curves assume $B = 0.3$, and gray shadings assume B varying in $[0, 1]$. Vertical dashed lines in (a) mark the inertial and semidiurnal frequencies. The dashed line in (b) marks the reference spectral slope -2 .

measurements between 80- and 150-m depth. Using nighttime observations, without sunlight induced velocity errors, we find the error of $U'_h(z, t)$ using only measurements deeper than 80 m is $<10\%$. Perturbation velocity U'_h has a standard deviation of 0.02 ms^{-1} . Horizontal gradients of ξ_z and $f \partial_z \eta$ are computed from float measurements (section 3). The daily average magnitude of $\nabla \text{PV} = \sqrt{(\partial_x \text{PV})^2 + (\partial_y \text{PV})^2}$ varies between 0.11 and $0.18 f \text{ km}^{-1}$. Therefore, $U'_h \cdot \nabla_h \text{PV}$ may have a standard deviation of $0.029\text{--}0.047 f^2$, explaining up to 49%–76% of the observed $(d_t \text{PV})_{\text{float}}$ standard deviation. Note that errors in estimates of vertical vorticity due to the unknown depth-independent constant velocity, and different transfer functions on estimates of vertical vorticity and vertical strain might also contribute to observed $(d_t \text{PV})_{\text{float}}$.

8. Summary

Simultaneous measurements of horizontal velocity, T , S , and density taken from a swarm of 20 EM-APEX floats in the upper-ocean thermocline of the summer Sargasso Sea provide the opportunity to compute potential vorticity averaged over horizontal scales of

$O(4\text{--}8)$ km and vertical scales of 7–70 m. The background internal wave field has 1/2 of the GM76 predicted energy level. The three-dimensional relative vorticity vector, horizontal divergence, isopycnal slopes, and vertical strain are estimated on isopycnal surfaces using EM velocity measurements and CTD measurements.

FIG. 19. Ship track of R/V *Oceanus* during the LatMix experiment (black lines) and perimeters of the float array (red curve). The blue curve follows the center of the float array.

FIG. 20. Horizontal wavenumber spectra of total energy (black curve), internal wave component (red curve), and vortical mode component (blue curve). Solid and dashed curves represent results from isotropic and anisotropic models, respectively.

In the internal wave frequency band, the observed vertical vorticity and vertical vortex stretching exhibit similar temporal and vertical fluctuations, with a correlation coefficient of 0.8. Frequency spectra of vertical vorticity and vortex stretching have nearly identical spectral shape and agree with the GM76 prediction at 1/2 GM76 energy level. They are coherent and in phase in the internal wave frequency band. Strong packets of vertical vorticity and vortex stretching are in close balance, yielding a weak net potential vorticity. Our observations provide a direct confirmation of the close balance between vertical vorticity and vortex stretching for internal waves, using direct oceanic measurements. Horizontal divergence and vertical vorticity are 90° out of phase in the internal wave frequency band, as expected for internal waves, further supporting the dominance of internal waves within the internal wave frequency band.

Burger number is estimated using time rates of change of vertical vorticity and vortex stretching of the vortical mode. The negative correlation between vertical vorticity and vortex stretching of the vortical mode is confirmed. Burger number estimated from two different methods ranges between 0.2 and 0.4, suggesting that the vortical mode has a KE/PE ratio of ~ 0.1 .

The linear vortical mode energy, averaged over a $O(4\text{--}8)$ -km horizontal scale and vertical scale of 7–70 m, is nearly two decades less than the observed energy in

the entire frequency. The rms velocity of the linear vortical mode is 0.004 m s^{-1} , which is half of current finestructure, $\sim 0.01 \text{ m s}^{-1}$ at vertical scales smaller than 10 m, inferred from IWEX measurements. Note that our measurements only capture PV in vertical scales of 7–70 m. We cannot confirm whether the IWEX current finestructure at vertical length scale < 10 m is the linear vortical mode. Furthermore, we likely underestimate the linear vortical mode energy by missing energy at vertical scales greater than 70 m. Our estimates of PV averaged horizontally at $O(4\text{--}8)$ km represent properties of small Ro vortical motions. The vortical motion at the observational site should have time scales longer than the inertial period. The observed PV in the internal frequency band is presumably due to advection of background PV by internal waves, weak vortical mode finestructure, or measurement errors. The vertical wavenumber energy spectrum of the linear vortical mode is 1–2 decades weaker than the observed vertical wavenumber energy spectrum. The vortical mode contribution to total energy increases with vertical wavenumber, suggesting its significant role at small vertical scales.

The horizontal wavenumber spectrum of the linear vortical mode, following the method of Bühler et al. (2014, 2017), is nearly 1/2 of internal waves on the order of one to tens of kilometers. Because the observed velocity field is horizontally anisotropic and does not support the underlying assumption of an uncorrelated velocity potential and streamfunction, caution is needed to interpret the derived linear vortical mode horizontal wavenumber spectrum.

Further investigation is needed to study the role of small-scale vortical motion on vertical shear instability and lateral dispersion. The oceanic vertical wavenumber spectrum of vertical shear often exhibits a transition from a white spectrum below 0.1 cpm to a -1 spectral slope between 0.1 cpm and the Ozmidov wavenumber. Both nonlinear internal waves and stratified turbulence have been suggested to reside in this “ -1 ” spectral regime, but no observational evidence has been reported. Kunze and Lien (2019), based on dimensional analysis, demonstrates that anisotropic turbulence, generated by internal wave breaking at horizontal scales much larger than the Ozmidov scale, can explain both the -1 vertical wavenumber spectral shape for vertical gradients of horizontal current and scalar, and the $1/3$ horizontal wavenumber spectral shape for horizontal gradients of horizontal current and scalar. However, it is unclear whether the anisotropic turbulence carries PV. Future studies should focus on PV at vertical scales less than 10 m and smaller horizontal scales in order to capture larger Ro-number flows when

the time scale of vortical motions overlaps that of internal waves.

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APPENDIX A

Properties of Internal Waves and the Linear Vortical Mode

The properties of linear internal waves and the linear vortical mode can be derived based on the normal mode decomposition of small-scale motions (Müller 1984). These properties are expressed in terms of horizontal divergence Γ , vertical relative vorticity ζ_z , and vortex stretching $f\partial_z\eta$ (Lien and Müller 1992a). For the linear vortical mode (VM), it is horizontally nondivergent and the ratio between relative vorticity and vortex stretching is related by the squared Burger number. These properties are expressed as

$$\nabla_h u_h^{\text{VM}} = 0, \quad \text{and} \quad (\text{A1})$$

$$\zeta_z^{\text{VM}} = -B^2 f \partial_z \eta^{\text{VM}}. \quad (\text{A2})$$

Here, the superscript *VM* denotes the vortical mode component. Burger number is defined as $B = N\alpha/f\beta$, where N is the buoyancy frequency, and α and β are magnitudes of horizontal and vertical wavenumbers, respectively. For linear internal waves, the relative vorticity equals the vortex stretching, such that the linear component of the Ertel potential vorticity anomaly vanishes, and the time rate of change of the Rossby number (ζ_z^{IW}/f) equals horizontal divergence. These properties are expressed as

$$\text{PV}^{\text{IW}} = \zeta_z^{\text{IW}} - f \partial_z \eta^{\text{IW}} = 0, \quad \text{and} \quad (\text{A3})$$

$$\partial_t \zeta_z^{\text{IW}} = -f \nabla_h u_h^{\text{IW}}. \quad (\text{A4})$$

Here, the superscript *IW* denotes the internal wave component. The linear perturbation potential vorticity anomaly is carried solely by the vortical mode.

Following properties expressed in (A1)–(A4), Γ , ζ_z , and $f\partial_z\eta$ are separated into linear internal wave and linear vortical mode components as

$$\Gamma^{\text{IW}} = \Gamma^{\text{tot}}, \quad (\text{A5})$$

$$\zeta_z^{\text{IW}} = -f \int dt \Gamma^{\text{tot}}, \quad (\text{A6})$$

$$f \partial_z \eta^{\text{IW}} = \zeta_z^{\text{IW}}, \quad (\text{A7})$$

$$\zeta_z^{\text{VM}} = \zeta_z^{\text{tot}} - \zeta_z^{\text{IW}}, \quad \text{and} \quad (\text{A8})$$

$$f \partial_z \eta^{\text{VM}} = f \partial_z \eta^{\text{tot}} - f \partial_z \eta^{\text{IW}}. \quad (\text{A9})$$

Müller et al. (1988) demonstrate that the total energy spectrum Φ_E^{VM} , kinetic energy spectrum $\Phi_{\text{KE}}^{\text{VM}}$, and potential energy spectrum $\Phi_{\text{PE}}^{\text{VM}}$ of the linear vortical mode can be expressed in terms of the potential vorticity spectrum Φ_{PV} and Burger number B as

$$\Phi_E^{\text{VM}} = \frac{1}{2} \frac{N^2}{f^2} (1 + B^2)^{-1} \beta^{-2} \Phi_{\text{PV}}, \quad (\text{A10})$$

$$\Phi_{\text{KE}}^{\text{VM}} = \frac{1}{2} \frac{N^2}{f^2} B^2 (1 + B^2)^{-2} \beta^{-2} \Phi_{\text{PV}}, \quad \text{and} \quad (\text{A11})$$

$$\Phi_{\text{PE}}^{\text{VM}} = \frac{1}{2} \frac{N^2}{f^2} (1 + B^2)^{-2} \beta^{-2} \Phi_{\text{PV}}. \quad (\text{A12})$$

APPENDIX B

Transfer Functions for Estimates of Vertical Vorticity, Horizontal Divergence, and Vertical Strain

Vertical wavenumber frequency spectra of estimated area-averaged vertical vorticity $\Phi_{\zeta_z}(\beta, \omega; R)$, horizontal divergence $\Phi_{\Gamma}(\beta, \omega; R)$, and vertical strain $\Phi_{\partial_z\eta}(\beta, \omega; R)$ can be expressed in terms of true spectra Φ_{ζ_z} , Φ_{Γ} , and $\Phi_{\partial_z\eta}$ and transfer functions as

$$\begin{aligned} \Phi_{\zeta_z}(\beta, \omega; R) = & \int_0^\infty d\alpha [\Phi_{\zeta_z}(\alpha, \beta, \omega) F(\alpha R) \\ & + \Phi_{\Gamma}(\alpha, \beta, \omega) G(\alpha R)] F_V(\beta H_{\text{vel}}), \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} \Phi_{\Gamma}(\beta, \omega; R) = & \int_0^\infty d\alpha [\Phi_{\Gamma}(\alpha, \beta, \omega) F(\alpha R) \\ & + \Phi_{\zeta_z}(\alpha, \beta, \omega) G(\alpha R)] F_V(\beta H_{\text{vel}}), \quad \text{and} \end{aligned} \quad (\text{B2})$$

FIG. B1. (a) Area averaging transfer function (black solid curve and thick gray) and vertical averaging/differencing transfer function (dashed curve), and (b) contamination function with 4 (thick black), 6 (medium black), and 8 (thin black) measurements around a circle of radius R .

$$\Phi_{\frac{\partial}{\partial z}\eta}(\beta, \omega; R) = \int_0^\infty d\alpha \Phi_{\frac{\partial}{\partial z}\eta}(\alpha, \beta, \omega) F(\alpha R) F_V(\beta H_{\text{ctd}}). \quad (\text{B3})$$

Here R is the approximate radius of the area enclosing the float array, and α and β are horizontal and vertical wavenumber magnitudes, respectively. Transfer functions F and G represent effects of horizontal area averaging and contamination for vertical vorticity and horizontal divergence, respectively (Fig. B1). Transfer function F_V describes the vertical averaging of velocity measurements and the vertical differencing effect for vertical strain (Fig. B1). Transfer functions for estimates of vertical vorticity and horizontal divergence using measurements around a circle were reported by Lien and Müller (1992a). Our estimates of vertical vorticity, horizontal divergence, and vertical strain are computed by averaging 20 measurements within a rounded area with an equivalent radius R of $O(2\text{--}4)$ km. The transfer function for the area averaging can be described approximately as $F(\alpha R) \sim [2J_1(\alpha R)/\alpha R]^2$, where J_1 is a Bessel function of first order. The contamination effect G results from the finite separation of measurements. We compute G by assuming six floats evenly distributed on the outer radius of R , which yields

$$G(\alpha R) = \frac{1}{3(\alpha R)^2} [1 - J_0(2\alpha R) - J_2(2\alpha R) - J_0(\sqrt{3}\alpha R) - 2J_2(\sqrt{3}\alpha R) + J_0(\alpha R) - 2J_2(\alpha R)].$$

Here, J_0 and J_2 are Bessel functions of zeroth and second order, respectively.

Estimates of vertical vorticity and horizontal divergence also suffer from the vertical averaging effect $F_V(\beta H_{\text{vel}}) = [\sin(\beta H_{\text{vel}})/\beta H_{\text{vel}}]^2$ of velocity measurements, where $H_{\text{vel}} \sim 3.5$ m is one-half of the vertical averaging scale of velocity measurements. Estimates of vertical strain suffer from the vertical finite differencing effect $F_V(\beta H_{\text{ctd}}) = [\sin(\beta H_{\text{ctd}})/\beta H_{\text{ctd}}]^2$, where $H_{\text{ctd}} \sim 1.5$ m is one-half of the vertical interval of CTD measurements.

APPENDIX C

GM76 Spectra of Horizontal Divergence and Vertical Relative Vorticity

Two-dimensional horizontal wavenumber-frequency spectra of horizontal divergence and relative vorticity of GM76 (Cairns and Williams 1976) are computed. Both are red in frequency and blue in horizontal wavenumber (Fig. C1a). However, the horizontal divergence has a bluer horizontal wavenumber spectrum than the vertical relative vorticity at each frequency. The vertical relative vorticity has a redder frequency spectrum than the horizontal divergence at each horizontal wavenumber. At the inertial frequency, the horizontal divergence and vertical vorticity frequency are identical at all horizontal wavenumbers. At higher frequencies, the ratio between the horizontal divergence and vertical vorticity increases with horizontal wavenumber. At 1 cph, for example, the horizontal divergence spectrum is 10^2 times vertical vorticity at 10^{-5} cpm and is 10^3 times at 10^{-4} cpm. Therefore, the GM76 horizontal divergence frequency spectrum is more sensitive to the horizontal area

FIG. C1. (a) GM76 frequency–horizontal wavenumber spectra of normalized vertical relative vorticity (RV) (black curves) and normalized horizontal divergence (HD) (red curves). Contours in (a) are logarithmic values of spectral levels ($\text{cph}^{-1} \text{cpm}^{-1}$). A vertical cutoff wavenumber of 0.1 cpm is applied. (b) Horizontal wavenumber spectra of relative vorticity (thick black curve) and horizontal divergence (thick red curve) without area averaging filter effect and contamination filter effect. In (b), blue curve represents the transfer function for area averaging effect and green curve represents the contamination effect. Thin black and red solid curves represent vertical vorticity and horizontal divergence spectra including area averaging effect. Thin dashed curves represent spectra including both the area averaging and contamination effects. (c) Frequency spectra of relative vorticity (thick black curve) and horizontal divergence (thick red curve). Thin solid curves represent spectra including area averaging effect, and thin dashed curves represent spectra including the additional contamination effect. The shading in (b) and (c) represents the contamination effect from horizontal divergence into estimates of relative vorticity. The blue solid curve in (c) shows the ratio of the relative vorticity spectrum to the horizontal divergence spectrum. The ratio is not affected by the area averaging effect. The blue dashed curve represents the ratio, including the contamination effect.

averaging effect than the vertical vorticity frequency spectrum.

Integrated over horizontal wavenumbers, the horizontal divergence has a white frequency spectrum and the vertical relative vorticity has ω^{-2} spectral shape (Fig. C1c). Integrated over the frequency domain, horizontal wavenumber spectra of horizontal divergence and vertical vorticity are blue below 10^{-3} cpm , and

become red at higher horizontal wavenumbers due to the GM76 prescribed cutoff vertical wavenumber, 0.1 cpm (Fig. C1b). In the present analysis, vertical vorticity and horizontal divergence are averaged over a horizontal scale of $O(4\text{--}8)$ km. These estimates suffer from area averaging and contamination effects. These effects are expressed by transfer functions (appendix B). The transfer function of the area averaging exhibits a low-pass

filter (Fig. B1b) with a -2 spectral slope for horizontal wavenumber beyond 10^{-4} cpm. The transfer function of the contamination effect exhibits a bandpass filter with the peak at 2×10^{-4} cpm. GM76 spectra of vertical vorticity and horizontal divergence, including the area averaging and contamination effects, are computed assuming a float array of radius 3 km (Fig. C1). Both vertical vorticity and horizontal divergence spectra are attenuated due to the area averaging effect. At frequencies beyond 0.1 cph, the estimated vertical vorticity spectrum is significantly contaminated by the horizontal divergence spectrum. The estimated horizontal divergence spectrum, however, does not suffer the vertical vorticity contamination effect. The relationship between relative vorticity and horizontal divergence spectra $\phi_{\zeta_c}(\omega)/\Phi_{\Gamma(\omega)} = f^2/\omega^2$ for linear internal waves is modified because estimates of vertical vorticity are contaminated by horizontal divergence above 0.1 cph (Fig. C1c).

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