Another Note on Rossby Wave Energy Flux

THEODORE S. DURLAND

College of Earth, Ocean and Atmospheric Sciences, Oregon State University, Corvallis, Oregon

J. THOMAS FARRAR

Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

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ABSTRACT

Longuet-Higgins in 1964 first pointed out that the Rossby wave energy flux as defined by the pressure work is not the same as that defined by the group velocity. The two definitions provide answers that differ by a nondivergent vector. Longuet-Higgins suggested that the problem arose from ambiguity in the definition of energy flux, which only impacts the energy equation through its divergence. Numerous authors have addressed this issue from various perspectives, and we offer one more approach that we feel is more succinct than previous ones, both mathematically and conceptually. We follow the work described by Cai and Huang in 2013 in concluding that there is no need to invoke the ambiguity offered by Longuet-Higgins. By working directly from the shallow-water equations (as opposed to the more involved quasigeostrophic treatment of Cai and Huang), we provide a concise derivation of the nondivergent pressure work and demonstrate that the two energy flux definitions are equivalent when only the divergent part of the pressure work is considered. The difference vector comes from the nondivergent part of the *geostrophic* pressure work, and the familiar westward component of the Rossby wave group velocity comes from the divergent part of the geostrophic pressure work. In a broadband wave field, the expression for energy flux in terms of a single group velocity is no longer meaningful, but the expression for energy flux in terms of the divergent pressure work is still valid.

1. Introduction

Longuet-Higgins (1964) showed that there is a discrepancy between the definitions of energy flux in planetary (Rossby) waves based on the group velocity viewpoint and the pressure work viewpoint. The product of perturbation pressure and the horizontal velocity vector (sometimes referred to as the "pressure work"[1]), is the natural energy flux definition in an energy equation for the linear, inviscid shallow-water equations. The derivation of group velocity, on the other hand, clearly establishes it as the transport rate for energy density within a narrow band of frequencies and wavenumbers. For superinertial oceanic waves, the two separately derived energy flux definitions provide the same result: the period average of the pressure work vector is equal to the product of the period-averaged energy density and the group velocity vector. In midlatitude Rossby waves at subinertial frequencies, however, the two vectors differ in both amplitude and direction, and the difference is a nondivergent vector. Longuet-Higgins suggested that the source of the discrepancy is the inherent ambiguity in the definition of energy flux, because any arbitrary nondivergent vector can be added to the definition of energy flux without changing its impact on the energy equation. He provided a mathematical derivation of the difference vector in the case of Rossby waves.

Over the years, numerous studies have speculated on the ambiguity in the Rossby wave energy flux definition and on the source of the nondivergent vector. Cai and Huang (2013, hereinafter CH13) summarized these studies nicely and went on to demonstrate that there really is no ambiguity when all of the divergent and nondivergent parts of the pressure work are identified. This note is inspired by CH13 and reaches a similar conclusion, but we feel that the resolution of the Longuet-Higgins (1964) paradox is much simpler both

[1] For example, Pedlosky (2003, p. 175) and CH13. From a technical perspective, the magnitude of this term is the *rate* of pressure work.

Corresponding author: Ted Durland, tdurland@coas.oregonstate.edu

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mathematically and conceptually than has been previously presented and thus is worth revisiting.

CH13 worked in the quasigeostrophic (qg) framework, and this required them to address the higher order terms in the perturbation expansion, including both the "unbalanced ageostrophic flow" that is normal to isobars, and a "balanced ageostrophic flow" that is parallel to isobars. This latter arises because the lowest order qg flow (the "geostrophic flow") is in exact geostrophic balance only at the reference latitude. The higher order "balanced ageostrophic flow" is key to resolving the discrepancy between the energy flux definitions. Evaluating this term was necessary for the qg energetics analysis done by CH13, but it resulted in more mathematical and conceptual complexity than is necessary to address only the Longuet-Higgins (1964) problem.

We work directly from the linear shallow water equations, as did Longuet-Higgins. Only the low-frequency approximation is made, the perturbation pressure is not expanded in a series, the velocity is only separated into geostrophic and ageostrophic parts, and the geostrophic velocity is evaluated with the local Coriolis acceleration, rather than with a single reference value that is used throughout the domain. With a single mathematical operation, the pressure work due to the geostrophic velocity is separated into a part that is always nondivergent and a part that is divergent in the presence of Rossby waves (i.e., when the geostrophic velocity is not along ambient potential vorticity contours). The nondivergent part cannot contribute to energy density evolution, and must be subtracted from the total pressure work for a comparison with the group velocity vector. The divergent part of the geostrophic pressure work produces the familiar "westward component"[2] of the Rossby wave group velocity.

The ideas above have been expressed in different words by previous researchers (e.g., Pedlosky 1987; CH13, and references therein), but we feel that the value of the present contribution lies in its simplicity of concept and mathematics. In just a few lines, the divergent part of the energy flux is derived directly from the linearized shallow water equations, using only the low-frequency approximation. An additional β-plane approximation is made to demonstrate the equivalence of the divergent pressure work with a familiar form of the qg energy flux, and hence the consistency with the group velocity concept. The original expression for the divergent energy flux, however, remains valid even when the β-plane approximation is not and/or when a unique group velocity cannot be defined.

[2] That is, normal to and counterclockwise from the ambient potential vorticity gradient.

2. Equations and analysis

The inviscid, linear, shallow-water equations for a barotropic fluid of variable depth on a rotating sphere are

$$\partial_t \mathbf{u} + f\hat{\mathbf{e}}_z \times \mathbf{u} + g\nabla\eta = 0 \quad \text{and} \tag{1}$$

$$\partial_t \eta + \nabla \cdot (h\mathbf{u}) = 0, \tag{2}$$

where $\mathbf{u}$ is the horizontal current, η is the surface displacement from equilibrium, $h(\mathbf{x})$ is the variable water depth, t is time, g is the gravitational acceleration, $f = 2\Omega\sin\theta$ is the Coriolis parameter, with θ being latitude and Ω being the sphere's rotation rate. The local vertical unit vector is $\hat{\mathbf{e}}_z$, gradients and divergences are strictly horizontal, and the operator $\hat{\mathbf{e}}_z\times$ rotates a vector 90° counterclockwise (viewed from above) in the horizontal plane. A time derivative is designated by ∂_t.

The energy equation derived by adding $h\mathbf{u}\cdot$ (1) to $g\eta \times$ (2) is

$$\partial_t E = -\nabla \cdot \mathbf{S}, \tag{3}$$

where

$$E \equiv (h|\mathbf{u}|^2 + g\eta^2)/2 \tag{4}$$

is the energy density and

$$\mathbf{S} \equiv gh\eta\mathbf{u} \tag{5}$$

is the vertically integrated energy flux. To avoid confusion with the energy flux defined by the group velocity, we will refer to $\mathbf{S}$ as the "pressure work." Longuet-Higgins (1964) worked with the nondivergent shallow-water equations on a flat bottom, and his equations can be retrieved from ours by letting $h = 1$, $g\eta = p/\rho$ and $\nabla\cdot(h\mathbf{u}) = 0$. The last condition eliminates the potential energy $(g\eta^2/2)$ from the definition of E in (4). Appropriate substitutions also extend our equations to a flat-bottom baroclinic mode.

Applying the operator $\partial_t - f\hat{\mathbf{e}}_z\times$ to the momentum equation in (1) yields

$$\mathbf{u} = \left(\frac{g}{f}\hat{\mathbf{e}}_z \times \nabla\eta - \frac{g}{f^2}\nabla\partial_t\eta\right)[1 + O(\omega_0^2/f^2)], \tag{6}$$

where ω_0 represents the high end of the range of frequencies that make a significant contribution to the motion of interest. We focus on motions for which $\omega_0^2/f^2 \ll 1$ and ignore the $O(\omega_0^2/f^2)$ correction in what follows. The first term on the right-hand side (rhs) of (6) is the geostrophic velocity, and the second term is the ageostrophic velocity. The pressure work is

$$\mathbf{S} = g^2\left[\hat{\mathbf{e}}_z \times \left(\frac{h}{f}\nabla\frac{\eta^2}{2}\right) - \frac{h}{f^2}\eta\nabla\partial_t\eta\right], \tag{7}$$

where the first term on the rhs of (7) is due solely to the geostrophic velocity. We identify this term as the geostrophic pressure work $\mathbf{S}_g$.

The chain rule $[a\nabla b = \nabla(ab) - b\nabla a]$ allows us to separate the geostrophic pressure work into a part that is absolutely nondivergent and a part that is divergent in the presence of Rossby waves:

$$\mathbf{S}_g = g^2\hat{\mathbf{e}}_z \times \left(\frac{h}{f}\nabla\frac{\eta^2}{2}\right) = g^2\left[\hat{\mathbf{e}}_z \times \nabla\left(\frac{h}{f}\frac{\eta^2}{2}\right) - \hat{\mathbf{e}}_z \times \left(\frac{\eta^2}{2}\nabla\frac{h}{f}\right)\right]. \tag{8}$$

The first term in the square brackets in (8) is nondivergent under all conditions, because the divergence of a 90°-rotated gradient is the curl of the gradient (which vanishes). The divergence of the second term is

$$\nabla\cdot\left[-g^2\hat{\mathbf{e}}_z \times \left(\frac{\eta^2}{2}\nabla\frac{h}{f}\right)\right] = g^2\hat{\mathbf{e}}_z\cdot\left[\nabla\times\left(\frac{\eta^2}{2}\nabla\frac{h}{f}\right)\right] = g^2\hat{\mathbf{e}}_z\cdot\left(\nabla\frac{\eta^2}{2}\times\nabla\frac{h}{f}\right) \tag{9}$$

This divergence is nonzero whenever $\nabla\eta$ does not vanish and is not parallel to the ambient potential vorticity gradient $\nabla(f/h)$—necessary conditions for the existence of Rossby waves. The nondivergent part of the geostrophic pressure work cannot contribute to the energy evolution, and we define a potentially divergent pressure work $\mathbf{S}_d$ by subtracting the nondivergent part of (8) from the total pressure work (7):

$$\mathbf{S}_d = \mathbf{S} - g^2\hat{\mathbf{e}}_z \times \nabla\left(\frac{h}{f}\frac{\eta^2}{2}\right) \tag{10}$$

$$= g^2\left[-\hat{\mathbf{e}}_z \times \left(\frac{\eta^2}{2}\nabla\frac{h}{f}\right) - \frac{h}{f^2}\eta\nabla\partial_t\eta\right] \tag{11}$$

$$= g^2\left[\frac{\eta^2}{2}\hat{\mathbf{e}}_z \times \left(\frac{h}{f}\right)^2\nabla\frac{f}{h} - \frac{h}{f^2}\eta\nabla\partial_t\eta\right]. \tag{12}$$

Equation (12) can be rewritten

$$\mathbf{S}_d = gL_d^2\left(\eta^2\hat{\mathbf{e}}_z \times \frac{\boldsymbol{\beta}}{2} - \eta\nabla\partial_t\eta\right), \tag{13}$$

where

$$L_d^2 \equiv gh/f^2 \tag{14}$$

is the squared local deformation radius and

$$\boldsymbol{\beta} \equiv h\nabla(f/h) \tag{15}$$

is the β vector (∇f when the bottom is flat). The β-plane approximation was not necessary for (13), but, to compare with previous work relevant to our present purpose, we now approximate the domain as a plane[3] with right-hand unit vectors ($\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, $\hat{\mathbf{e}}_z$), in which $\hat{\mathbf{e}}_y$ is aligned with $\boldsymbol{\beta}$. The divergent pressure work is then

$$\mathbf{S}_d = gL_d^2\left(-\eta^2\frac{|\boldsymbol{\beta}|}{2}\hat{\mathbf{e}}_x - \eta\nabla\partial_t\eta\right). \tag{16}$$

This is equivalent to a vertical integral of Eq. (16.5b) in Pedlosky (2003), which defines the energy flux in an energy equation derived from the linear qg potential vorticity equation. Pedlosky (2003, 174–175) demonstrates that the period average of this flux for a plane Rossby wave is equal to the product of the group velocity and the period-averaged energy density.

In our notation, and for a wave train of frequency ω, slowly varying amplitude, and spatial phase dependence $\exp(i\mathbf{K}\cdot\mathbf{x})$, the period average of (16) is

$$\langle\mathbf{S}_d\rangle = g\omega L_d^2\frac{|\eta|^2}{2}\left(-\frac{|\boldsymbol{\beta}|}{2\omega}\hat{\mathbf{e}}_x - \mathbf{K}\right), \tag{17}$$

where $|\eta|$ is the amplitude of the surface-level oscillations. The period-averaged divergent pressure work $\langle\mathbf{S}_d\rangle$ is equal to the product of the period-averaged, vertically integrated energy density and the group velocity calculated from the dispersion relation

$$\omega = \frac{\hat{\mathbf{e}}_z\cdot\boldsymbol{\beta}\times\mathbf{K}}{|\mathbf{K}|^2 + L_d^{-2}}. \tag{18}$$

Note that the "westward" component of the divergent pressure work—the first term on the rhs of (12), (13), (16), or (17)—is the divergent part of the geostrophic pressure work.

3. Comments

In agreement with CH13, there is no discrepancy between the group velocity view of energy flux and the pressure work view, as long as we recognize that the geostrophic pressure work contains an easily determined nondivergent component that cannot be included in a comparison of the two viewpoints.

As noted by Longuet-Higgins (1964), any arbitrary nondivergent vector can be added to the energy flux definition without changing the energy evolution. This

[3] This implies a constant $|\nabla f|$.

situation, however, is not unique to Rossby waves, and for the question at hand we adopt the view that there is no need to appeal to this arbitrariness in the energy flux definition. The pressure work expression (7) is the mathematically defined energy flux, and the fact that a well defined part of it is nondivergent under all conditions simply means that this part cannot contribute to energy evolution, and therefore cannot be included in a comparison of the pressure work and group velocity vectors. It also cannot be included in a more general assessment of how an energy field evolves in response to Rossby wave radiation. There is, however, no arbitrariness involved.

Equation (13) expresses the potentially divergent energy flux $\mathbf{S}_d$ as the sum of the ageostrophic pressure work and the divergent part of the geostrophic pressure work. This is the most convenient form for comparing with the group velocity. For the purpose of calculating $\mathbf{S}_d$ from model output or data, it may be more convenient to subtract the nondivergent part of the geostrophic pressure work from the total pressure work, because this can be done from a snapshot of pressure and velocity. This option is expressed in (10), and is rewritten more explicitly here:

$$\mathbf{S}_d = gh\eta\mathbf{u} - g^2\hat{\mathbf{e}}_z \times \nabla\left(\frac{h}{f}\frac{\eta^2}{2}\right). \tag{19}$$

[Note that $\mathbf{u}$ in (19) must include both the geostrophic and ageostrophic velocities.]

A unique group velocity applies only to a process that is narrow banded in both frequency and wavenumber. Many important processes in the ocean are not narrow banded, however, and we may wish to understand energy evolution at times and in places for which dispersion has not yet been able to separate frequency and wavenumber components. In such cases, (13), (16), and (19) are still valid expressions for the potentially divergent energy flux.

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