An Analytical Spectral Model for Infragravity Waves over Topography in Intermediate and Shallow Water under Nonbreaking Conditions

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ABSTRACT: The theoretical model for group-forced infragravity (IG) waves in shallow water is not well established for nonbreaking conditions. In the present study, analytical solutions of the group-forced IG waves at $O(\beta_1)$ ($\beta_1 = h_x/(\Delta kh)$, h_x = bottom slope, Δk = group wavenumber, h = depth) in intermediate water and at $O(\beta_1^{-1})$ in shallow water are derived separately. In case of off-resonance [$\beta_1\mu^{-1} = O(\beta_1)$], where $\mu = 1 - c_g^2/(gh)$ is the resonant departure parameter, c_g = group speed] in intermediate water, additional IG waves in quadrature with the wave group forcing (hereinafter, the nonequilibrium response or component) are induced at $O(\beta_1)$ relative to the equilibrium bound IG wave solution of Longuet-Higgins and Stewart (1962) in phase with the wave group. The present theory indicates that the nonequilibrium response is mainly attributed to the spatial variation of the equilibrium bound IG wave amplitude instead of group-forcing. In case of near-resonance [$\beta_1\mu^{-1} = O(1)$] in shallow water; however, both the equilibrium and nonequilibrium components are $\sim O(\beta_1^{-1})$ at the leading order. Based on the nearly-resonant solution, the shallow water limit of the local shoaling rate of bound IG waves over a plane sloping beach is derived to be $\sim h^{-1}$ for the first time. The theoretical predictions compare favorably with the laboratory experiment by Van Noorloos (2003) and the present numerical model results generated using SWASH. Based on the proposed solution, the group-forced IG waves over a symmetric shoal are investigated. In case of off-resonance, the solution predicts a roughly symmetric reversible spatial evolution of the IG wave amplitude, while in cases of near to full resonance the IG wave is significantly amplified over the shoal with asymmetric irreversible spatial evolution.

KEYWORDS: Gravity waves; Waves, oceanic

1. Introduction

Infragravity (IG) waves are ocean surface waves of low frequencies, typically between 0.004 and 0.04 Hz, that distinct from the wind waves or swells of frequencies between 0.04 and 1 Hz (Bertin et al. 2018). IG wave plays an important role in many coastal processes, such as resonance in harbors (Miles 1974; Bowers 1977; Okiihiro et al. 1993; Maa et al. 2010; Thotagamuwege and Pattiaratchi 2014a; Gao et al. 2019; Gao et al. 2020), morphological evolution (Roelvink et al. 2009; Mendes et al. 2020; Ruffini et al. 2020), and hydrodynamics in surf zones (Guedes et al. 2013; Padilla and Alsina 2018; Contardo et al. 2019) and on coral-reef flats (Nwogu and Demirbilek 2010; Pomeroy et al. 2012; Liu and Li 2018). The generation, evolution, and dissipation of IG waves in intermediate and shallow water have therefore been the focus of numerous studies in the past decades (Longuet-Higgins and Stewart 1962, hereinafter LHS62; Symonds et al. 1982; Herbers et al. 1994; Sheremet et al. 2002; Janssen et al. 2003; Battjes et al. 2004; Henderson et al. 2006; Thomson et al. 2006; Baldock 2012; De Bakker et al. 2015; Li et al. 2020). One of the main generation mechanisms of IG waves is the nonlinear wave group forcing, and the relevant analytical frameworks are

well established in intermediate water where the resonance between forcing and IG waves is generally weak (Bowers 1992; Janssen et al. 2003; Zou 2011, hereinafter referred to as ZOU11). However, in shallow water where strong resonance occurs, the nonlinear wave group forcing becomes more significant, but currently there is a lack of analytical models. This limits the physical understanding of many important behaviors of IG waves in shallow water, such as IG waves shoaling on nearshore beaches and the recently reported offshore shoal-induced amplification of IG waves (Paniagua-Arroyave et al. 2019; Li et al. 2020).

Under nonbreaking conditions, IG waves can be forced by wave groups through difference interactions. Their evolution is governed by the shallow water equation with a forcing term of radiation stress (Phillips 1977; Mei and Benmoussa 1984). For bichromatic wave groups over a flat bottom, LHS62 derived the following solution for the surface elevation of group-forced IG wave [$\eta^{(2)}(x,t)$] known as the equilibrium bound IG wave,

$$\eta^{(2)}(x,t) = -\frac{\tilde{S}(x,t)}{\rho g \mu h}, \quad (1)$$

where $\tilde{S}(x,t)$ denotes the radiation stress that slowly varies at the wave group scale; ρ and g are the fluid density and gravitational acceleration, respectively; and h is the water depth; $\mu = 1 - c_g^2/(gh)$ the departure from resonance with c_g the group speed (Janssen et al. 2003) (see appendix C for a list of

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variables). The LHS62 solution describes an antiphase equilibrium response of IG waves to the group forcing. However, the equilibrium response is broken in the shoaling zone, where the IG waves start to lag behind the equilibrium response of LHS62 (List 1992; Van Leeuwen 1992; Janssen et al. 2003) and gain energy from the primary waves. This process is normally manifested by the bulk shoaling rate of group-forced IG waves being greater than Green's law $\sim h^{-0.25}$ and negatively correlated with the normalized bottom slope $\beta_1 = h_x/(\Delta kh)$ (Battjes et al. 2004; van Dongeren et al. 2007; De Bakker et al. 2016; Zhang et al. 2020), where the subscript x denotes the horizontal gradient in the cross-shore direction and Δk denotes the difference wavenumber of a bichromatic wave group.

For nonbreaking waves over a slowly varying topography ($|\beta_1| \ll 1$), the existing theoretical works on group-forced IG waves (edge wave excluded) may be categorized into two types. The first type focuses on waves over sloping beaches, where the resonance intensity between group-forcing and IG waves increases from $\mu = O(1)$ in intermediate depth to $\mu \ll 1$ in shallow water. Accordingly, the relative importance between effects of depth variation and local resonance may evolve from off-resonance [$\beta_1 \mu^{-1} = O(\beta_1)$] to near-resonance [$\beta_1 \mu^{-1} = O(1)$] or even greater. In the off-resonant condition, the effect of depth variation has been incorporated analytically as perturbations to the LHS62 solution (Bowers 1992; Van Leeuwen 1992; Janssen et al. 2003). It was found that the topography-induced perturbation at the leading order $O(\beta_1)$ is in quadrature with the LHS62 solution, and therefore causes the phase shift of bound IG waves. However, in the full-resonant condition in shallow water, the resonance is greatly enhanced ($\mu \rightarrow 0$) so that the LHS62 solution may diverge and the perturbation method fails. In this situation, by numerically solving the governing equation accurate to $O(\beta_1)$, Janssen et al. (2003) concluded that it is not appropriate to employ the asymptotic shallow water limit of the LHS62 solution ($\sim h^{-2.5}$) as the shoaling trend.

The second type focuses on the group-forced IG waves over a finite topography in intermediate depth where concomitant free IG waves are scattered from the topography. With $\mu = O(1)$ at this region, the second type problem is often the off-resonant case. Starting with a one-dimensional topography in deep water, Molin (1982) reported the free IG waves induced by a bichromatic wave group for the first time. Later on, Mei and Benmoussa (1984) investigated obliquely incident bichromatic waves on two-dimensional topographies with shore parallel contours in intermediate depths. They found that the free IG wave is scattered in a direction different from the incident wave direction. The governing equation of group-forced IG waves additional to the LHS62 solution over variable depth was derived and numerically solved. Liu (1989) corrected the boundary conditions of Mei and Benmoussa (1984) and obtained different numerical results. ZOU11 investigated the IG waves generated by a fully modulated bichromatic wave group propagating over a 1D topography using a multiscale Wentzel–Kramers–Brillouin (WKB) expansion method. Assuming $\beta_2 = h_{xx}/(\Delta k^2 h) = O(\beta_1^2)$, a closed-form analytical solution accurate to the second order $O(\beta_1^2)$ was proposed to consider the contributions of both bottom

slope h_x and bottom curvature h_{xx} . The topography-induced nonequilibrium bound IG wave at the first order of $O(\beta_1)$ was also reported by ZOU11 to be in quadrature with the LHS62 solution and contribute to the phase lag and the associated nonlinear energy transfer. It was also observed that the spatial gradients of the equilibrium response described by the LHS62 solution and group-forcing jointly induce the non-equilibrium IG wave response at the leading order of $O(\beta_1)$. However, the relative importance of these two contributors remains unclear.

As illustrated in Fig. 1, as wave groups propagate toward the shoreline from deep to shallow water, they would pass through several zones with different resonant intensities. Apart from the shallow water over a sloping beach close to the shoreline, strong resonance [$\beta_1 \mu^{-1} = O(1)$] may also occur at an offshore topography with relatively shallow water depth on the top and could generate remarkable IG waves. For example, we propose the strong resonance at the crest of a large-scale submarine shoal is responsible for the considerably amplified IG waves reported by several observational and numerical studies listed in Table 1. However, the existing off-resonant IG wave solutions cannot explain this phenomenon since it is a single-valued function of wave radiation stress, depth, and bottom slope. Therefore, it predicts the same order of magnitude of group-forced IG waves on both sides of the topography.

The objective of the present work is to derive theoretical solutions of group-forced IG waves under off- and nearly-resonant conditions, from intermediate to shallow water depth. The off-resonant solution provides new insight for the relative importance of different generation mechanisms of the non-equilibrium bound IG waves in intermediate depths, while the nearly-resonant solution predicts the shallow water shoaling rate of bound IG waves on a plane sloping beach to be $\sim h^{-1}$ for the first time. More importantly, it indicates that in shallow water where strong resonance occurs, the group-forced IG wave is not locally determined but relies on its spatial evolution history. We will also be able to theoretically demonstrate for the first time that on a symmetric underwater shoal topography, the presence of strong resonance near the shoal crest at low tide is responsible for the irreversible asymmetric spatial evolution of group-forced IG waves. The governing equation of the IG wave response to group-forcing is derived in section 2. Analytical solutions for off- and nearly-resonant conditions are obtained and discussed in section 3. Boundary condition of topography-scattered free IG wave is derived in section 4. In section 5, examples of bichromatic and irregular waves on a plane beach are presented. In section 6, group-forced IG waves over a symmetric topography are investigated. Conclusions and main findings are summarized in section 7.

2. Governing equations

Consider bichromatic wave groups over a 2D topography of variable depth at both intermediate and shallow water with a surface displacement (see Fig. 1 for the coordinates and variables),

$$\eta^{(1)}(x, t) = \frac{1}{2} A(x, t) e^{i\theta(x, t)} + *, \quad (2)$$

FIG. 1. Definition sketch of the coordinate system and variables for bichromatic waves and group-forced IG wave propagating over an offshore shoal and a nearshore beach. Different colors schematically indicate zones of different resonant intensities: $\beta_1 \mu^{-1} = O(\beta_1)$ (off-resonant, gray), $\beta_1 \mu^{-1} = O(1)$ (nearly-resonant, blue), and $\beta_1 \mu^{-1} = O(\beta_1^{-1})$ (fully-resonant, red).

$$A(x, t) = a_1(x) e^{i\Delta\theta(x, t)/2} + a_2(x) e^{-i\Delta\theta(x, t)/2}, \quad (3)$$

where $\theta(x, t) = \int^x k(x') dx' - \omega t$ and $\Delta\theta = \int^x \Delta k(x') dx' - \Delta\omega t$ are the mean and difference phase of the bichromatic wave group, respectively; $A(x, t)$ denotes the modulated complex amplitude that slowly varies with space and time at the wave group scale; and a_1 and a_2 are the real amplitudes of the two monochromatic wave components with similar frequencies and wavenumbers. The asterisk denotes the conjugate of the preceding term and will be omitted hereinafter.

The evolution of group-forced IG waves is governed by the linearized shallow water equation with the wave radiation stress as the forcing term (Mei and Benmoussa 1984; Schäffer 1993; Janssen et al. 2003; ZOU11),

$$\eta_{tt}^{(2)} - g(h\eta_x^{(2)})_x = \frac{1}{\rho} S_{xx}, \quad (4)$$

where the superscript “(2)” will be omitted hereafter, and the subscripts t and x denote temporal and spatial partial derivative, respectively.

The radiation stress is given by (LHS62)

$$S(x, t) = \frac{1}{2} \rho g |A(x, t)|^2 \left(\frac{2c_g}{c} - \frac{1}{2} \right), \quad (5)$$

where $c = \omega/k$ is the phase speed of the carrier wave. Substituting Eq. (3) into Eq. (5) and only retaining the slowly varying radiation stress component yields

$$\tilde{S}(x, t) = \frac{1}{2} \rho g \hat{S}(x) e^{-i\Delta\omega t}, \quad (6)$$

where $\hat{S}(x) = a_1 a_2 (2c_g/c - 1/2) e^{i\varphi_g(x)}$ describes the spatial variation of the complex amplitude with $\varphi_g(x) = \int^x \Delta k(x') dx'$ being the spatial wave phase of the wave group.

Since the governing equation [Eq. (4)] is linear, the solution of $\eta(x, t)$ takes the same form as the forcing term $\tilde{S}(x, t)$, i.e., $\eta(x, t) = (1/2) \hat{\eta}(x) e^{-i\Delta\omega t}$. The complex amplitude $\hat{\eta}$ may be decomposed into forced ($\hat{\eta}_B$) and free ($\hat{\eta}_f$) wave components as

$$\hat{\eta}(x) = \hat{\eta}_B + \hat{\eta}_f, \quad (7)$$

corresponding to the particular and homogeneous solutions to the governing equation [Eq. (4)], respectively. Different from previous studies, a complex response function F is introduced here to describe the group-forced IG wave induced by the radiation stress

$$\hat{\eta}_B(x) = -F \frac{\hat{S}}{h}, \quad (8)$$

where the negative sign is introduced since the leading order IG wave response (LHS62) is in antiphase with the wave group. The real and imaginary parts of F represent the equilibrium and nonequilibrium responses that are in phase/antiphase and in quadrature with the wave group forcing, respectively. According to Eqs. (7) and (8), the phase lag of IG wave relative to the equilibrium response of LHS62 is

$$\Delta\varphi(x) = \text{Arg} \left[F - \frac{\hat{\eta}_f}{\hat{S}/h} \right], \quad (9)$$

where $\text{Arg}[\cdot]$ denotes the principal value of the argument of the complex variable. Substituting Eqs. (6) and (8) into Eq. (4), and introducing the normalized coordinate $dX = \Delta k dx$ [X is by

TABLE 1. IG wave amplification induced by an offshore shoal and the resonance parameters $\mu = 1 - c_g^2/(gh)$ and normalized bottom slope $\beta_1 = h_x/(\Delta kh)$ at the crest of the shoal.

Previous study	Location of the shoal	Methodology	$ \beta_1 $	μ	$ \beta_1/\mu $
Thotagamuwage and Pattiaratchi (2014b)	Offshore of Two Rocks Marina, West Australia	Boussinesq wave model	0.18	0.14	1.25
Paniagua-Arroyave et al. (2019)	Offshore of Cape Canaveral, Florida	Field observation	0.02–0.05	0.14–0.16	0.15–0.31
Li et al. (2020)	Offshore	Nonhydrostatic wave model	0.16–0.41	0.24	0.68–1.7

definition equivalent to $\varphi_g(x)$] yields the governing equation of $F(X)$

$$F_{XX} + [(2\Lambda_1 + \delta - 1)\beta_1 + 2i]F_X + [(\Lambda_1 - 1)\beta_2 + (\Lambda_2 - \Lambda_1 - 3)\beta_1^2 + (2\Lambda_1 + \delta - 1)\beta_1 i - \mu]F - [\Lambda_1\beta_2 + \Lambda_2\beta_1^2 + (2\Lambda_1 + \delta)\beta_1 i - 1] = 0, \quad (10)$$

where $\beta_2 = h_{xx}/(\Delta k^2 h)$ is the normalized bottom curvature; $\Lambda_1(kh) = |\hat{S}|_h h / |\hat{S}|$ and $\Lambda_2(kh) = |\hat{S}|_{hh} h^2 / |\hat{S}|$ are the first and second normalized derivatives of the magnitude of group-forcing relative to water depth h , respectively; $\delta(kh) = \Delta k_h / (\Delta k / h) = -(h/c_g) dc_g / dh$ is the topography-induced normalized variation in the propagating speed of the group forcing.

3. Analytical solutions

a. Bichromatic waves

Assuming that the topography varies slowly at the scale of group lengths, i.e., $|\beta_1| \ll 1$, and $\beta_2 = O(\beta_1^2)$, and that the variation of group-forced IG waves response function F with respect to X is of order $F_X/F = O(\beta_1)$ and $F_{XX}/F = O(\beta_1^2)$ (see appendix A), the governing Eq. (10) is reduced to the first-order differential equation in the form depending on the resonance intensity $(\beta_1 \mu^{-1})$ in the following subsections.

1) OFF-RESONANT SOLUTION

In the intermediate depth where off-resonance occurs, i.e., $\mu = O(1)$ and $\beta_1 \mu^{-1} = O(\beta_1)$, at the lowest order of β_1 the response function and its derivatives are of order $F = O(\mu^{-1}) = O(1)$, $F_X = O(\beta_1)$ and $F_{XX} = O(\beta_1^2)$, the governing equation, Eq. (10), is reduced to $O(\beta_1)$ as

$$F_X + \left[\left(\Lambda_1 - \frac{1}{2} + \frac{\delta}{2} \right) \beta_1 + \frac{i\mu}{2} \right] F - \left[\left(\Lambda_1 + \frac{\delta}{2} \right) \beta_1 + \frac{i}{2} \right] = O(\beta_1^2). \quad (11)$$

Substitute the asymptotic expansion of the response function $F = \sum_{n=0}^{\infty} F^{(n)}$ [$F^{(n)} = O(\beta_1^n)$] into Eq. (11), and the solution accurate to $O(\beta_1)$ is derived as

$$F(X) = \frac{1}{\mu} \left\{ [1 - i\beta_1(2\Lambda_1 + \delta)] + 2i\beta_1 |F_H| h \frac{d}{dh} \left(\frac{\mu^{-1}}{|F_H|} \right) + O(\beta_1^2) \right\}, \quad (12)$$

where $F_H(X)$ is the homogeneous solution to Eq. (11)

$$F_H(X) = \sqrt{\frac{h}{\Delta k}} |\hat{S}|^{-1} e^{i\theta_H}, \quad (13)$$

where $\theta_H(X) = -(1/2) \int \mu dX'$ is the phase of $F_H(X)$. Note that in the off-resonant case $F_H(X)$ is not the approximate homogeneous solution of the original governing Eq. (10), because for freely propagating IG waves, the phase coupling between the group forcing and free waves propagating in the same direction varies with depth as $h \left(\frac{d}{dh} \right) [\text{Arg}(\hat{\eta}_f) - \varphi_g] = \beta_1^{-1} [\sqrt{1 - \mu} - 1 + O(h_x)]$, which is of order $O(\beta_1^{-1})$ in intermediate depth. Hence, the assumption $F_X = O(\beta_1)$ is not valid, and Eq. (10) cannot be reduced to Eq. (11).

FIG. 2. Normalized amplitudes of IG waves induced by the spatial gradient of group-forcing (F_I , dashed line), and the gradient of IG wave amplitude ($F_{II} + F_{III}$, black solid line). The latter is the sum of the equilibrium bound IG wave described by LHS62 (F_{II} , red dash-dotted line) and the IG wave described by F_H (F_{III} , blue dotted line).

Equation (12) is the off-resonant linear solution (OLS) of the response function. It is verified to be equivalent to the local response correction solution of Janssen et al. (2003) as well as the higher-order bound IG wave solution of ZOU11 by omitting the terms at $O(h_x^2)$ and $O(h_{xx})$ therein. To interpret the underlying physical mechanisms for the group-forced IG wave, Eq. (12) is rewritten as

$$F = F_{LHS62} + F_{Slope} + O(\beta_1^2), \quad (14)$$

where $F_{LHS62} = \mu^{-1}$ is the LHS62 solution for a flat bottom; F_{Slope} is the additional component induced by the variable depth which can be decomposed further into three components

$$F_{Slope} = F_I + F_{II} + F_{III}, \quad (15)$$

where F_I , F_{II} , and F_{III} are the additional IG response functions induced by spatial gradients of the group-forcing, bound IG wave amplitude described by LHS62 solution, and amplitude of the IG wave described by F_H , i.e.,

$$\begin{Bmatrix} F_I \\ F_{II} \\ F_{III} \end{Bmatrix} = F_{LHS62} \left(\frac{2\beta_1}{i\mu} \right) \begin{Bmatrix} \mu(\Lambda_1 + 0.5\delta) \\ -\frac{h}{|\hat{\eta}_{LHS62}|} \frac{d|\hat{\eta}_{LHS62}|}{dh} \\ \frac{h}{|\hat{\eta}_H|} \frac{d|\hat{\eta}_H|}{dh} \end{Bmatrix}, \quad (16)$$

where $\hat{\eta}_H = -F_H \hat{S} / h$ is the IG wave described by F_H . All three topography-induced additional bound IG wave responses in Eq. (16) are proportional to the bottom slope and in quadrature with the flat bottom equilibrium solution F_{LHS62} , thus constitute the nonequilibrium responses at the leading order of $O(\beta_1)$. Moreover, $F_{Slope} = F_I + F_{II} + F_{III}$ can be grouped into two parts. The first part F_I is induced by the spatial variation of both the amplitude (Λ_1) and propagating speed (0.5δ) of the group-forcing, and the second part ($F_{II} + F_{III}$) is induced by the

spatial variation of IG wave amplitudes. Figure 2 shows that the first part F_I is much smaller than the second part $F_{II} + F_{III}$ at all water depths, indicating that the bottom slope-induced spatial variation of bound IG wave amplitudes is the main mechanism for the nonequilibrium response. Furthermore, Fig. 2 shows that F_{II} is much larger than F_{III} , therefore, the predominant contributor to the second part. In summary, for the off-resonant condition, the spatial variation of the amplitude of the equilibrium bound IG wave (LHS62) is the controlling factor for the nonequilibrium response.

For freely propagating IG waves, to the leading order of $O(\beta_1)$, the complex amplitude reduces to (ZOU11)

$$\begin{aligned} \hat{\eta}_f(x) &= \hat{\eta}_f(x_0) \left(\frac{h_0}{h} \right)^{0.25} \\ &\times \left\{ 1 + \frac{h^{-0.5}}{2ik_f} \int_{x_0}^x h^{-0.25} \frac{d}{dx} \left[h \frac{d}{dx} (h^{-0.25}) \right] dx \right\} \\ &\times e^{i \int_{x_0}^x k_f dx'} + O(\beta_1^2), \end{aligned} \quad (17)$$

where the subscript 0 denotes quantities at the offshore edge of the topography. For shoreward and seaward propagating free IG waves, the wavenumber k_f takes the expression $\Delta\omega/\sqrt{gh}$ and $-\Delta\omega/\sqrt{gh}$, respectively.

2) NEARLY-RESONANT SOLUTION

In shallow water where near-resonance occurs, i.e., $\beta_1\mu^{-1} = O(1)$, at the lowest order the response function and its derivatives are of order $F = O(\mu^{-1}) = O(\beta_1^{-1})$, $F_X = O(1)$, and $F_{XX} = O(\beta_1)$. The governing Eq. (10) is truncated to $O(1)$ as

$$F_X + \left[\left(\Lambda_1 - \frac{1}{2} + \frac{\delta}{2} \right) \beta_1 + \frac{i\mu}{2} \right] F - \frac{i}{2} = O(\beta_1). \quad (18)$$

Substitute the asymptotic expansion of the response function $F = \sum_{n=-1}^{\infty} F^{(n)}$ into Eq. (18), and the solution accurate to the leading order $O(\beta_1^{-1})$ is

$$F(X) = \frac{i}{2} F_H(X) \int F_H^{-1}(X') dX' + O(1). \quad (19)$$

Equation (19) is the nearly-resonant solution (NRS) of group-forced IG wave response, and it is neither purely real nor imaginary, meaning that both the equilibrium and nonequilibrium responses contribute to the total response at the lowest order. The integral form of the NRS [Eq. (19)] highlights that the group-forced IG wave response is the accumulation result of its spatial evolution history in nearly-resonant conditions. This provides the theoretical explanation to the numerical results of Li et al. (2020). They reported that nonlinear energy transfer from primary waves to IG waves persistently occurs as waves propagate over the shallow flat plateau of an idealized shoal topography, a phenomenon that cannot be accounted for by the LHS62 solution (cf. section 4.2 therein).

Note that the phase of $F_H(X)$ is given by $\theta_H(X) = -(1/2) \int \mu \beta_1^{-1} h^{-1} dh' = O(\mu \beta_1^{-1})$. On a uniform sloping beach, as $h \rightarrow 0$, it follows $\theta_H(X) = O(\mu \beta_1^{-1}) = O(\Delta k^2 k h^3) \rightarrow 0$.

In this case, with Taylor expansion of $e^{i\theta_H} = 1 + i\theta_H + O(\theta_H^2)$, the response in Eq. (19) at the leading order is nonequilibrium (purely imaginary) while the equilibrium response becomes secondary, which is exactly opposite to the conclusion in the off-resonant condition in section 3a(1).

Substituting the expression of $F_H(X)$ into Eq. (19) and applying the shallow water approximations of $\Lambda_1 = \delta = -0.5$, $\mu \doteq (\omega^2/g)h$, and $c_g \doteq \sqrt{gh}$, as $h \rightarrow 0$, the asymptotic expression of Eq. (19) on a uniform sloping beach in shallow water becomes

$$F(h) = -\frac{2}{3} i \frac{\Delta\omega}{h_x} \sqrt{\frac{h}{g}} + O(1), \quad (20)$$

which is inversely proportional to the relative bottom slope $\beta_1 = (h_x/\Delta\omega)\sqrt{g/h}$. Equation (20) indicates that the group-forced IG wave response $F(h)$ varies approximately as $\sim h^{0.5}$ when the system is fully-resonant in shallow water on a uniform sloping beach. Invoking the relationship between $\hat{\eta}_B$ and F in Eq. (8) and the shoaling rate of radiation stress $|\hat{S}| \propto h^{-0.5}$ in shallow water, this result implies that the shallow water asymptotic limit of the shoaling rate of group-forced IG wave is $\sim h^{-1}$. Similar shallow water shoaling rate of $\sim h^{-1.3}$ was obtained by Janssen et al. (2003) on a plane sloping beach of $h_x = -1/40$ for $|\beta_1|\mu \sim 1.9$ (cf. Fig. 8 therein). This result also implies the group-forced IG wave amplitude is dependent on the inverse of the relative bottom slope $(h_x/\Delta\omega)\sqrt{g/h}$, consistent with the conclusion of previous studies that the bulk shoaling rate of group-forced IG waves on a beach is negatively correlated with β_1 (Battjes et al. 2004; van Dongeren et al. 2007; De Bakker et al. 2016; Zhang et al. 2020).

For the freely propagating IG wave, in this case, with the expansion $\sqrt{1-\mu} - 1 = -(1/2)\mu + O(\mu^2)$ the phase coupling between free IG waves and group forcing varies with respect to depth as $h \frac{d}{dh} [\text{Arg}(\hat{\eta}_f) - \phi_g] = O(\mu\beta_1^{-1}) = O(1)$, thus the assumption $F_X = O(F\beta_1)$ is also valid for free IG waves. Therefore, the spatial evolution of free IG wave is described by $F_H(X)$ instead of Eq. (17) as

$$\hat{\eta}_f(x) = \hat{\eta}_f(x_0) \frac{F_H(x)}{F_H(x_0)} \frac{\hat{S}(x)/h}{\hat{S}(x_0)/h_0} + O(\beta_1^2). \quad (21)$$

b. Irregular waves

For irregular waves, the first-order surface displacement can be written in the form of Fourier series,

$$\eta^{(1)}(x, t) = \frac{1}{2} \sum_{m=1+m_c}^{N+m_c} \hat{a}_m(x) e^{-i\omega_m t}, \quad (22)$$

where $\hat{a}_m(x) = |\hat{a}_m(x)| e^{i \left(\int_{x_0}^x k_m dx' + \psi_m \right)}$ is the single-side complex amplitude of the m th frequency component of the primary wave with radian frequency ω_m , wavenumber k_m and phase angle ψ_m , respectively; m_c is the rank of the cutoff frequency between primary wave and IG waves; ω_{1+m_c} and ω_{N+m_c} are the lower and upper radian frequencies of the first-order wave components, respectively. For a unimodal narrow-band spectrum, we may choose $\omega_{1+m_c} = 0.5\omega_p$ and $\omega_{N+m_c} = 3\omega_p$ where ω_p

is the peak radian frequency. Let the wave component of frequency ω_p be the carrier wave in Eq. (2) and substituting Eq. (22) into Eqs. (2) and (5) yields

$$\begin{cases} \tilde{S}(x, t) = \frac{1}{2} \rho g \sum_{n=1}^{m_c} \hat{S}_n(x) e^{-i\omega_n t} \\ \hat{S}_n(x) = \left(\frac{2c_g}{c} - \frac{1}{2} \right) \sum_{m=m_c+1+N}^{m_c+N} \hat{a}_m \hat{a}_{m-n}^* \end{cases}, \quad (23)$$

where the upper limit of summation $N - 1$ has been cut off at m_c in order to consider the IG band only. Equation (23) shows that each frequency component of the group forcing is constructed by multiple pairs of bichromatic waves with the same difference frequency but different central frequencies. For narrow-banded waves, the difference wavenumber $k_m - k_{m-n}$ is approximately equal to $k_m - k_{m-n} = (\partial k / \partial \omega)|_{\omega=\omega_p} (\omega_m - \omega_{m-n}) + O(\omega_m - \omega_{m-n})^2$. Because $\omega_n = \omega_m - \omega_{m-n}$ and $c_g^{-1} = (\partial k / \partial \omega)|_{\omega=\omega_p}$, it equals to $k_m - k_{m-n} = \omega_n / c_g + O(\omega_n)^2$ which is not related to m , therefore, shall be noted as Δk_n . Assuming a conservative shoaling, i.e., $c_g |\hat{a}_m \hat{a}_{m-n}| = \text{const.}$, Eq. (23) can be rewritten in the frequency domain as

$$\hat{S}_n(x) = \hat{S}_n(x_0) \left(\frac{2}{c} - \frac{1}{2c_g} \right) \bigg/ \left(\frac{2}{c} - \frac{1}{2c_g} \right)_0 e^{i \int_{x_0}^x \Delta k_n dx'}. \quad (24)$$

Thus, the IG wave solution (8) becomes

$$\eta(x, t) = \frac{1}{2} \sum_{n=1}^{m_c} \left[\hat{\eta}_{f,n} - F_n \frac{\hat{S}_n}{h} \right] e^{-i\omega_n t}, \quad (25)$$

where F_n is the response function for the n th frequency component, $\hat{\eta}_{f,n}$ is the single-side complex amplitude for the n th frequency component of free IG waves. Therefore, for the n th frequency component of IG waves, the response function F_n may be obtained in the same manner as in section 3a for a pair of bichromatic waves with central radian frequency ω_p and

difference radian frequency ω_n . Furthermore, the phase lag of IG waves relative to the equilibrium response can also be derived from the response function (see the appendix B for details) as

$$\Delta \varphi_{\text{IG}}(x) = \text{Arg} \sum_{n=1}^{m_c} \left[|\hat{S}_n|^2 \left(F_n - \frac{\hat{\eta}_{f,n}}{\hat{S}_n / h} \right) \right]. \quad (26)$$

4. Boundary condition

Gravity waves and freely propagating IG waves may be scattered away from the offshore edge of the topography x_0 when the bottom slope h_x suddenly changes (Liu 1989; Miles and Zou 1993; ZOU11). By integrating Eq. (4) in the neighborhood of x_0 and then taking a limit, the matching conditions at x_0 are (Schäffer 1993)

$$\begin{cases} [\hat{\eta}]_{-}^{+} = 0 \\ [h_0 \hat{\eta}_x + \hat{S}]_{-}^{+} = 0 \end{cases}, \quad (27)$$

where $[\cdot]_{-}^{+}$ denotes the difference between the left (−) or right (+) limits at x_0 . Considering the intermediate depth at the offshore edge of the topography and substituting Eqs. (7), (8), and (17) into Eq. (27), we obtain

$$\begin{cases} \left[\hat{\eta}_f - F \frac{\hat{S}}{h} \right]_{-}^{+} = 0 \\ \left[\hat{\eta}_f [ik_f h - 0.25h_x + O(h_x^2)] + \hat{S} - h_0 \left(F \frac{\hat{S}}{h} \right) \right]_{x-}^{+} = 0 \end{cases}. \quad (28)$$

Assuming $h_x = 0$ for $x < x_0$ so that $F = \mu_0^{-1}$, while the right limit of F at x_0 is described by the OLS [Eq. (12)]. The complex amplitude of the scattered shoreward propagating free waves at x_0 can thus be derived from Eq. (28) as

$$-\frac{\hat{\eta}_f(x_0)}{\hat{S}(x_0)/h_0} = \left\{ \frac{[(\mu^{-1} - 1)\Lambda_1 - \mu^{-1} + h \frac{d\mu^{-1}}{dh}] \beta_1 + i F_{\text{slope}} (1 + \sqrt{1 - \mu})}{0.25\beta_1 - 2i\sqrt{1 - \mu}} \right\}_{0^+} + O(\beta_1^2), \quad (29)$$

where $[\cdot]_{0^+}$ denotes taking limit at $x = x_0$ from the shoreward side ($x > x_0$) of the topography, F_{slope} is the topography-induced additional bound IG wave response in Eq. (14).

5. Theoretical results

a. Bichromatic waves over a sloping bottom

1) OFFSHORE RAMPS

Evolution of the IG waves induced by a bichromatic wave group propagating over downward and upward sloping offshore ramps (Figs. 3a,b) in the intermediate depth range $k_\infty h \in [1, 2]$

($k_\infty = \omega^2/g$) are given in Figs. 3c and 3d. The predictions of the present OLS [Eqs. (12) and (17)] compare well with the ZOU11 solution.

Figure 3c shows that on the downward sloping ramp, compared with OLS, the NRS [Eqs. (19) and (21)] overestimates the amplitude of additional bound IG wave, while the predicted overall amplitude by OLS and NRS is similar to each other. This is because with increasing depth the scattered free IG wave becomes the predominant component of the additional IG wave due to its much lower shoaling rate, and the predicted amplitude of scattered free waves by NRS is close to that by OLS. On the upward sloping ramp (Fig. 3d), however, the predicted amplitudes of the additional bound IG wave by

FIG. 3. Topographies of a (a) downward- and (b) upward-sloping ramp connecting two constant depths. Amplitudes of LHS62 solution (thin solid lines) and topography-induced additional IG waves (dashed lines) on (c) downward- and (d) upward-sloping ramps in intermediate depth ($|\beta_1 \mu^{-1}| \in [0.03, 0.09]$). The topography-induced additional IG waves are further decomposed into forced (thick solid lines) and free (dash-dotted lines) components. Predictions of ZOU11 (black), the off-resonant linear solution [OLS, Eqs. (12) and (17), green lines], and nearly-resonant solution [NRS, Eqs. (19) and (21), red lines] are given. The width of the ramp is $L = 10\omega/(\Delta\omega k_\infty)$, with $k_\infty = \omega^2/g$ the deep-water wavenumber.

OLS and NRS are nearly the same. The additional group-forced IG wave becomes dominant over the scattered free wave with decreasing depth due to its greater shoaling rate. As a result, predictions of amplitudes of total additional IG wave by OLS and NRS are similar.

2) PLANE SLOPING BEACH

For wave groups over a plane sloping beach, predictions of the analytical solutions are validated against the results of the Van Noorloos (2003) experiment, which generated high-quality datasets of IG waves induced by bichromatic waves on a sloping beach with an offshore depth of 0.7 m and bottom slope of 1/35. In addition to the OLS and NRS, the exact solution of the governing Eq. (10) was also numerically calculated without decomposition into forced and free components. As shown in Fig. 4, predictions of the exact solution and NRS are in good agreement with the experimental results. Performance of the OLS is promoted with increasing wave group frequency, i.e., weakening resonant intensity.

To illustrate the evolution of group-forced IG waves through zones of different resonance intensities, examples of bichromatic waves on three sloping beaches ($h_x = -1/20, -1/35, -1/100$) are demonstrated in Fig. 5. The physical parameters of the second case ($h_x = -1/35$) were chosen to be the same as those of the A-4 case in Fig. 4.

For the nearly- to fully-resonant cases of $h_x = -1/20$ and $-1/35$ [$\beta_1 \mu^{-1} = O(1)$], (Figs. 5a,b), compared with the exact solution predictions, both the amplitude (Figs. 5a,b) and phase lag (Figs. 5d,e) of group-forced IG waves are overestimated by the OLS. The NRS agrees well with the exact solution except for slight overestimations of the phase lag in the vicinity of the breakpoint ($kh \approx 0.5$) where the assumption $|\beta_1| \ll 1$ is becoming invalid with decreasing depth. In the case of $h_x = -1/100$, the resonance intensity is generally weak ($|\beta_1 \mu^{-1}| \in [0.06, 1.2]$) and the system is mainly off-resonant in the shoaling zone. Accordingly, the OLS agrees with the exact solution in terms of predictions of the amplitude and phase lag in $kh \geq 0.8$ (Figs. 5c,f). Meanwhile, predictions of the NRS are nearly the same as the exact solution,

FIG. 4. Spatial evolution of the amplitude and phase lag with respect to the forcing of group-forced infragravity waves for bichromatic wave groups on a sloping beach of $h_x = -1/35$: the series A cases of Van Noorloos (2003) physical experiment (Exp., circles), the exact solution (ES, solid lines), off-resonant linear solution [OLS, Eqs. (12) and (17), dashed lines], and nearly-resonant linear solution [NRS, Eqs. (19) and (21), dotted lines] are presented. Wave parameters are $h_0 = 0.7$ m, $a_1(x_0) = 5a_2(x_0) = 0.06$ m.

suggesting that the NRS might be practically effective over the whole beach. Furthermore, amplitudes of topography-induced additional bound IG wave of two parts [section 3a(1)] are calculated based on the OLS. Amplitudes of additional bound IG waves induced by the enhanced group-forcing (blue line) and by the varying IG wave amplitude with decreasing water depth (red line) are demonstrated in Fig. 5c. Consistent with the theoretical calculations displayed in Fig. 2, the results confirm the dominance of the latter type.

The local shoaling rates of IG waves, $(h/|\hat{\eta}|)d|\hat{\eta}|/dh = [h(h_x|\hat{\eta}|)]d|\hat{\eta}|/dx$, are also presented for the three cases (Figs. 5g–i). The spatial evolutions of local shoaling rate predicted by the exact solution, OLS, and NRS all display a single-peaked pattern, with the greater peak value occurring in deeper depth on a milder bottom slope, i.e., weaker resonance intensity. The predicted local shoaling rate by OLS is in agreement with the exact solution only near the offshore edge of the topography for $kh \geq 1$ on the mildest slope of $h_x = -1/100$ (Fig. 5i), and severely exceeds the exact solution predictions for the steeper slopes $h_x = -1/20$ and $h_x = -1/35$ (Figs. 5g,h). In all three cases, the NRS is in good agreement with the exact solution in shoaling rate predictions. According to the shallow water limit of NRS [Eq. (20)], the local shoaling rate would approach $\sim h^{-1}$ asymptotically with decreasing water depth in shallow water. This trend can be seen from the predictions of the exact solution and NRS (Figs. 5g–i). Figures 5g–i also show that the prebreaking shoaling rate is closer to $\sim h^{-1}$ on steeper slopes, implying that the group-forced IG wave response approaches the shallow water limit of NRS faster when the resonance is stronger on a steeper beach.

b. Irregular waves on a plane sloping beach

For irregular waves propagating over a plane beach of bottom slope $h_x = -1/80$, comparison among predictions of the exact solution, NRS, and the nonhydrostatic wave model SWASH (Zijlema et al. 2011) is given in Fig. 6. The SWASH model is essentially a Reynolds-averaged Navier–Stokes (RANS) equation solver capable of describing the processes of wave motion with strong nonlinearity (Kirby 2017). The model has been extensively validated against both laboratory (De Bakker et al. 2016) and field data (Rijnse et al. 2015) of nearshore wave evolution with strong nonlinearity. Detailed descriptions of the model can be found in Rijnse et al. (2014) and Smit et al. (2014).

All parameters are the same as the A3 case in the numerical simulation by De Bakker et al. (2016), which has been validated against the laboratory experiment of Ruessink et al. (2013). Irregular waves described by JONSWAP spectrum (Hasselmann et al. 1973) were generated with the significant wave height $H_s = 0.1$ m, the peak period $T_p = 2.25$ s, and peak enhancement factor $\gamma = 20$. Time series of the surface elevation predicted by the SWASH model at the toe of the slope was bandpass filtered through the frequency range $0.5\omega_p \leq \omega \leq 3\omega_p$, and then transformed into the single-side complex Fourier amplitudes. Then the spectrum of wave radiation stress was calculated according to Eqs. (23) and (24), and employed as the input for the theoretical solution Eq. (25). Note that since the boundary condition Eq. (29) is derived based on the off-resonant assumption ($|\beta_1\mu^{-1}| = O(\beta_1) \ll 1$), it does not apply to the wave frequency components close to zero frequency. Therefore, we applied the boundary condition only to the wave frequency components that satisfy $|\beta_1\mu^{-1}| \leq 0.1$ at the toe of

FIG. 5. Spatial evolution of the (a)–(c) amplitude, (d)–(f) phase lag with respect to the forcing, and (g)–(i) local shoaling rate ($h|\hat{\eta}|_h/|\hat{\eta}|$) of group-forced IG waves for a bichromatic wave group on a sloping beach of (left) $h_x = -1/20$, (center) $h_x = -1/35$, and (right) $h_x = -1/100$: the exact solution (ES, solid lines), off-resonant linear solution [OLS, Eqs. (12) and (17), dashed lines], and nearly-resonant solution [NRS, Eqs. (19) and (21), dotted lines] are presented. Amplitudes of the additional nonequilibrium bound IG waves (F_{slope}) induced by the spatial variations of group-forcing (F_I , blue line) and IG wave amplitude ($F_{II} + F_{III}$, red line) are demonstrated. Wave parameters are the same as the A-4 case in the laboratory experiment of Van Noorloos (2003) [$h_0 = 0.7$ m, $a_1(x_0) = 5a_2(x_0) = 0.06$ m, $[\omega, \Delta\omega] = [3.605, 0.614]$ rad s $^{-1}$].

the bottom slope, while the response function of the rest of the lower-frequency wave spectral components was calculated using the LHS62 solution neglecting the scattering effect of free IG waves.

Figure 6a shows the spatial evolution of normalized integrated IG wave height over the frequency band $0 \leq \omega \leq 0.5\omega_p$. The NRS and exact solution predictions are nearly the same, and both agree with those predicted by the SWASH model reasonably well in the shoaling zone. In addition, the predicted phase lag by exact solution is in excellent agreement with that by the SWASH model (Fig. 6b), whereas the NRS gradually overpredicts the phase lag with decreasing depth. The local shoaling rate of the significant wave height predicted by the SWASH model approximately reaches $\sim h^{-1}$ near the breakpoint, while the exact solution and NRS predict greater value (Fig. 6c), possibly due to the fact that the narrow-band

assumption may become less effective as the bandwidth increases when waves propagate shoreward.

Figure 7a shows the variance density spectrum of group-forced IG waves as a function of depth and frequency by the SWASH model in the shoaling zone using a horizontal resolution of 1 m. The spectrum was generated using the surface elevation of the incident IG waves of 1 h duration, sampled at 10 Hz, and smoothed with a moving window of 50 bins width. Figure 7b shows the comparison between the predicted variance density spectrum by the theoretical model OLS, NRS, and SWASH at the wave breaking depth ($k_p h = 0.51$). Both the NRS and exact solution well captured the shape of the spectrum (Fig. 7b) prior to wave breaking, but overestimated the IG wave energy near the breaking depth as discussed in Fig. 6 earlier. Figure 7b also shows that at extremely low frequencies ($0 \leq \omega/\omega_p \leq 0.1$), the NRS predicts

FIG. 6. Spatial evolution of the (a) normalized wave height, (b) phase lag with respect to the group-forcing, and (c) local shoaling rate (hH_{IGh}/H_{IG}) of IG waves forced by irregular waves on a plane sloping beach: the SWASH model (black solid lines), the nearly-resonant solution (NRS, green dot-dashed lines), and the exact solution (ES, red dashed lines). Vertical dashed lines denote the breaking depth. JONSWAP spectrum was employed to generate the irregular waves.

lower wave energy than SWASH while the exact solution agrees reasonably well with SWASH, indicating that the second-order terms of $O(\beta_1^2)$ and $O(\beta_2)$ could be important for components in this band.

Figure 7c indicates that the local shoaling rate of IG component is greater for higher frequencies, i.e., lower-resonance intensities. In the band of $0.1 \leq \omega/\omega_p \leq 0.5$, the single-peaked pattern of the spatial evolution of local shoaling rate can be recognized as in Fig. 6, and for components in lower-resonance intensities the peak is greater and the depth for its occurrence is deeper (Fig. 7c). With decreasing depth, the local shoaling rates in this frequency band eventually approaches the theoretical shallow limit of $NRS \sim h^{-1}$. At extremely low frequencies ($0 \leq \omega/\omega_p \leq 0.1$), the local shoaling rates are evidently lower (Figs. 7c,d). The predicted local shoaling rate by the exact solution decays with decreasing frequency in the same trend as the result of SWASH model, while that predicted by the NRS decays

faster. The aforementioned significant second-order effect of $O(\beta_1^2)$ in this band may be partially responsible for this difference.

6. Discussions on group-forced infragravity waves over a symmetric topography

To theoretically diagnose the mechanisms for the shoal-induced amplification of IG waves mentioned in the Introduction (Table 1), the behavior of IG waves in the area of near-resonance on a symmetric topography, where $h(x) = h(L - x)$, with L the finite horizontal length of the topography, is analyzed. The topography is assumed to be smooth, so no scattering of free IG wave due to discontinuity in bottom slope is considered except at the offshore edge.

Assuming that the system is nearly-resonant near the crest of the shoal, the spatial evolution of group-forced IG wave

FIG. 7. (a) SWASH predictions of spatial evolution of the variance density spectrum of IG waves induced by wave groups in the shoaling zone over a plane sloping beach and (b) comparison between the SWASH model predictions (black) and the exact solution [ES, Eq. (10), red] and nearly-resonant solution [NRS, Eqs. (19) and (21), green] predictions at the breaking depth (horizontal dashed line). (c), (d) As in (a) and (b), but for the local shoaling rate of each frequency component. Vertical dashed lines denote the frequency where the NRS predicts the amplitude lower than the ES does by 10%.

response is described by the NRS [Eq. (19)], which is restated as (for the convenience of description the X coordinate is converted to x coordinate)

$$F(x) = \frac{F_H(x)}{F_H(L/2)} F(L/2) + F_f(x), \quad (30)$$

where $F_f(x) = (i/2) F_H(x) \int_{L/2}^x F_H^{-1}(x') \Delta k dx'$ is introduced to represent the symmetrically conjugate part of $F(x)$ that satisfies $F_f(x) = F_f^*(L-x)$, where the asterisk denotes the complex conjugate. Also, one can verify that the ratio $F_H(x)/F_H(L/2)$ is also symmetrically conjugated. Therefore, as long as the phase angle of $F(L/2)$ satisfies

$$\varphi_F(L/2) = n\pi, \quad n \in \mathbb{Z}, \quad (31)$$

then $F(L/2)$ becomes purely real and the NRS [Eq. (30)] would be symmetrically conjugated. Equation (31) corresponds to the

situation that the IG waves are in phase/antiphase with the radiation stress at the shoal crest $x = L/2$. In this situation, the response function satisfies $F(x) = F^*(L-x)$, and it follows that in the nearly-resonant zone near the crest, the spatial evolution of the group-forced IG wave amplitude on the first half ($x \leq L/2$) of the topography is simply the mirror image of that on the second half ($L/2 \leq x$), whereas the phase coupling between IG wave and group-forcing is in antiphase with that on the second half, i.e., $\varphi_F(x) = -\varphi_F(L-x)$. In this sense, the condition Eq. (31) stands for the reversible condition for the spatial evolution of the group-forced IG waves in the nearly-to fully-resonant area.

Examples of bichromatic waves propagating across a shoal-mimic topography (Fig. 8a) with different tide levels are given in Fig. 8. For $k_x h_0 = 2$ the system is off-resonant over the entire region, the OLS agrees with the exact solution, while the NRS overpredicts the amplitude of the additional bound

FIG. 8. Group-forced IG waves at three different tide levels and water depth $k_\infty h_0$ for bichromatic waves over (a) a symmetric shoal (shaded area), which was generated by smoothing the trapezoid profile (dot-dashed line) with a moving window of $L/2$ width. (b)–(d) Normalized amplitudes of additional IG waves and (e)–(g) phase lag relative to the equilibrium response predicted by the exact solution (ES, black solid lines), the off-resonant solution (OLS, green solid lines), and the nearly-resonant solution (NRS, red solid lines). The additional IG waves predicted by the OLS (green) and NRS (red) are further decomposed into forced (dashed lines) and free (dash-dotted lines) components. The length of the shoal is $L = 13\omega/(k_\infty \Delta\omega)$.

IG wave (Fig. 8b) and phase lag (Fig. 8e). In this situation, the scattered free wave is of the same order as the additional group-forced wave, therefore the behavior of the total additional wave deviates from that of the additional forced IG wave. Consequently, the predictions of exact solution and OLS of the spatial evolution of additional IG wave only show rough symmetric conjugacy.

For $k_\infty h_0 = 1$, nearly- to fully-resonant area emerges near the crest of the shoal. In this case, the OLS remains in agreement with the exact solution in the range $0 \leq x/L \leq 0.29$, while the NRS agrees with the exact solution reasonably well over the entire domain (Figs. 8c,f). Additional bound IG wave is considerably amplified over the shoal according to the predictions of both the NRS and exact solution. As the tidal level decreases to $k_\infty h_0 = 0.5$, the prediction by the NRS is almost exactly the same as that by the exact solution, while the OLS deviates from the exact solution significantly

(Figs. 8d,g). It is shown that with increasing resonant intensity the maximum phase lag occurs closer to the crest of the shoal where resonant intensity attains maximum (Figs. 8e–g). Since the additional forced IG wave dominates over the scattered free wave in these two strong resonant cases (Figs. 8c,d), the nonequilibrium response contributes to the total response at the leading order $O(\beta_1^{-1})$ according to the NRS in Eq. (19). As a result, the reversible condition Eq. (31) cannot be satisfied, and the spatial evolution of the IG wave magnitude changes from roughly symmetric (reversible) in off-resonant condition to being amplified continuously across the topography (irreversible) in nearly- to fully-resonant conditions near the crest of the shoal. Physically, due to the nearly symmetric distribution of the phase relative to the crest, the phase lag remains positive across the topography, and consequently the energy is persistently transferring from the primary waves to IG waves as waves propagate over the

entire region. The irreversible asymmetric spatial evolution of additional forced IG wave involved in Fig. 8 is consistent with the numerical results by SWASH model of irregular wave group-forced IG waves on shoals with shallow water on the top (Li et al. 2020). The present analytical solution allows us to elucidate the mechanism for IG waves to be greatly amplified as waves propagate over asymmetric offshore shoals (Table 1) for the first time.

Theoretically, for the more general cases other than a single shoal, the presence of the nearly-resonant area does not always rule out the possibility for the reversible spatial variation of the group-forced IG waves to occur. The reversible condition Eq. (31) suggests that in the nearly-resonant area, the group-forced IG wave would start to evolve reversely from the location where the depth starts to vary symmetrically, and meanwhile the group forcing and group-forced IG waves are nearly in phase/antiphase with each other. Regarding the bathymetry of a single shoal, however, this condition is expected to be unlikely to be satisfied so that the amplification of IG waves should always occur.

For the wave group with significant directional spreading in the field condition, group-forced IG waves propagating in a direction different from the main direction of primary waves may be induced. Future works of extending the present solution to accommodate the 2D wave field are needed to theoretically model both the frequency and directional spreading of group-forced IG waves.

7. Conclusions

The spatial evolution of infragravity (IG) waves induced by wave group over topography in a wide range of resonance intensity and water depth under nonbreaking conditions has been investigated with a newly defined complex response function. The analytical solution for the governing equation at the first order of the relative bottom slope is proposed and validated against the laboratory, theoretical and numerical results. In intermediate water, the off-resonance solution is consistent with the previous second-order nonlinear theory of ZOU11 when the higher-order terms therein are negligible. In shallow water, a novel nearly-resonance solution is obtained and is the first of its kind according to the authors' knowledge.

The nonequilibrium bound IG wave in quadrature with the wave group forcing and the equilibrium component on flat bottoms is generated by the off-resonant $[\beta_1\mu^{-1} = O(\beta_1)]$ wave groups in the intermediate depth as indicated by the off-resonant linear solution [Eqs. (12) and (17)]. At the leading order, it is proportional to the normalized bottom slope β_1 , therefore, secondary compared with the equilibrium component at $O(1)$. It is jointly induced by the spatial variation of magnitude and transmitting speed of group-forcing and the spatial variation of the bound IG wave amplitude [cf. Eq. (15)]. The latter is dominant over the former, and therefore, is the primary contributor to the spatially varying phase coupling between bound IG waves and group-forcing and the energy transfer from primary waves to IG waves.

Opposite to the aforementioned off-resonant case, for the nearly-resonant $[\beta_1\mu^{-1} = O(1)]$ wave groups in shallow water, the nearly-resonant solution [NRS, Eqs. (19) and (21)] indicates that both the nonequilibrium and equilibrium group-forced IG waves are of the leading order $O(\beta_1^{-1})$. For nearly-resonant wave groups propagating over a plane sloping beach in shallow water, it was found that the local shoaling rate of the group-forced IG wave approaches $\sim h^{-1}$ asymptotically with decreasing water depth toward the shore, which is much lower than the widely known shallow water limit of the shoaling rate of the LHS62 solution $\sim h^{-2.5}$. The numerical results by SWASH of irregular waves over a plane beach of bottom slope $h_x = -1/80$ confirm that the local shoaling rates of most frequency components ($0.1 \leq \omega/\omega_p \leq 0.5$) approach $\sim h^{-1}$ in the shoaling zone with decreasing depth. For the remaining extremely low frequencies of $0 < \omega/\omega_p \leq 0.1$, however, the shoaling rates are lower, possibly because the linear assumption $|\beta_1| \ll 1$ is no longer valid and the higher-order $O(\beta_1^2)$ effect of the bottom slope and bottom curvature is important.

The solutions proposed also enable a theoretical investigation of the spatial evolution of group-forced IG waves over a symmetric shoal in intermediate and shallow water to reveal the influences of the strong resonance near the topography crest. When the resonance is weak across the whole topography at high tide, the solution indicates that the spatial evolution of the IG wave amplitude on the shoreward slope of the shoal is roughly the mirror image of that of its counterpart on the seaward slope. As the water level drops sufficiently at low tide, however, strong resonance begins to occur near the crest of the shoal, the group-forced IG wave dominates over the topography-scattered freely propagating IG wave, with the maximum phase lag relative to the equilibrium response occurring near the crest. The spatial distribution of the phase lag becomes increasingly symmetric and remains positive with decreasing water level, which leads to the persistent nonlinear energy transfer from primary waves to IG waves, and therefore, significant amplification of IG wave amplitude across the shoal.

According to the NRS [Eqs. (19) and (21)], the irreversible amplification in the strong resonance area would occur when the group-forced IG wave and group-forcing are not in phase/antiphase at the shoal crest. This finding provides the theoretical explanation for the previously reported IG wave amplification induced by a shoal in field (Paniagua-Arroyave et al. 2019) and numerical models (Li et al. 2020), or a trapezoid low-crest man-made structure such as breakwater (Peng et al. 2009; Zou and Peng 2011). Generally, the NRS suggests that in the nearly-resonant area, the group-forced IG wave would start to evolve reversely from any location where the depth starts to vary symmetrically and meanwhile the forcing and IG wave are in phase/antiphase with each other, regardless of the history of the spatial evolution of these waves.

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Data availability statement. The input files and outputs of the numerical wave model SWASH presented in [section 5b](#) and the MATLAB codes of the exact solution, off-resonant linear solution, and nearly-resonant solution are openly available on [Zenodo.org](https://zenodo.org/record/3951926) (10.5281/zenodo.3951926).

APPENDIX A

Spatial Variation of Response Function

Equation (10) indicates that given the central and difference frequencies of a bichromatic wave group, the group-forced IG wave response is the function of the horizontal coordinate, depth,

bottom slope, and bottom curvature, i.e., $F = F(x, h, h_x, h_{xx})$. The response function varies with X as

$$\begin{aligned} F_X &= \frac{dx}{dX} \left(\frac{\partial F}{\partial x} + h_x \frac{\partial F}{\partial h} + h_{xx} \frac{\partial F}{\partial h_x} + h_{xxx} \frac{\partial F}{\partial h_{xx}} \right) \\ &= \beta_1 \left(h \frac{\partial F}{\partial h} \right) \left[1 + h_x^{-1} \left(\frac{\partial F}{\partial x} / \frac{\partial F}{\partial h} \right) + \frac{\beta_2}{\beta_1} \left(\Delta k \frac{\partial F}{\partial h_x} / \frac{\partial F}{\partial h} \right) \right. \\ &\quad \left. + \frac{\beta_3}{\beta_1} \left(\Delta k^2 \frac{\partial F}{\partial h_{xx}} / \frac{\partial F}{\partial h} \right) \right], \end{aligned} \quad (\text{A1})$$

and similarly

$$\begin{aligned} F_{XX} &= -\delta \beta_1^2 \left(h \frac{\partial F}{\partial h} \right) \left[1 + \left(h_x^{-1} \frac{\partial F}{\partial x} / \frac{\partial F}{\partial h} \right) + \frac{\beta_2}{\beta_1} \left(\Delta k \frac{\partial F}{\partial h_x} / \frac{\partial F}{\partial h} \right) + \frac{\beta_3}{\beta_1} \left(\Delta k^2 \frac{\partial F}{\partial h_{xx}} / \frac{\partial F}{\partial h} \right) \right] \\ &\quad + \beta_1^2 \left(h^2 \frac{\partial^2 F}{\partial h^2} \right) \left[1 + h_x^{-2} \left(\frac{\partial^2 F}{\partial x^2} / \frac{\partial^2 F}{\partial h^2} \right) + \frac{\beta_2^2}{\beta_1^2} \left(\Delta k^2 \frac{\partial^2 F}{\partial h_x^2} / \frac{\partial^2 F}{\partial h^2} \right) + \frac{\beta_3^2}{\beta_1^2} \left(\Delta k^4 \frac{\partial^2 F}{\partial h_{xx}^2} / \frac{\partial^2 F}{\partial h^2} \right) \right. \\ &\quad \left. + 2h_x^{-1} \left(\frac{\partial^2 F}{\partial h \partial x} / \frac{\partial^2 F}{\partial h^2} \right) + 2h_x^{-1} \frac{\beta_2}{\beta_1} \left(\Delta k \frac{\partial^2 F}{\partial h_x \partial x} / \frac{\partial^2 F}{\partial h^2} \right) + 2h_x^{-1} \frac{\beta_3}{\beta_1} \left(\Delta k^2 \frac{\partial^2 F}{\partial h_{xx} \partial x} / \frac{\partial^2 F}{\partial h^2} \right) \right. \\ &\quad \left. + 2 \frac{\beta_2}{\beta_1} \left(\Delta k \frac{\partial^2 F}{\partial h \partial h_x} / \frac{\partial^2 F}{\partial h^2} \right) + 2 \frac{\beta_3}{\beta_1} \left(\Delta k^2 \frac{\partial^2 F}{\partial h \partial h_{xx}} / \frac{\partial^2 F}{\partial h^2} \right) + 2 \frac{\beta_2 \beta_3}{\beta_1^2} \left(\Delta k^3 \frac{\partial^2 F}{\partial h_x \partial h_{xx}} / \frac{\partial^2 F}{\partial h^2} \right) \right], \end{aligned} \quad (\text{A2})$$

where $\beta_3 = h_{xxx}/\Delta k^3 h$. We require that the variation of the response function is mainly induced by variable depth by assuming the magnitudes of the terms in the square brackets of Eqs. (A1) and (A2) do not exceed $O(1)$, so that $F_X = O(\beta_1 h \partial F / \partial h)$ and $F_{XX} = \beta_1^2 [O(h \partial F / \partial h) + O(h^2 \partial^2 F / \partial h^2)]$. Furthermore, since the response function varies with depth at the order $h \partial F / \partial h = O(F)$ and $h^2 \partial^2 F / \partial h^2 = O(F)$ [Eq. (3.7) in [ZOU11](#)], therefore $F_X/F = O(\beta_1)$ and $F_{XX}/F = O(\beta_1^2)$.

Substituting the complex amplitudes of the radiation stress $\hat{S}_n^* \propto \sum_{m=m_c+1+n}^{m_c+N} \hat{a}_m \hat{a}_{m-n}^*$ [Eq. (23)] and the IG wave $\hat{a}_n = \hat{\eta}_{f,n} - F_n \hat{S}_n/h$ [Eq. (25)] into Eq. (B2) leads to

$$\Delta \varphi_{\text{IG}}(x) = \text{Arg} \left[\sum_{n=1}^{m_c} \left(F_n - \frac{\hat{\eta}_{f,n}}{\hat{S}_n/h} \right) |\hat{S}_n^2| \right]. \quad (\text{B3})$$

APPENDIX C

Notation

Integrated Biphas from the Response Function

For irregular waves, the integrated biphas of all pairs of first-order bichromatic waves and their subharmonics in addition to π measures the phase lag of the group-forced IG waves with respect to the equilibrium response of [LHS62](#), and it writes as

$$\Delta \varphi_{\text{IG}}(x) = \text{Arg} \left[\sum_{n=1}^{m_c} \sum_{m=m_c+1}^{m_c+N-n} B_{m,n}(x) \right] - \pi, \quad (\text{B1})$$

where $B_{m,n} = (1/8) \mathbf{E}[\hat{a}_m \hat{a}_n \hat{a}_{n+m}^*]$ is the discrete bispectrum, $\hat{a}_m(x)$ is the single-side complex amplitude, and $\mathbf{E}[\cdot]$ denotes ensemble average. Equation (B1) can be rewritten into

$$\Delta \varphi_{\text{IG}}(x) = \text{Arg} \left[\sum_{n=1}^{m_c} \hat{a}_n(x) \sum_{m=m_c+1+n}^{m_c+N} \hat{a}_{m-n}(x) \hat{a}_m^*(x) \right] - \pi. \quad (\text{B2})$$

$\eta^{(1)}$	Primary wave surface elevation
$\eta^{(2)}, \eta$	IG wave surface elevation
A	Modulated amplitude of primary wave
a_1, a_2	Real amplitudes of bichromatic waves
k	Central wavenumber of bichromatic waves
ω	Central radian frequency of bichromatic waves
θ	Phase of carrier wave
$\Delta \theta$	Phase of wave group
Δk	Wavenumber of wave group
$\Delta \omega$	Radian frequency of wave group
φ_g	Spatial phase of wave group
g	Gravitational acceleration
c_g	Wave group speed
c	Wave speed
S	Wave radiation stress
$\tilde{S}$	Slowly varying part of wave radiation stress
$\hat{S}$	Single-side complex amplitude of slowly varying wave radiation stress

$\hat{\eta}$	Single-side complex amplitude of IG waves
$\hat{\eta}_B$	Single-side complex amplitude of group-forced IG waves
$\hat{\eta}_f$	Single-side complex amplitude of freely propagating IG waves
$\Delta\varphi$	Phase lag of IG waves relative to the equilibrium bound IG wave described by LHS62
F	Group-forced IG wave response to radiation stress for bichromatic waves, $F(x) = -\hat{\eta}_B(x)/[\hat{S}(x)]/h$
φ_F	Phase of group-forced IG wave response function F
F_n	Group-forced IG wave response of the n th frequency component to radiation stress for irregular waves
x	Horizontal coordinate, positive onshore
z	Vertical coordinate, positive upward
t	Time
X	Slowly varying horizontal coordinate at group scale, $dX = \Delta k dx$
h	Water depth
h_x	Bottom slope
h_{xx}	Bottom curvature
β_1	Relative bottom slope, $\beta_1 = h_x/(\Delta kh)$
β_2	Relative bottom curvature, $\beta_2 = h_{xx}/(\Delta k^2 h)$
Λ_1	First-order normalized derivative of the amplitude of radiation stress with respect to depth, $\Lambda_1 = h \hat{S} _h/ \hat{S} $
Λ_2	Second-order normalized derivative of the amplitude of radiation stress with respect to depth, $\Lambda_2 = h^2 \hat{S} _{hh}/ \hat{S} $
δ	First-order normalized derivative of the speed of radiation stress with respect to depth, $\delta = -(h/c_g)dc_g/dh$
μ	Degree of departure from resonance, $\mu = 1 - c_g^2/(gh)$
ϕ_0	ϕ at the offshore incident edge x_0 of the topography, ϕ is an arbitrary quantity
F_H	Homogeneous solution to the truncated governing equation of the response function F
θ_H	Phase of F_H
ZOU11	Second-order analytical solution of group-forced IG waves of Zou (2011)
ES	Exact solution of the group-forced IG wave response
OLS	Off-resonant [$\beta_1\mu^{-1} = O(1)$] linear solution of the group-forced IG wave
NRS	Nearly-resonant [$\beta_1\mu^{-1} = O(\beta_1^{-1})$] solution of the group-forced IG wave
F_{LHS62}	Bound IG wave response corresponding to the LHS62 solution, $F_{\text{LHS62}} = \mu^{-1}$
F_{Slope}	Nonequilibrium additional bound IG wave response induced by the variable depth at $O(\beta_1)$ in off-resonant condition, $F_{\text{Slope}} = F_I + F_{II} + F_{III}$
F_I	Additional bound IG wave response induced by variable group-forcing
F_{II}	Additional bound IG wave response induced by variable amplitude of equilibrium bound IG wave described by LHS62
F_{III}	Additional subharmonic response induced by variable amplitude of IG wave described by F_H
ω_p	Peak radian frequency
k_p	Peak wavenumber
m_c	Rank of the cutoff frequency between IG waves and primary waves

$\hat{a}_m$	Single-side complex amplitude of the m th frequency component of primary wave
H_{IG}	Significant wave height of IG waves
$\Delta\varphi_{\text{IG}}$	Integrated phase lag of IG waves with respect to group-forcing in addition to π for irregular waves
k_∞	Wavenumber in deep water, $k_\infty = \omega^2/g$
L	Horizontal length of finite topographies
$F_J(x)$	Symmetrically conjugate part of the nearly-resonant solution of $F(x)$ on symmetric topographies, $F_J(x) = F_J^*(L - x)$

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Authors Zhiling Liao, Shaowu Li, Ye Liu, and Qingping Zou have noted a graphic error with Fig. 4 and a typographical error with Eq. (10) in their article “An analytical spectral model for infragravity waves over topography in intermediate and shallow water under nonbreaking conditions” that appeared in the September 2021 issue of the *Journal of Physical Oceanography*, Vol. 51, No. 9, pages 2749–2765, <https://doi.org/10.1175/JPO-D-20-0164.1> (Liao et al. 2021). The corrected Fig. 4 appears here.

The corrected Fig. 4 shows that the exact solution and the near-resonant solution agree with the experimental results better as the difference frequency of incident bichromatic waves decreases. The errors are graphic and do not change the conclusions of the paper.

The corrected Eq. (10) reads

$$F_{XX} + [(2\Lambda_1 + \delta - 1)\beta_1 + 2i]F_X + [(\Lambda_1 - 1)\beta_2 + (\Lambda_2 - \Lambda_1 + 1)\beta_1^2 + (2\Lambda_1 + \delta - 1)\beta_1 i - \mu]F - [\Lambda_1\beta_2 + \Lambda_2\beta_1^2 + (2\Lambda_1 + \delta)\beta_1 i - 1] = 0, \quad (10)$$

where the coefficient of $\beta_1^2 F$ should be $\Lambda_2 - \Lambda_1 + 1$. The error is typographical and the correct equation was indeed solved in the original paper.

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FIG. 4. Theory–data comparisons of the spatial evolution of the amplitude and phase lag with respect to group forcing of group-forced infragravity waves for bichromatic wave groups with radian frequencies ω_1 and ω_2 on a sloping beach of $h_x = -1/35$: the series A cases, Case A-1 to Case A-4, of [Van Noorloos's \(2003\)](#) physical experiment (Exp.; circles), the exact solution (ES; solid lines), off-resonant linear solution [OLS; Eqs. (12) and (17) in the original article; dashed lines], and nearly resonant linear solution [NRS; Eqs. (19) and (21) in the original article; dotted lines] are presented. Wave parameters are $h_0 = 0.7$ m and $a_1(x_0) = 5a_2(x_0) = 0.06$ m.