The Drag on the Barotropic Tide due to the Generation of Baroclinic Motion

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ABSTRACT: The interaction of a barotropic flow with topography generates baroclinic motion that exerts a stress on the barotropic flow. Here, explicit solutions are calculated for the spatial-mean flow (i.e., the barotropic tide) resulting from a spatially uniform but time-varying body force (i.e., astronomical forcing) acting over rough topography. This approach of prescribing the force contrasts with that of previous authors who have prescribed the barotropic flow. It is found that the topographic stress, and thus the impact on the spatial-mean flow, depend on the nature of the baroclinic motion that is generated. Two types of stress are identified: (i) a “wave drag” force associated with propagating wave motion, which extracts energy from the spatial-mean flow, and (ii) a topographic “spring” force associated with standing motion at the seafloor, including bottom-trapped internal tides and propagating low-mode internal tides, which significantly damps the time-mean kinetic energy of the spatial-mean flow but extracts no energy in the time-mean. The topographic spring force is shown to be analogous to the force exerted by a mechanical spring in a forced-dissipative harmonic oscillator. Expressions for the topographic stresses appropriate for implementation as baroclinic drag parameterizations in global models are presented.

KEYWORDS: Internal waves; Tides; Parameterization

1. Introduction

The ocean is subject to periodically varying body forces at various frequencies due to the gravitational attraction of celestial bodies. In the absence of rough bathymetry these forces would drive a periodic, purely barotropic flow at the scale of ocean basins—the “barotropic tide” (e.g., Egbert and Erofeeva 2002). In reality, the presence of rough bathymetry results in the generation of smaller-scale (100 km or less in the horizontal) baroclinic motion¹ that imposes additional stresses on the system and thus modifies the kinetic energy of the barotropic flow. Such baroclinic motion may include internal waves (e.g., Garrett and Kunze 2007), bottom turbulence and hydraulic effects (e.g., Winters and Armi 2014), bottom-trapped internal tides (e.g., Falahat and Nycander 2015), or a combination thereof. Much of this baroclinic motion is unresolved—horizontally and/or vertically—in global (and even regional) tidally forced ocean models, and its effects must therefore be parameterized. Such parameterizations are vital to the realistic representation of the barotropic tide (Arbic et al. 2004) and internal tide (Ansong et al. 2015) in ocean models.

The scale separation between the spatial scales of the barotropic tide (comparable to ocean basins) and the spatial scales

of the baroclinic motion (100 km or smaller) makes it reasonable to assume that the barotropic tide is a spatially uniform flow over topographic scales that generate the baroclinic motion. Thus, from a theoretical perspective, the impact of topography on the barotropic tide reduces to the problem of the stress exerted on a time-varying but spatial-mean flow over topography. Substantial previous work on this problem (e.g., Bell 1975; Baines 1982; Llewellyn Smith and Young 2002; Khatiwala 2003; Nycander 2006) has focused on one particular baroclinic motion: the generation of internal waves/tides, and the associated energy loss (sometimes called “baroclinic conversion”) from the spatial-mean flow ($W = \bar{\mathbf{u}} \cdot \boldsymbol{\tau}$, where $\bar{\mathbf{u}}$ is the spatial-mean flow and $\boldsymbol{\tau}$ is the topographic stress). Furthermore, since the time-mean stress for a purely periodic spatial-mean flow generating linear internal waves is zero (e.g., Bell 1975; Shakespeare and Hogg 2019), the primary focus has been on the magnitude of the energy flux, which is thought to be a significant source of energy for mixing in the ocean interior (Wunsch and Ferrari 2004; St Laurent and Garrett 2002). This energy flux is dependent on the behavior of internal waves subsequent to their generation. Models treating the ocean as infinite depth (e.g., Bell 1975) yield a continuous spectrum of energy flux with wavenumber, and a different total magnitude, as compared with those incorporating the finite-depth of the ocean (e.g., Llewellyn Smith and Young 2002; Khatiwala 2003), which yield net energy flux only at resonant wavenumbers.² However, none of these works directly considered the impact of these energy fluxes and associated stresses on the

¹ Baroclinic motion is associated with smaller horizontal scales than barotropic motion because it is *by definition* associated with smaller vertical scales (i.e., barotropic is vertical “mode zero”), and the ratio of horizontal and vertical scales is fixed by the properties of the fluid (e.g., Coriolis and stratification) and the frequency of the motion.

² Wavenumbers for which integer multiples of the half-vertical-wavelength fit into the ocean depth.

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spatial-mean flow itself (and thus the feedback on the magnitude of the generated flux).

Previous work has focused almost exclusively on the role of radiating internal waves in contributing to the energy flux (baroclinic conversion) from the barotropic tide. However, topographic interactions also lead to significant nonradiating motion such as bottom-trapped tides. Bottom-trapped internal tides are generated poleward of certain “critical” latitudes where the tidal frequency becomes subinertial, and thus tidal-frequency internal waves are unable to propagate. Instead, the baroclinic motion decays exponentially with height above the topography. For the diurnal tides the critical latitude is around 30° , implying the very large region of ocean poleward of 30° will exhibit bottom-trapped diurnal internal tides. Multiple studies have implicated bottom-trapped tides as important sources of near-bottom mixing in certain locations (e.g., D’Asaro and Morison 1992; Nakamura et al. 2000; Falahat and Nycander 2015; Musgrave et al. 2017). Falahat and Nycander (2015) estimate the energy dissipation from bottom-trapped tides as about 10 GW—far smaller than the energy fluxes associated with the freely propagating internal tide (~ 1 TW). This small flux is because bottom-trapped tides—unlike propagating internal tides—extract no energy from the barotropic tide, *unless* they somehow exchange energy with other flows or have sufficiently large amplitudes to drive small-scale turbulence and mixing (e.g., Nakamura et al. 2000). The question to be addressed here is whether this small energy flux necessarily implies a small topographic stress and therefore negligible impact on the strength of the barotropic flow.

Many different baroclinic drag parameterizations have been proposed and implemented in a variety of global ocean models (e.g., Jayne and St. Laurent 2001; Egbert et al. 2004; Garner 2005; Zaron and Egbert 2006; Green and Nycander 2013). Often these schemes are called “wave drag” or “tidal conversion” parameterizations because they are exclusively based on processes that extract energy from the barotropic tide. Some of these parameterizations are based purely on dimensional arguments (Jayne and St. Laurent 2001), others on linear wave theory (Nycander 2005), and still others on some combination of the two (Zaron and Egbert 2006). Green and Nycander (2013) present a comparison of the three different tidal parameterizations in terms of their representation of the M_2 barotropic tide in a single-layer ocean model. Their results suggest that the schemes based more strongly on dynamical theory perform the best overall. Of course, the complexity of certain schemes and thus the numerical expense, must always be weighed against the potential for improved accuracy.

In this paper we investigate the drag on the barotropic tide due to both subinertial bottom-trapped tides and superinertial, propagating internal tides. The paper is organized as follows. Section 2 describes a simple mechanical system analogous to the forced-dissipative barotropic tide system to aid with the interpretation of the results that follow. This analogy shows that previous investigations of the forces on barotropic tides—which focused only on the energy conversion—have omitted an important time-dependent stress which does no work, but can significantly impact the kinetic energy of the barotropic tide. With this analogy in hand, we derive (in section 3) the coupled equations for the spatial-mean flow and baroclinic perturbation

FIG. 1. A forced and damped harmonic oscillator system comprising a mass m attached to a solid wall by a spring (constant κ) subject to a time-dependent forcing $F(t)$ and a frictional drag (constant r). The displacement of the system from the rest position is x and the velocity $u = dx/dt$. This system is an example of a situation in which forces exist (i.e., the spring force) that can modify the kinetic energy without doing any time-mean work.

flow associated with a spatially uniform body force acting over rough topography. The spatial-mean flow generates the baroclinic flow through the boundary interaction with the topography, and the baroclinic flow feeds back on the spatial-mean flow through the topographic stress. Novel explicit solutions for the mean flow in the coupled problem are presented in section 4. Solutions are presented in three limits: locally dissipating internal tides, propagating internal tide modes, and bottom-trapped internal tides. In each limit, the topographic stress takes a different form, and thus has a different impact on the spatial-mean flow. The stresses, and their physical implications, are summarized in section 5. Readers who are not interested in the detail of calculations may wish to skip sections 3 and 4 and go straight to the summary (section 5). Using the equations summarized therein, the impact of the topographic stress on the barotropic tide is evaluated for northern mid-Atlantic ridge topography (section 6). Further discussion and conclusions appear in section 7.

2. An important analogy

Before discussing the barotropic tide problem, we consider a simple mechanical system that displays important similarities. The system of interest is the forced and damped harmonic oscillator shown in Fig. 1. The oscillator comprises a mass m attached to a solid wall by a spring (constant κ) subject to a time-dependent forcing $F(t) = F_0 \sin \omega t$ and a frictional drag (constant r). The displacement x of the system from the rest position of the spring is thus given by

$$m \frac{d^2 x}{dt^2} = F(t) - \tau = \underbrace{F_0 \sin \omega t}_{\text{forcing}} - \underbrace{r \frac{dx}{dt}}_{\text{drag}} - \underbrace{\kappa x}_{\text{spring}}. \quad (1)$$

The velocity of the mass is $u = dx/dt$ and τ is the stress generated by the forced displacement of the system from rest: it is composed of a component that is in-phase with the velocity (the frictional drag) and a component that is out of phase (the spring force). The solution to (1) is well known. The time-mean kinetic energy once the oscillator reaches a phase-averaged steady state is

$$K = \frac{1}{T} \int_0^T \frac{1}{2} m u^2 dt = \frac{m \omega^2 F_0^2}{4[r^2 \omega^2 + (m \omega^2 - \kappa)^2]}. \quad (2)$$

The time-mean work done by the stress τ at phase-averaged steady state is

$$W = \frac{1}{T} \int_0^T u\tau \, dt = \frac{1}{T} \int_0^T ru^2 \, dt, \quad (3)$$

and is equal to the time-mean energy input from the forcing F . Both the drag force and spring force modify the kinetic energy of the system in Eq. (2). However, only the drag force does work on the system in Eq. (3). *The harmonic oscillator thus provides an example of a situation in which forces exist (i.e., the spring force) that can modify time-mean kinetic energy of the forced system without doing any time-mean work.*

In the following sections we will show that the stress on the barotropic tide due to the generation of baroclinic motion (e.g., waves) is exactly analogous. However, existing descriptions of the wave stress only include the drag force component and ignore the spring force. The reason is that previous authors (e.g., Jayne and St. Laurent 2001) have sought to deduce the wave stress from the wave energy flux (the work done on the topography). The pitfall of such an approach is immediately obvious from the above example: if we look only at the time-mean work Eq. (3) without considering the mechanics of the forced oscillator system, we might be tempted to *falsely* conclude that the stress is simply $\tau = ru$.

3. The coupled barotropic–baroclinic equations

In this section we derive general equations for the interaction of a barotropic flow with rough bathymetry to generate baroclinic flow. Unlike the conventional internal wave generation problem (e.g., Bell 1975; Llewellyn Smith and Young 2002; Khatiwala 2003) we do not prescribe the barotropic flow: instead we assume that the barotropic flow (e.g., tide) can be represented as a spatial-mean flow that is determined by the balance between a spatially uniform body force (e.g., astronomical forcing) and the stress due to the topographic interaction. The assumption of a spatially uniform barotropic flow ignores the possibility of basin-scale resonances that are important in setting the magnitude of the barotropic tide in the ocean, but will not affect the form of the topographic stress since the scale of topography in the open ocean is typically far smaller than the basin-scale amplitude variation of the barotropic tide. The relevant equations are for a body-forced, inviscid fluid on an f plane:

$$\frac{D}{Dt} \mathbf{u} + f \hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + \mathbf{F} + b \hat{\mathbf{z}}, \quad (4a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{and} \quad (4b)$$

$$\frac{D}{Dt} b = 0, \quad (4c)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \quad (4d)$$

(u, v, w) are the velocities in the Cartesian (x, y, z) directions, p is the dynamic pressure, b is the buoyancy, f is the Coriolis parameter, and $\mathbf{F} = (F_x, F_y, 0)$ is the spatially uniform body force. Let

$$u = \bar{u}(t) + u'(x, y, z, t) \quad \text{and} \quad v = \bar{v}(t) + v'(x, y, z, t), \quad (5)$$

where $(\bar{u}, \bar{v}, 0)$ is the spatial-mean flow, and let

$$b = \int_{-H}^z N^2(z') \, dz' + b'(x, y, z, t). \quad (6)$$

The perturbation fields (primed) are assumed to vanish at the domain edges (or at infinity). The domain is shown in Fig. 2. It is bounded by a rigid lid at the surface, $z = 0$, and topography at the bottom, $z = -H + h(x, y)$, where H is the mean depth such that $\int_{-L_x/2}^{L_x/2} \int_{-L_y/2}^{L_y/2} h \, dx \, dy = 0$. The perturbation velocity boundary conditions are no-normal-flow through the surface and bottom; hence, $w = 0$ at the surface and $w = \bar{\mathbf{u}} \cdot \nabla h$ at the bottom. The perturbation buoyancy boundary conditions then follow from Eq. (4c); that is, $b' = 0$ at the surface, and $b' = -N^2 h$ at the bottom.

a. The mean flow equations

The spatial-mean flow is here defined as the average over the fluid volume

$$\begin{aligned} \bar{\mathbf{u}}_h &= \frac{1}{V} \int_V \mathbf{u}_h \, dV \\ &= \frac{1}{HL_x L_y} \int_{-L_x/2}^{L_x/2} \int_{-L_y/2}^{L_y/2} \int_{-H+h}^0 \mathbf{u}_h \, dz \, dy \, dx \\ &= \frac{1}{H} \left\langle \int_{-H+h}^0 \mathbf{u}_h \, dz \right\rangle, \end{aligned} \quad (7)$$

where the angle brackets are used to denote the horizontal average, and $\mathbf{u}_h = (u, v, 0)$. Assuming that the topography is of small amplitude we may write

$$\int_{-H+h}^0 \mathbf{u}_h \, dz = \int_{-H}^0 \mathbf{u}_h \, dz + \int_{-H+h}^{-H} \mathbf{u}_h \, dz \approx \int_{-H}^0 \mathbf{u}_h \, dz - h \mathbf{u}_h|_{z=-H}, \quad (8)$$

and similarly for any other variable of interest.³ To determine the equations for the mean flow we apply the averaging (bar)

³ The particular choice of spatial-mean operator in Eq. (7) is significant: the mean has been defined to incorporate the impact of the topography in constricting the flow. The mean velocity may also be written as

$$\begin{aligned} \bar{\mathbf{u}}_h &= \left\langle \frac{1}{H} \int_{-H}^0 \mathbf{u}_h \, dz - \frac{h}{H} \mathbf{u}_h \Big|_{z=-H} \right\rangle \\ &= \left\langle \frac{1}{H} \int_{-H}^0 \left[\mathbf{u}_h - \frac{\partial}{\partial z} \left(\frac{\mathbf{u}_h b'}{N^2} \right) \right] dz \right\rangle, \end{aligned}$$

which may be recognized as the spatial-mean of the *residual* flow (Andrews and McIntyre 1976). The second line in the above equation follows from the first by applying the small-amplitude boundary condition of $b' = -N^2 h$ at the topography (i.e., the topography is an isopycnal surface) and boundary condition $b' = 0$ at the ocean surface.

FIG. 2. Schematic of the model domain. The domain is bounded by a rigid lid at the surface, $z = 0$, and topography at the bottom, $z = -H + h(x, y)$. The system is forced by a spatially uniform body force $\mathbf{F}(t)$. The resultant flow is separated into the spatial-mean component $\bar{\mathbf{u}}(t)$ and the spatially varying perturbation flow $\mathbf{u}'(x, y, z, t)$ induced by the presence of topography. Horizontally, the domain extends over $-L_x/2 < x < L_x/2$ and $-L_y/2 < y < L_y/2$, and perturbation flow is assumed to vanish in the far field. The domain may be considered as periodic in x and y , or we can let $L_x \rightarrow \infty$ and $L_y \rightarrow \infty$ for the unbounded case. The baroclinic perturbation velocity shown in color is a schematic representation of the u' velocity of a decaying mode-1 internal tide generated at the ridge in the center of the domain.

operator to the horizontal momentum Eq. (4a). For the x -momentum equation this yields

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} + \frac{1}{V} \int_V \nabla \cdot (\mathbf{u}\mathbf{u}) dV - f\bar{v} &= - \left\langle \frac{1}{H} \int_{-H+h}^0 \frac{\partial p}{\partial x} dz \right\rangle + F_x \\ &= - \left\langle \frac{1}{H} \int_{-H+h}^{-H} \frac{\partial p}{\partial x} dz \right\rangle + F_x, \end{aligned} \quad (9)$$

where the simplification assumes that the pressure vanishes at infinity. Applying the small-amplitude approximation, Eq. (9) may be simplified to

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} + \frac{1}{V} \int_S (\mathbf{u}\mathbf{u}) \cdot d\mathbf{S} - f\bar{v} &= \left\langle \frac{\partial p_{z=-H}}{\partial x} \frac{h}{H} \right\rangle + F_x \\ &= - \frac{1}{H} \left\langle p_{z=-H} \frac{\partial h}{\partial x} \right\rangle + F_x, \end{aligned} \quad (10)$$

where we have applied Gauss's theorem with S as the surface bounding the domain and integrated the pressure term by parts. It may be shown that the small amplitude approximation is asymptotically valid when the vertical decay scale of the pressure gradient ($1/m$ for wavenumber m) is much greater than the height of the topography, or $mh \ll 1$. Since the flow is always parallel to the rigid upper and lower boundaries, and is assumed to vanish at the horizontal edges (or be periodic), the surface integral goes to zero. The final expression is thus

$$\frac{\partial \bar{u}}{\partial t} - f\bar{v} = - \frac{1}{H} \tau_x + F_x. \quad (11a)$$

A similar equation may be derived for the y momentum:

$$\frac{\partial \bar{v}}{\partial t} + f\bar{u} = - \frac{1}{H} \tau_y + F_y. \quad (11b)$$

The stress may be written from Eq. (10) as

$$\boldsymbol{\tau} = (\tau_x, \tau_y) = \langle p'_{z=-H} \nabla h \rangle. \quad (12)$$

The spatial-mean flow $\bar{\mathbf{u}}_h$ is thus determined by the balance between the body force $\mathbf{F}$ and the topographic stress $\boldsymbol{\tau}$. The kinetic energy of the spatial-mean flow, $\overline{\mathbf{KE}} = (\bar{u}^2 + \bar{v}^2)/2$, evolves according to

$$\frac{\partial \overline{\mathbf{KE}}}{\partial t} = -W + \bar{\mathbf{u}}_h \cdot \mathbf{F}, \quad (13)$$

where $W = \bar{\mathbf{u}}_h \cdot \boldsymbol{\tau}/H$ is the work done by the topographic stress on the spatial-mean flow, which must balance the work done by the body force $\bar{\mathbf{u}}_h \cdot \mathbf{F}$ in the time average.

b. The baroclinic flow equations

The baroclinic flow equations are obtained by linearizing the full Eqs. (4) about the mean buoyancy in Eq. (6) and spatial-mean flow defined above:

$$\frac{\overline{D}}{Dt} \mathbf{u}' + f\hat{\mathbf{z}} \times \mathbf{u}' = -\nabla p' + b'\hat{\mathbf{z}}, \quad (14a)$$

$$\nabla \cdot \mathbf{u}' = 0, \quad \text{and} \quad (14b)$$

$$\frac{\overline{D}}{Dt}b' + N^2w' = 0, \quad (14c)$$

where

$$\frac{\overline{D}}{Dt} = \frac{\partial}{\partial t} + \bar{\mathbf{u}} \cdot \nabla. \quad (15)$$

These equations are valid assuming that the topography is of small height, $h \ll H$, and the perturbation flow is weak relative to the spatial-mean flow, $|\mathbf{u}'| \ll |\bar{\mathbf{u}}|$. To determine under what condition this latter assumption holds, consider that the boundary condition at the bottom is $w' = \bar{\mathbf{u}} \cdot \nabla h$. Thus the vertical velocity scales as $W' \sim \bar{U}kh$, and the horizontal baroclinic velocity as $U' \sim \bar{U}mh$ from Eq. (14b), for baroclinic horizontal length scale $1/k$ and vertical length scale $1/m$. Thus, the assumption that $U' \ll \bar{U}$ is valid if the vertical length scale greatly exceeds the height of topography, or $mh \ll 1$, consistent with our previous use of the small amplitude approximation in Eq. (10). For the largest-scale motions (e.g., low-mode waves) $m \sim 1/H$ and the assumption is valid; however, it becomes an increasingly stringent constraint on topographic height h for smaller scale motions.

4. Solution to the coupled problem

Here we wish to determine how a given spatial-mean flow is modified by the presence of topography and therefore the generation—and possible dissipation—of wave and nonwave baroclinic perturbation flows. We therefore modify the baroclinic flow equations in Eq. (14) to add a drag term as a zeroth-order model of dissipation; the final linearized equations are thus

$$\frac{\overline{D}}{Dt}\mathbf{u}' + f\hat{\mathbf{z}} \times \mathbf{u}' = -\nabla p' + b'\hat{\mathbf{z}} - \alpha\mathbf{u}'_h, \quad (16a)$$

$$\nabla \cdot \mathbf{u}' = 0, \quad \text{and} \quad (16b)$$

$$\frac{\overline{D}}{Dt}b' + N^2w' = -\alpha b', \quad (16c)$$

where α is the decay rate (here assumed to only act on the buoyancy and horizontal velocities). This constant decay rate is not intended as a realistic model of wave dissipation. Instead, it is utilized to allow the representation of both locally dissipating waves (α finite and a deep ocean; $H \rightarrow \infty$) and propagating modes ($\alpha \rightarrow 0$) simultaneously; the final expressions for the topographic stress in each of these limits will be independent of the value of α , and thus the exact form of α is not important to the problem (e.g., the same solutions result if a Laplacian dissipation is assumed). For the sake of mathematical convenience, we will assume in this section that the buoyancy frequency N is a constant.

Our primary interest in the present work is the case of periodic body forcing (e.g., astronomical tidal forcing, and hence periodic spatial-mean flow) acting across larger-scale bathymetry. In this limit, the excursion parameter $\varepsilon = k\bar{u}/\omega \ll 1$, and it is reasonable to neglect advection by the spatial-mean

flow in (16); that is, we make the small excursion approximation. With this approximation, we take a Fourier transform of the above equations in xyt and combine into a single equation for vertical velocity,

$$\left[\frac{\partial^2}{\partial z^2} + \frac{K^2(N^2 - \omega^2 - i\alpha\omega)}{\omega^2 - (f^2 + \alpha^2) + 2i\alpha\omega} \right] \hat{w}' = 0, \quad (17)$$

where $(k, l, 0)$ is the horizontal wavevector, $K = (k^2 + l^2)^{1/2}$ is the modulus thereof, ω is the frequency, and the caret denotes the Fourier transform (e.g., $\hat{w}$ is the transform of w). The boundary conditions are now $\hat{w}'|_{-H} = -i\mathbf{k} \cdot \hat{\mathbf{u}}'\hat{h}$ and $\hat{w}'|_0 = 0$. The solution to Eq. (17) in the limit of weak dissipation, $\alpha \ll f$, is

$$\hat{w}' = \hat{w}' \Big|_{-H} \frac{e^{-imz - \gamma z} - e^{imz + \gamma z}}{e^{imH + \gamma H} - e^{-imH - \gamma H}}, \quad (18)$$

where

$$m = \text{sgn}(\omega)K \sqrt{\frac{N^2 - \omega^2}{\omega^2 - f^2}} \quad \text{and} \quad \gamma = \alpha m \frac{\omega(2N^2 - \omega^2 - f^2)}{2(N^2 - \omega^2)(\omega^2 - f^2)}. \quad (19)$$

The pressure may be determined from Eq. (16) as

$$\frac{\partial \hat{p}'}{\partial z} = \left(i\omega + \frac{N^2}{i\omega - \alpha} \right) \hat{w}' = \frac{N^2 - \omega^2 - i\alpha\omega}{i\omega - \alpha} \hat{w}', \quad (20)$$

or, for small α ,

$$\hat{p}' = \frac{N^2 - \omega^2}{i\omega} \hat{w}' \Big|_{-H} \frac{1}{-im} \frac{e^{-imz - \gamma z} - e^{imz + \gamma z}}{e^{imH + \gamma H} - e^{-imH - \gamma H}}, \quad (21)$$

$$= \frac{N^2 - \omega^2}{m\omega} \hat{w}' \Big|_{-H} \frac{e^{-imz - \gamma z} + e^{imz + \gamma z}}{e^{imH + \gamma H} - e^{-imH - \gamma H}}. \quad (22)$$

The pressure at $z = -H$ is therefore

$$\hat{p}' \Big|_{z=-H} = \frac{N^2 - \omega^2}{m\omega} \hat{w}' \Big|_{-H} \coth(imH + \gamma H). \quad (23)$$

The spatial-mean topographic stress as defined by Eq. (12), expressed as a Fourier transform in time, is

$$\hat{\tau} = \frac{1}{4\pi^2 A} \iint \mathbf{k} \hat{h}^* \hat{p}' \Big|_{-H} dk dl, \quad (24)$$

applying Parseval's theorem, where asterisk denotes the complex conjugate. Substituting for the pressure from Eq. (23) we obtain

$$\hat{\tau} = \frac{1}{4\pi^2 A} \iint |\mathbf{k}| \hat{h}^2 \frac{N^2 - \omega^2}{m\omega} \coth(imH + \gamma H) (\mathbf{k} \cdot \hat{\mathbf{u}}) dk dl. \quad (25)$$

It is convenient to further define

$$\hat{\tau} \equiv H(\sigma_{xx} \hat{u} + \sigma_{xy} \hat{v}, \sigma_{xy} \hat{u} + \sigma_{yy} \hat{v}), \quad (26)$$

where the σ are drag coefficients:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{1}{4\pi^2 HA} \iint |\hat{h}|^2 \frac{N^2 - \omega^2}{m\omega} \coth(imH + \gamma H) \begin{Bmatrix} k^2 \\ l^2 \\ kl \end{Bmatrix} dk dl. \quad (27)$$

With these definitions, the mean flow equations in Eq. (11), Fourier transformed in time, are

$$-i\omega\hat{u} - f\hat{v} = \hat{F}_x - \sigma_{xx}\hat{u} - \sigma_{xy}\hat{v} \quad \text{and} \quad (28a)$$

$$-i\omega\hat{v} + f\hat{u} = \hat{F}_y - \sigma_{yy}\hat{v} - \sigma_{xy}\hat{u} \quad (28b)$$

Equations (28a) and (28b) make clear that the effect of topographic interaction on the barotropic flow is *not* a simple drag opposite to the flow direction, since the stress depends on both components of the velocity, and the drag coefficients σ may be complex. The general solution to Eq. (28) for the barotropic tidal flow is

$$\hat{u} = \frac{\hat{F}_x(\sigma_{yy} - i\omega) - \hat{F}_y(\sigma_{xy} - f)}{f^2 - \omega^2 + \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 - i\omega(\sigma_{xx} + \sigma_{yy})} \quad \text{and} \quad (29a)$$

$$\hat{v} = \frac{\hat{F}_y(\sigma_{xx} - i\omega) - \hat{F}_x(f + \sigma_{xy})}{f^2 - \omega^2 + \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 - i\omega(\sigma_{xx} + \sigma_{yy})}. \quad (29b)$$

Isotropic topography

Let $\mathbf{k} = K(\cos\phi, \sin\phi)$, and the drag coefficients in Eq. (27) may be written as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{1}{4\pi^2 HA} \iint |\hat{h}|^2 K^2 \frac{N^2 - \omega^2}{m\omega} \coth(imH + \gamma H) \begin{Bmatrix} c \cos^2\phi \\ \sin^2\phi \\ \frac{1}{2} \sin 2\phi \end{Bmatrix} d\phi. \quad (30)$$

Assuming $|\hat{h}|$ is a function of K only (i.e., isotropic), Eq. (30) may be integrated in ϕ to obtain

$$\begin{aligned} \sigma = \sigma_{xx} = \sigma_{yy} &= \frac{1}{4\pi HA} \int_0^\infty |\hat{h}(K)|^2 \frac{K^2}{|\omega|} \sqrt{(N^2 - \omega^2)(\omega^2 - f^2)} \\ &\times \coth(imH + \gamma H) dK, \\ \sigma_{xy} &= 0, \end{aligned} \quad (31)$$

where we have substituted the expression for vertical wave-number Eq. (19). The spatial-mean kinetic energy for isotropic topography is thus, from Eq. (29),

$$\begin{aligned} \overline{\text{KE}} &= \frac{1}{2}(|\hat{u}|^2 + |\hat{v}|^2) \simeq \overline{\text{KE}}_0 \left[1 + \Im(\sigma) \frac{2\omega(3f^2 + \omega^2)}{\omega^4 - f^4} \right. \\ &\quad \left. - \Re(\sigma)^2 \frac{\omega^4 + 6f^2\omega^2 + f^4}{(\omega^2 + f^2)(\omega^2 - f^2)^2} \right], \end{aligned} \quad (32)$$

at first order in σ , where $\overline{\text{KE}}_0$ is the kinetic energy in the absence of a topographic stress. Thus, the real component

of σ always acts to reduce the kinetic energy, but the imaginary component can (in theory) act to increase or decrease the kinetic energy depending on the sign of $\Im(\sigma)\omega/(\omega - f)$. The time-mean work done on the spatial-mean flow in Eq. (13) is

$$W = \Re(\sigma) \overline{\text{KE}}. \quad (33)$$

Thus, work is only done when there exists a nonzero real component of the drag coefficient. By analogy with the harmonic oscillator described in section 2, the real component of the drag coefficient is the frictional drag force, and the imaginary component is the spring force.

It is helpful to consider the topographic drag in three separate limits, shown in Fig. 3, corresponding to three distinct perturbation flows: (i) locally dissipating internal tides, (ii) propagating internal tide modes, and (iii) bottom-trapped internal tides. We consider each limit in detail below.

1) LOCALLY DISSIPATING INTERNAL TIDES

First, let us assume that dissipation is sufficiently strong that topographically generated motion (waves) dissipate before interacting with the ocean surface, or $\gamma H \gg 1$, as might be expected for large horizontal and vertical wavenumbers. The limit $\gamma H \gg 1$ is satisfied for a large ocean depth, $H \rightarrow \infty$, and finite dissipation (i.e., $\alpha \ll f$, which was assumed in the above derivation). In this limit,

$$\lim_{\gamma H \rightarrow \infty} \coth(imH + \gamma H) = \lim_{\gamma H \rightarrow \infty} \frac{-2i \sin(2mH) + e^{2\gamma H} - e^{-2\gamma H}}{e^{2\gamma H} + e^{-2\gamma H} - 2 \cos(2mH)} = 1, \quad (34)$$

and thus the drag coefficient from Eq. (31) is real and positive:

$$\begin{aligned} \sigma_d &= \frac{\sqrt{(N^2 - \omega^2)(\omega^2 - f^2)}}{4\pi HA |\omega|} \int_0^\infty |\hat{h}(K)|^2 K^2 dK \\ &= \frac{1}{2H} \frac{\sqrt{(N^2 - \omega^2)(\omega^2 - f^2)}}{|\omega|} h_{\text{rms}} \overline{K}, \end{aligned} \quad (35)$$

where h_{rms} is the root-mean-square topographic height,

$$h_{\text{rms}} = \sqrt{\frac{1}{4A\pi^2} \int_{-\infty}^\infty \int_{-\infty}^\infty |\hat{h}|^2 dk dl}, \quad (36)$$

and $\overline{K}$ is the height-weighted-mean wavenumber,

$$\overline{K} = \frac{1}{h_{\text{rms}}^2} \left(\frac{1}{4A\pi^2} \int_{-\infty}^\infty \int_{-\infty}^\infty K |\hat{h}|^2 dk dl \right). \quad (37)$$

We denote this drag coefficient in Eq. (35) with the subscript d for dissipating. Since σ_d is real and positive, the topographic stress decelerates the spatial-mean flow, and also extracts energy from the flow (which is then dissipated by the waves). If we introduce the additional assumption that the forcing (tidal) frequency obeys $f \ll \omega \ll N$ (e.g., near the equator) then Eq. (35) becomes

FIG. 3. Schematic of the three dynamical regimes for the interaction of an astronomically forced barotropic tide with topography: (a) The *locally dissipating wave regime*, likely dominated by smaller topographic scales, where waves are generated and dissipate near to their generation site without encountering the ocean surface. The energy carried vertically by the waves is extracted from the barotropic tide by the work $W = \bar{\mathbf{u}} \cdot \boldsymbol{\tau}$. (b) The *propagating mode regime*, likely dominated by larger topographic scales, where internal tide modes are generated and dissipate remotely away from their generation site. The energy carried horizontally by the waves is extracted from the barotropic tide by the work $W = \bar{\mathbf{u}} \cdot \boldsymbol{\tau}$. (c) The *bottom-trapped internal tide regime*, associated with both small and large topographic scales, where non-propagating, energy-conserving periodic motions are generated. No net energy is extracted from the barotropic tide ($W = 0$) since the stress is out of phase with the tidal velocity. However, the kinetic energy of the barotropic tide is significantly reduced by the generation of these motions. The contours/shading in the plots show snapshots of the in-page horizontal perturbation velocity (as a fraction of the velocity magnitude without topography; see the color bar) from the solutions to the baroclinic flow equations in Eq. (17). The equations shown give the stress in the limit $f \ll \omega \ll N$ for (a) and (b) and $\omega \ll f \ll N$ for (c)—the general expressions valid for arbitrary frequency are given in the text (also see Fig. 4, below).

$$\sigma_d = \frac{Nh_{rms}^2 \bar{K}}{2H}, \quad (38)$$

and the stress exerted on the topography by locally dissipating internal tides is

$$\boldsymbol{\tau}_d \simeq \frac{Nh_{rms}^2 \bar{K}}{2H} \bar{\mathbf{u}}. \quad (39)$$

The form of this stress is identical to the Jayne and St. Laurent (2001) parameterization currently used to represent baroclinic wave drag in some global models (e.g., Ansong et al. 2015). However, here we have introduced a rigorous definition in Eq. (37) of the wavenumber $\bar{K}$, which was treated as a globally uniform scaling parameter by Jayne and St. Laurent (2001).

2) PROPAGATING INTERNAL TIDE MODES

Second, let us assume that dissipation is sufficiently weak that topographically generated motion (waves) interact strongly with the ocean surface, or $\gamma H \rightarrow 0$, as might be expected for small horizontal and vertical wavenumbers that form low-mode internal tides. In this limit,

$$\lim_{\gamma H \rightarrow 0} \coth(imH + \gamma H) = -i \frac{\sin(2mH)}{1 - \cos(2mH)} + \lim_{\gamma H \rightarrow 0} \frac{e^{2\gamma H} - e^{-2\gamma H}}{e^{2\gamma H} + e^{-2\gamma H} - 2\cos(2mH)} \quad (40)$$

$$= -i \frac{\sin(2mH)}{1 - \cos(2mH)} + \lim_{\gamma H \rightarrow 0} \frac{\gamma H}{(\gamma H)^2 + \frac{1}{2}[1 - \cos(2mH)]} \quad (41)$$

$$= -i \frac{\sin(2mH)}{1 - \cos(2mH)} + \pi \sum_{n=0}^{\infty} \delta(mH - n\pi), \quad (42)$$

where Eq. (41) is obtained from Eq. (40) by Taylor expanding about $\gamma H = 0$, and then simplified to the final expression in Eq. (42) by utilizing the definition of the Dirac delta function,

$$\delta(\nu) = \frac{1}{\pi} \lim_{\eta \rightarrow 0} \frac{\eta}{\eta^2 + \nu^2}.$$

In other words, the solution for vanishing dissipation collapses to a form that has large amplitude for wavelengths where half-integer multiples fit into the ocean depth (i.e., $n\lambda_\nu/2 = H \rightarrow mH = n\pi$). The real component of the drag coefficient from Eq. (31) is thus

$$\begin{aligned}\Re(\sigma_p) &= \frac{1}{4\pi HA} \frac{\sqrt{(N^2 - \omega^2)(\omega^2 - f^2)}}{|\omega|} \int_0^\infty K^2 |\hat{h}(K)|^2 \pi \\ &\quad \times \sum_{n=0}^\infty \delta(mH - n\pi) dK \\ &= \frac{1}{4H^2 A} \frac{\omega^2 - f^2}{\omega} \sum_{n=1}^\infty [K^2 |\hat{h}(K)|^2]_{mH=n\pi},\end{aligned}\quad (43)$$

where the Dirac delta function collapses the integral. The drag-coefficient is denoted by a subscript p for propagating. The horizontal wavenumbers corresponding to a given vertical mode n can be determined from Eq. (19) as

$$\max[0, (n - 1/2)\pi] \leq mH = KH \sqrt{\frac{N^2 - \omega^2}{\omega^2 - f^2}} < (n + 1/2)\pi. \quad (44)$$

Thus, we can define the rms height associated with a given vertical mode n as

$$\begin{aligned}(h_{\text{rms}}^n)^2 &= \frac{1}{2\pi A} \int_{mH=\max[0, (n-1/2)\pi]}^{mH=(n+1/2)\pi} K |\hat{h}(K)|^2 dK \\ &= \frac{\pi |\hat{h}_n|^2}{4AH^2} \left| \frac{\omega^2 - f^2}{N^2 - \omega^2} \right| \begin{cases} \frac{1}{4} & n=0 \\ 2n & n \neq 0 \end{cases},\end{aligned}\quad (45)$$

where we have assumed that the topographic spectrum $|\hat{h}(K)|^2$ is sufficiently flat over the range of a given mode number n that it may be approximated by a constant, $|\hat{h}_n|^2$, over the range of that mode number.⁴ With this result, Eq. (43) becomes simply

$$\begin{aligned}\Im(\sigma_p) &= \sum_{n=0}^\infty \frac{-(h_{\text{rms}}^n)^2}{\pi^2 H^2} \frac{\omega^2 - f^2}{\omega} \begin{cases} 4 & n=0 \\ \frac{1}{2n} & n \neq 0 \end{cases} \int_{\min[0, (n-1/2)\pi]}^{(n+1/2)\pi} (mH)^2 \frac{\sin(2mH)}{1 - \cos(2mH)} d(mH) \\ &= - \sum_{n=0}^\infty \left(\frac{h_{\text{rms}}^n}{H} \right)^2 \frac{\omega^2 - f^2}{\omega} \begin{cases} \frac{2\pi^2 \ln 2 - 7\zeta(3)}{2\pi^2} \simeq 0.27 & n=0 \\ \ln 2 & n \neq 0 \end{cases},\end{aligned}\quad (50)$$

where ζ is the Riemann zeta function. Equations (46) and (50) combined thus give a simple expression for the drag on the spatial-mean flow due to the generation of each individual internal tide mode.

Since $\omega \Im(\sigma_p)$ is negative, both the real and imaginary parts of the drag coefficient cause a reduction in the kinetic energy of the spatial-mean flow [see Eq. (32)]. Furthermore, since the drag coefficient has a real component, the propagating mode also extracts energy from the spatial-mean flow in Eq. (33). It is notable that the imaginary component of the drag (for a given frequency) depends purely on the square of the topographic amplitude at scales corresponding to that mode, whereas the real component scales with the mode (or wave) number.

⁴ This assumption may be shown to be valid for power-law spectra $|\hat{h}(K)|^2 \sim K^{2p}$ where the slope p at mode number n satisfies $n \gg [2p(2p + 1)/3]^{1/2}$ (see the appendix for the derivation of this result).

$$\Re(\sigma_p) = \frac{\pi \omega^2 - f^2}{2} \sum_{n=1}^\infty n \left(\frac{h_{\text{rms}}^n}{H} \right)^2 = \frac{\pi}{2H^2} \frac{\omega^2 - f^2}{\omega} \bar{n} h_{\text{rms}}^2, \quad (46)$$

where $\bar{n}$ is the height-weighted-mean mode number,

$$\bar{n} = \sum_{n=1}^\infty n (h_{\text{rms}}^n / h_{\text{rms}})^2. \quad (47)$$

Noting that

$$\bar{n} \equiv \frac{\bar{m}H}{\pi} = \frac{H}{\pi} \text{sgn}(\omega) \sqrt{\frac{N^2 - \omega^2}{\omega^2 - f^2}} \bar{K}, \quad (48)$$

from the definition of vertical wavenumber in Eq. (19), an identical expression to the dissipating wave drag in Eq. (35) may be recovered for the propagating wave drag from Eq. (46).

The imaginary component of the drag coefficient for propagating waves is

$$\begin{aligned}\Im(\sigma_p) &= \frac{-1}{4\pi HA} \frac{\sqrt{(N^2 - \omega^2)(\omega^2 - f^2)}}{|\omega|} \\ &\quad \times \int_0^\infty K^2 |\hat{h}(K)|^2 \frac{\sin(2mH)}{1 - \cos(2mH)} dK.\end{aligned}\quad (49)$$

Large contributions to $\Im(\sigma_p)$ also occur for wavenumbers close to $mH = n\pi$, where n is an integer. It thus makes sense to also decompose Eq. (49) into modes, whereby

Therefore, we anticipate that the imaginary component will dominate the stress for lower modes generated at larger-scale topography that typically have greater topographic heights.

In the additional limit $f \ll \omega \ll N$ (e.g., near the equator), the stress exerted on the topography by propagating internal tide modes will simply be

$$\tau_p \simeq \left(\frac{h_{\text{rms}}}{H} \right)^2 \ln(2) \frac{\partial \bar{\mathbf{u}}}{\partial t} + \frac{N h_{\text{rms}}^2 \bar{\mathbf{K}}}{2H} \bar{\mathbf{u}}. \quad (51)$$

By analogy with the harmonic oscillator example (section 2; Fig. 1), propagating internal tide modes act as a combination of a frictional drag and a spring in damping the barotropic tide.

3) BOTTOM-TRAPPED INTERNAL TIDES

The previous two limits considered regimes where the forcing (tidal) frequency exceeds the inertial frequency and thus internal waves are generated. We now consider the case in which $\omega < f$ and topographically induced tidal-frequency

motion no longer propagates away but is confined near the bottom boundary as a “bottom-trapped tide.” The corresponding perturbation vertical velocity is

$$\hat{w}' = -i\mathbf{k} \cdot \hat{\mathbf{u}} h e^{-|m|(z+H)}, \quad (52)$$

where m is now an imaginary number, and $1/|m|$ gives the e -folding distance of the bottom-trapped motion (assuming $|m|H \gg 1$, and surface effects are therefore not important). This inviscid decay rate generally far exceeds any decay due to the nonconservative effects and the explicit dissipation is thus neglected here, $\alpha \rightarrow 0$. The bottom pressure corresponding to Eq. (52) is

$$\hat{p}'|_{-H} = \frac{N^2 - \omega^2}{|m|\omega} \mathbf{k} \cdot \hat{\mathbf{u}} h. \quad (53)$$

Substituting this bottom pressure into the stress expression in Eq. (24) and simplifying for isotropic topography yields a drag coefficient of

$$\sigma_b = \frac{i}{2H} \frac{\sqrt{(N^2 - \omega^2)(f^2 - \omega^2)}}{\omega} h_{\text{rms}}^2 \bar{K}, \quad (54)$$

with the mean wavenumber and rms height defined as previously in Eqs. (36) and (37). The subscript b denotes bottom-trapped. Since $\omega\sigma_b$ is positive and imaginary, and $\omega < f$ in this limit, the bottom-trapped internal tide stress reduces the spatial-mean kinetic energy [Eq. (32)]. However, since the drag coefficient is purely imaginary, no energy is extracted from the spatial-mean flow [i.e., Eq. (33)]. The modulus of the bottom-trapped drag coefficient $|\sigma_b|$ is identical to the modulus of the dissipating wave drag coefficient in Eq. (35), but the resultant stress has a different phase with respect to the spatial-mean flow. This result is most evident in the quasigeostrophic limit, $\omega \ll f \ll N$, (e.g., the high-latitude diurnal tide) where the stress exerted on the topography by bottom-trapped, nonpropagating internal tides reduces to

$$\tau_b \simeq \frac{N}{2H} \bar{K} h_{\text{rms}}^2 |f| \int \bar{\mathbf{u}} dt, \quad (55)$$

showing that the stress is 90° out of phase with the spatial-mean flow. We note that $\int \bar{\mathbf{u}} dt$ is a displacement of a fluid parcel, and thus the stress (55) has precisely the same form as the spring force in the harmonic oscillator example (section 2; Fig. 1). In other words, bottom-trapped tides act as a spring in damping the amplitude of the barotropic tide.

5. Summary of solutions and limits

Here we present a summary of the results of the calculations in the sections 3 and 4, and their physical implications. The results comprise a general formulation for the drag experienced by a spatially averaged oscillatory flow when it interacts with ocean bathymetry, which may be simplified under various approximations and in various limits. A flowchart of the equations from the most general solution to its various limiting forms is shown in Fig. 4.

The most general formulation (Fig. 4a) only requires small-amplitude bathymetry and a small-excursion distance $\varepsilon = k\bar{u}/\omega \ll 1$ to be valid. The topographic stress is expressed in terms of a matrix of linear drag coefficients σ (i.e., the stress is not in general parallel to the tidal flow) multiplied

by the Fourier transform in time of the barotropic velocity ($\hat{\mathbf{u}}$). These drag coefficients are in general complex numbers, and depend on the dissipation (through the nondimensional decay rate γH) and the degree of resonance (through the nondimensional wavenumber mH) of internal tides. The real part of the drag coefficient corresponds to a frictional drag force that is associated with a time-mean energy flux and the complex part to a spring force (by analogy with the harmonic oscillator; see Fig. 1) that is not associated with a time-mean flux. The description of this spring force is a novel feature of the present work.

Moving down the flow chart, Fig. 4b shows the simplified solution when isotropic topography is assumed. In this case, the matrix of drag coefficients collapses to a single drag coefficient and the topographic stress is always oriented parallel to the barotropic tide. The other features of the solution are unchanged from the general formulation.

At the next level (Fig. 4c) of the chart, the solution is simplified in three limits to remove the explicit dependence on the dissipation rate γH . This simplification is helpful because the dissipation rate is generally unknown, and further, the representation of wave dissipation used here (i.e., a constant decay rate α) is not realistic. Let us now consider the three limits:

- 1) The “dissipating wave” limit applies to the internal tide regime ($f < \omega < N$) where the generated waves dissipate before reflecting back to the bathymetry, consistent with the infinite depth model of internal tide generation (e.g., Bell 1975). It is thought that smaller-scale internal tides break above the generating topography (e.g., St Laurent and Nash 2004) and thus fall into this dynamical regime. This nonreflective internal tide regime is characterized by a purely real drag coefficient; that is, a frictional drag force (in phase with the barotropic tide) that removes energy from the barotropic tide.
- 2) The “reflecting wave” limit applies to the internal tide regime ($f < \omega < N$) where the generated waves reflect back to the bathymetry without dissipating, consistent with the finite depth model of internal tide generation (e.g., Llewellyn Smith and Young 2002). The reflecting waves provide an additional pressure at the bathymetry and thus an additional topographic stress, over and above the frictional drag force associated with the wave generation (which is identical to that in the dissipating limit). For a sufficiently flat topographic height spectrum (assumed in the derivation), this additional stress acts as a spring force that damps the barotropic tide without removing energy in the time-mean.
- 3) The “bottom trapped” limit applies to the evanescent regime ($\omega < f$)—poleward of critical latitudes—where bottom-intensified, nonpropagating tides are generated at rough bathymetry. Since these bottom-trapped tides naturally decay with height above the bottom, the dissipation is not important to the topographic stress. This regime is characterized by a purely imaginary drag coefficient; that is, a spring force (out of phase with the barotropic tide) that removes no energy in the time-mean. This is consistent with bottom-trapped tides having zero energy flux.

In all of the above limits, the drag coefficients depend on the frequency of the barotropic tide. This dependence has been separated out in Fig. 4 in terms of functions $F_{1-3}(\omega)$. We use

FIG. 4. Tabular summary of forces exerted when a periodic flow interacts with topography, in various solution limits: (a) The most general solution. (b) The solution for isotropic topography. (c) The solution for the three limits of (i) dissipating waves, (ii) propagating waves, and (iii) bottom-trapped tides. (d) The solutions from (c) simplified to remove frequency dependence.

these expressions for our calculations in the following section (section 6).

While the frequency dependence does not present a problem for analytic calculations, it can present serious complications for numerical model implementation (i.e., frequency dependence in the drag coefficient implies that the topographic stress expressed in time is a convolution of the inverse Fourier

transform of the drag coefficient with the flow velocity). As such, we present a further simplification (Fig. 4d) of the stress formulation in various frequency limits where the frequency dependence vanishes, $F_{1-3}(\omega) = 1$, allowing the topographic stress to be written as an explicit function of time. This simplification is also useful in making the qualitative impact of the stresses (i.e., frictional drag or spring force) more apparent.

FIG. 5. Northern mid-Atlantic ridge bathymetry: (a) Ocean depth variation (m) relative to mean ocean depth (windowed, from the GEBCO 30 arc s bathymetry product). Root-mean-square depth variation is $h_{\text{rms}} = 447$ m, and mean depth is $H = 3582$ m. (b) Equivalent isotropic spectra from bathymetry (dotted), smoothed analytic representation (red), Goff (2010) abyssal hill spectra for the region (green), combined abyssal hill and analytic (black dashes).

For dissipating waves, the resulting topographic stress for $\omega \gg f$ collapses to the Jayne and St. Laurent (2001) formulation (i.e., a frictional drag). For propagating/reflecting waves, the resulting stress collapses to the Jayne and St. Laurent (2001) formulation, plus a term that depends on the time rate of change of the barotropic tide (and hence removes no energy in the time-mean). For bottom trapped tides, the stress collapses to a form that depends on the time-integral of the barotropic tide (and hence removes no energy in the time-mean). We emphasize that these expressions (Fig. 4d) are only valid in the very restricted limits of $\omega \gg f$ for the waves (e.g., very near the equator) and $\omega \ll f$ for the bottom-trapped tides (e.g., high-latitude oceans). Thus, we do not advocate their use in the general case and they will not be employed further in the present work.

6. Solutions for North Atlantic topography

Using the results developed in the previous sections we now consider the effect of topographic drag on the barotropic tide over a section of the northern mid-Atlantic ridge (Fig. 5a), a site of significant tidal conversion. At these latitudes the semidiurnal tide is superinertial and the diurnal tide is sub-inertial, allowing investigation of both the wave and nonwave limits discussed previously. The relevant formulas for the drag coefficients in each limit are summarized in Fig. 4c, and may be substituted into the general solution Eq. (29) to determine the barotropic tide flow speeds (and kinetic energy). The equivalent isotropic bottom-roughness spectrum $|\hat{h}(K)|^2$ computed

for this North Atlantic Ocean region is shown in Fig. 5b, along with a smoother analytic representation (that we use in our calculations in order to avoid the high amplitudes at very large scales in the original spectrum, which are a product of the spatial windowing). Topographic datasets only contain horizontal wavelengths down to a few tens of 10 km, but scales smaller than this may be important to the topographic stress. At smaller wavelengths abyssal hills dominate the bathymetry and we therefore combine the resolved topographic spectrum with an appropriate abyssal hill spectrum for this region from Goff (2010). The combined spectrum is shown as a black dashed curve on Fig. 5b. The theory presented in earlier sections was derived in the small-excision limit that assumes $\varepsilon = ku/\omega \ll 1$. For a typical tidal flow speed in this region of ($\sim 1\text{--}2$ cm s $^{-1}$) and the semidiurnal tidal frequency, $\varepsilon \ll 1$ implies $k \ll 0.0071$ m $^{-1}$ for validity of the theory (or equivalently a wavelength $\lambda = 2\pi/k \gg 1$ km). We will check this assumption a posteriori.

We first consider the topographic stress associated with the semidiurnal tide in the locally dissipating (limit i; blue line in Fig. 6a) and propagating mode (limit ii; black) regimes. The limits are considered independently: in limit (i) we assume that 100% of the waves dissipate locally, whereas in limit (ii) we assume that 100% of the waves propagate away. The real situation will be some combination of the two, with larger scales propagating away, and smaller scales dissipating locally. The modulus of the drag coefficient (for each of the real and imaginary components, if they exist) is shown in Fig. 6a for each limit, cumulatively summed from small to large wave (or

FIG. 6. Topographic drag on the semidiurnal barotropic tide ($\omega = 1.41 \times 10^{-4} \text{ s}^{-1}$) due to the Atlantic topography shown in Fig. 5: (a) Cumulative sum over mode/wavenumbers of the modulus of the drag coefficient for 100% propagating modes (black; real component is dashed; imaginary component is solid) and 100% dissipating waves (blue). (b) Resulting spatial-mean kinetic energy associated with the drag coefficient shown in (a), normalized by the value obtained in the absence of drag. The kinetic energy decreases as higher wavenumbers (more drag) are included. Black circles show modes $n \geq 1$. Parameter values are $f = 10^{-4} \text{ s}^{-1}$ and $N = 3 \times 10^{-6} \text{ s}^{-1}$.

mode) number. For propagating modes, the largest contributions to both the imaginary component (the spring force) and the real component (the drag force) come from the lowest modes, with essentially all the force arising from mode numbers less than 10 (scales exceeding 5 km). The total spatial-mean kinetic energy (Fig. 6b) is reduced by 7% by the propagating mode drag, with about 6% from above 10 km and only 1% from smaller than 10 km. The biggest effect comes from the first two modes each contributing about 1% reduction each in spatial-mean kinetic energy. The dissipating wave drag, in contrast, has almost no amplitude at the largest scales. Cumulatively, dissipating wave drag at sub-10-km scales reduces the kinetic energy by about 1%, as compared with 2% for the above-10-km topographic scales. Thus, the total reduction in spatial-mean kinetic energy associated with locally dissipating waves is 3%—significantly less than the 7% due to remotely dissipating,

propagating modes. In reality, the total drag will be a combination of propagating mode drag, for scales that generate such modes, and dissipating wave drag for scales that dissipate locally.

We now consider the drag on the diurnal tide, which is subinertial in the North Atlantic sample region (and also significantly weaker than the semidiurnal tide), and will therefore generate bottom-trapped internal tides (limit iii). The drag coefficient and spatial-mean kinetic energy are shown in Fig. 7. The bottom-trapped tides cause a 40% reduction in the spatial-mean kinetic energy, predominantly due to intermediate (10–200 km) spatial scales. The impact of the bottom-trapped internal tide stress is about 4 times as large as would be the equivalent dissipating wave drag, if internal waves were generated instead (shown as dashed line for comparison). The drag coefficients for both stresses are identical in modulus but the bottom-trapped stress acts out of phase with the barotropic

FIG. 7. Topographic drag on the diurnal barotropic tide ($\omega = 7.0 \times 10^{-5} \text{ s}^{-1}$) due to the Atlantic topography shown in Fig. 5. (a) Cumulative sum over wavenumbers of the modulus of the drag coefficient due to bottom-trapped tides. (b) Resulting spatial-mean kinetic energy associated with the drag coefficient shown in (a), normalized by the value obtained in the absence of drag. The dissipating wave solution is shown for comparison as a dashed line (but is not valid since $\omega < f$). The modulus of the drag coefficient for dissipating waves is the same as for bottom-trapped tides, but the drag is phased differently, resulting in a lesser impact on the spatial-mean kinetic energy. Parameter values are $f = 10^{-4} \text{ s}^{-1}$ and $N = 3 \times 10^{-6} \text{ s}^{-1}$.

flow, as a spring force, whereas the dissipating wave stress acts in-phase as a frictional drag. Thus, the generation of bottom-trapped internal tides—and the associated topographic spring force—is particularly effective at damping the barotropic tide, as compared with other topographic interactions.

In both the diurnal and semidiurnal tide examples, significant drag is only generated by scales greatly exceeding 1 km, thus satisfying the small-excursion approximation made in the derivation of the theory.

7. Discussion

Here we have investigated the impact of topographic interactions on the time-periodic spatial-mean flow that evolves in response to periodic body forcing. The interaction of the spatial-mean flow with topography gives rise to a variety of propagating (wave) and bottom-trapped (nonwave) motions, which are each associated with a different magnitude and phase of stress relative to the spatial-mean flow. For wave

motion, the stress is further modified depending on whether the waves dissipate above the generating topography without reflecting, or reflect back from the ocean surface. We have identified two “types” of topographic stress: the wave drag force and the topographic spring force. Previous theoretical investigations have only identified the former of the two stresses (e.g., Bell 1975; Jayne and St. Laurent 2001; Llewellyn Smith and Young 2002; Khaliwala 2003).

The wave drag force is always in phase with the spatial-mean flow. This stress is thus associated with a net energy extraction from that flow (work $W = \int \bar{\mathbf{u}} \cdot \boldsymbol{\tau} dt > 0$). This type of stress occurs when waves (propagating motions) are generated by the interaction of the spatial-mean flow with topography.

The spring force is out of phase with the spatial-mean flow. This stress is therefore not associated with any net energy extraction from that flow (work $W = \int \bar{\mathbf{u}} \cdot \boldsymbol{\tau} dt = 0$). Stresses of this kind have been observed in the interaction of tidal flows with headlands (e.g., Warner and MacCready 2009; Warner et al. 2013, call the spring force “inertial drag”). Even without net energy extraction, this

stress can have a very significant impact on the amplitude (kinetic energy) of the spatial-mean flow. This stress occurs whenever periodic standing motions are generated, including bottom-trapped internal tides (generated in regions where the astronomical forcing frequency is less than the local Coriolis parameter, $\omega < f$) and propagating low-mode internal tides (generated where $\omega > f$).

Stresses associated with the interaction of resolved flows with topography are implemented as parameterizations in many large-scale global and regional ocean models. Indeed, some sort of parameterized stress is vital in global tidal models in the order to achieve realistic amplitudes of the barotropic tides (e.g., [Arbic et al. 2004](#); [Ansong et al. 2015](#)). A major motivation of the present work was to investigate the physical validity of the parameterizations currently used in such models and whether any improvements are necessary. Currently, many tide models employ the relatively simple [Jayne and St. Laurent \(2001\)](#) parameterization to represent the stress associated with unresolved internal tides. While [Jayne and St. Laurent \(2001\)](#) derived this parameterization as a “scale relation,” here we have shown that it arises directly from the physical problem as the correct form of the stress in the limit of a sufficiently superinertial tidal frequency ($\omega \gg f$) generating locally dissipating internal tides at isotropic topography [see Eq. (39), or Fig. 4]. Given its limitation on frequency, the [Jayne and St. Laurent \(2001\)](#) parameterization is only strictly valid near the equator, but is nonetheless applied globally in numerical models (e.g., [Arbic et al. 2004](#); [Ansong et al. 2015](#)). Such an approach removes any frequency dependence (e.g., see Fig. 4) and thus makes the parameterization much easier and numerically efficient to implement in global tidal models (which solve the fluid equations in the time domain, not the frequency domain), but raises questions as to the physical validity worthy of further inquiry. Here we have also clarified the physically correct definition of the wavenumber scale (37) that appears in the [Jayne and St. Laurent \(2001\)](#) parameterization. Furthermore, even if the internal tide does not dissipate locally, and instead propagates away, the [Jayne and St. Laurent \(2001\)](#) expression gives the correct (for $\omega \gg f$) magnitude of the wave drag force.

However, while arguably satisfactory for wave drag forces (for $\omega \gg f$), the [Jayne and St. Laurent \(2001\)](#) parameterization does not account for the additional topographic spring forces associated with propagating internal tide modes and bottom-trapped internal tides. Under analogous assumptions to those used by [Jayne and St. Laurent \(2001\)](#)—that is, isotropic and frequency limited—these stresses have very simple forms [Eqs. (51) and (55)], and introduce no new physical parameters beyond those used by [Jayne and St. Laurent \(2001\)](#). At the very least, the current [Jayne and St. Laurent \(2001\)](#)

parameterization based on internal wave generation, which is applied globally in models, should be replaced by the bottom-trapped tide spring force in Eq. (55) poleward of the critical latitudes, where no waves are generated. In the example considered here (section 6; Fig. 7), we found that the spring force resulting from the bottom-trapped tide had a very significant ($\sim 40\%$) impact on amplitude on the barotropic tide.

More investigation is undoubtedly needed to clarify the impact of stresses at abyssal hill scales where mean and tidal motion become coupled ([Shakespeare 2020](#)), and to explore the spatial distribution of the wave stresses. The present model considers the impact of topographic interactions on the *spatial-mean flow* and is therefore unable to identify the spatial location at which these stresses are applied: at the topography? In a layer just above the topography? At height in the water column where the waves dissipate? Some combination of these options? These questions need to be addressed in future work. In addition, we have not considered stresses associated with nonlinear interactions at the ocean bottom, including bottom turbulence and hydraulic effects (e.g., [Garner 2005](#); [Winters and Armi 2014](#)), which may substantially impact the mean flow. Nevertheless, the present work has made an incremental step forward in quantifying the strength and nature of the stresses that act when a periodic flow encounters rough topography, and suggesting how these stresses may be more completely represented in numerical models. It remains to be determined how significant the results are in practice in improving the veracity of barotropic tide models.

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APPENDIX

Approximation for Smooth Topographic Spectra

This appendix explains the assumption (and limits thereof) of the approximation made in Eq. (45) to neglect the variation in the spectral amplitude $|\hat{h}(K)|^2$ over the range of a mode number n . The motivation for doing this step is so the expression for the drag can be written as a sum over mode numbers. The assumption may be justified as follows. Suppose the spectrum $|\hat{h}(K)|^2 \simeq K^{2p}$ near some mode number (i.e., a power-law scaling) as is typical of realistic spectra; then the integral in Eq. (45) may be expanded as a power series:

$$\begin{aligned} \int_{mH=(n-1/2)\pi}^{mH=(n+1/2)\pi} K |\hat{h}(K)|^2 dK &= \int_{mH=(n-1/2)\pi}^{mH=(n+1/2)\pi} K^{2p+1} dK \\ &= \left[K_n^{2p+1} K + \frac{1}{2} (2p+1) K_n^{2p} (K - K_n)^2 + \frac{1}{3} p(2p+1) K_n^{2p-1} (K - K_n)^3 + \dots \right]_{mH=(n-1/2)\pi}^{mH=(n+1/2)\pi} \\ &= K_n^{2p+1} \Delta K + \frac{2}{3} p(2p+1) K_n^{2p-1} \Delta K^3 + \dots, \end{aligned} \quad (\text{A1})$$

where $\Delta K = \pi(\omega^2 - f^2)^{1/2}/(NH)$ is the wavenumber spacing between mode numbers and $K_n = n\Delta K$ is the horizontal wavenumber of the n th mode. The second- and higher-order terms (associated with the slope of the spectrum) in Eq. (A1) may be neglected if

$$K_n^{2p+1}\Delta K \gg \frac{2}{3}p(2p+1)K_n^{2p-1}\Delta K^3, \quad (\text{A2})$$

or, equivalently,

$$n \gg \sqrt{\frac{2p(2p+1)}{3}}. \quad (\text{A3})$$

Thus, the assumption is valid as long as the slope of the spectrum (i.e., p) is not too great. For the first mode $n = 1$, the spectrum needs to be essentially flat ($p \ll 0.6$) for the approximation to be valid—but this is typically the case (e.g., Fig. 5). Beyond the corner wavenumber of the spectra, the spectral slope limits to $2p = 4$ (or $p = 2$; see Fig. 5), which implies $n \gg 2.6$, but the modes here are ~ 6 and higher, so again the assumption is satisfied.

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