NOTES AND CORRESPONDENCE

On the Effect of Barotropic Current Fluctuations on the Two-Layer Transport Capacity of a Constriction

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ABSTRACT

A simple theory shows that the two-layer transport capacity of a constriction may be increased considerably by barotropic current fluctuations. This is confirmed by laboratory experiments. The effect may be of great importance for the deep-water renewal process in some sill fjords and for the hydrographic conditions in some overmixed estuaries.

1. Introduction

Water exchange between an estuary and the sea has received considerable attention during the last 25 years. A great step forward in the understanding of this process was taken by Stommel and Farmer (1952, 1953) who treated the transport capacity of a constriction between an estuary and the sea. They showed, both theoretically and experimentally, that the constriction can control the salinity in the estuary, thereby imposing an upper limit on that parameter. This state they called overmixing. No matter how much mixing energy is put into the estuary the salinity cannot increase beyond this limit. The reason is that the transport capacity of the constriction has an upper limit with respect to two-layer exchange. For small fresh-water supply to the estuary the transport limit is reached when

$$\left.\begin{aligned} F_{d1}^2+F_{d2}^2&=1 \\ H_1\approx H_2&\approx \tfrac{1}{2}H \end{aligned}\right\}, \qquad (1)$$

where $F_{di}^2=u_i^2/g'H_i$ is a densimetric Froude number; u_i and H_i are velocity and layer depth; $i=1.2$ for upper and lower layer, respectively; and g' is the reduced gravity [$g'=g(\rho_2-\rho_1)/\rho_2$, where $\rho_1(\rho_2)$ is the density of the upper (lower) layer]. The only possibility for increasing the transport capacity of the constriction in this stationary case is to change the vertical area of the constriction by making it wider and/or deeper.

Under natural conditions the constriction often has a considerable length. Frictional effects can then be of importance and the transport capacity of the constriction can be expected to decrease. This case has been discussed by Assaf and Hecht (1974). Stommel and Farmer (1952) and later Welander (1974) pointed out that barotropic current fluctuations can influence the state of overmixing, but none of them made a quantitative estimation of the effect. The barotropic currents could be generated by astronomical or meteorological tides.

For stationary currents in a constriction, conditions (1) are satisfied when the fresh-water supply is small and frictional effects are neglibigle. The constriction acts as a hydraulic control. Now if the periodic barotropic currents with amplitude $u_b>u_1,u_2$ are superimposed, the hydraulic control will be eliminated for shorter or longer periods during which the currents in the constriction will be nearly purely barotropic and therefore directed one way.

It will be shown by a simple mathematical treatment that the transport capacity of the constriction with respect to the two water masses will increase. This is confirmed by laboratory experiments. Finally, an application of the results on the process of exchange of bottom water in a sill fjord, with special reference to the Oslofjord, is discussed. It is shown that a proposed increase of the width of the sill (the Drøbak sill) will have a much smaller effect on the exchange time of bottom water than previously supposed.

2. Theory

Consider a constriction connecting two basins, each containing essentially homogeneous water of different densities. We will assume that the width of this constriction is smaller than the internal radius of deformation so that effects of the Coriolis force are negligible in the transition. Later we will describe a laboratory experiment simulating exchange of deep water in a sill fjord. Therefore, we have this case especially in mind, but the theory should be equally applicable on the case of an overmixed estuary. We consider a small basin (the fjord) with horizontal area Y containing water of density ρ_1, which, through a narrow constriction of width B and depth H, is connected to a large basin (the sea) containing water of density ρ_2. At the start of the process the constriction is closed by a gate. When the gate is opened heavy sea water enters the fjord through the constriction as an undercurrent and the light water from the fjord flows out into the sea on top of the sea water. The transports in the constriction are studied only during an initial state when the outflowing fjord water has not been contaminated by inflowing sea water. During this state the currents can be looked upon as quasi-stationary and they obey (1) so that $u_1=-u_2=\frac{1}{2}(g'H)^{\frac{1}{2}}$ and $H_1=H_2=\frac{1}{2}H$. When a fluctuating barotropic current with frequency ω and amplitude u_b is forced to flow through the constriction we can formally superimpose the barotropic and baroclinic currents. With currents into the fjord taken positive and out of the fjord negative we then have

$$u_1^*=u_b\sin\omega t-\tfrac{1}{2}(g'H)^{\frac{1}{2}}, \tag{2}$$

$$u_2^*=u_b\sin\omega t+\tfrac{1}{2}(g'H)^{\frac{1}{2}}, \tag{3}$$

$$H_1=H_2=\tfrac{1}{2}H. \tag{4}$$

It is clear that (2) and (3) can represent the real velocities u_1 and u_2 only when $u_1^*\leqslant 0$ and $u_2^*\geqslant 0$. When $u_b>\frac{1}{2}(g'H)^{\frac{1}{2}}$, $u_1^*>0$ and $u_2^*<0$ intermittently. During these occasions there must be one way flow and only one water mass in the constriction, which means that the current is then barotropic. Eq. (4) must be revised under these circumstances, permitting the interface to intersect the surface or the sill. For those cases when the interface does not lie at mid-depth in the constriction the control of the two-layer flow is eliminated from the constriction. Eqs. (2)–(4) are replaced by the following expressions for velocities and layer

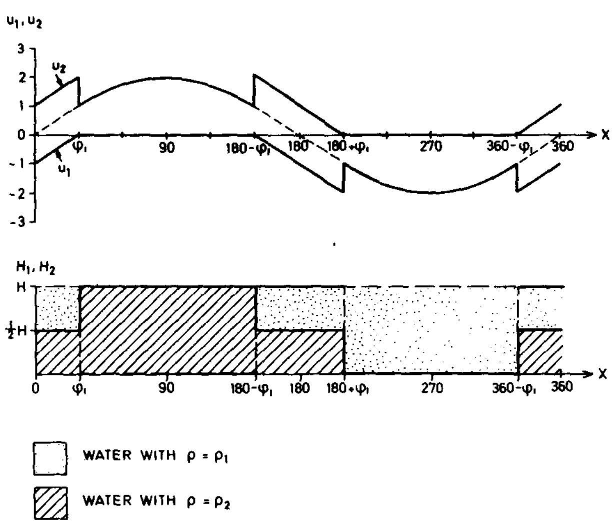

FIG. 1. Behavior of density and velocity fields in the model according to (5).

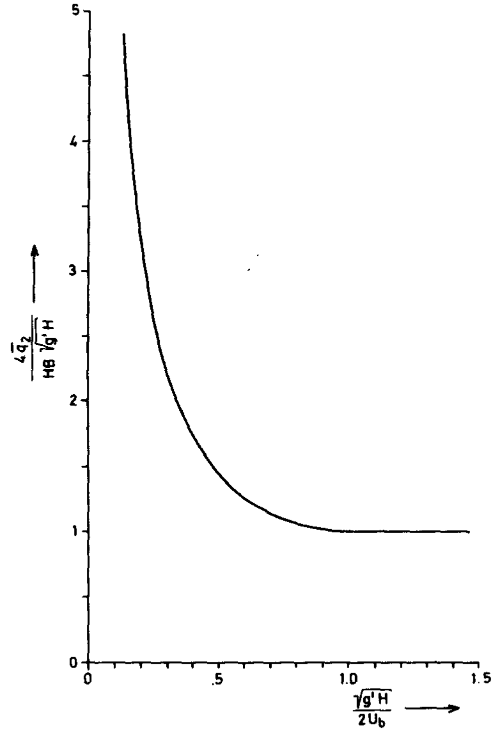

FIG. 2. Normalized transport capacity of the constriction as a function of F^{-1}.

thicknesses in the constriction:

$$\left.\begin{aligned} u_1&=u_b\sin\omega t-\tfrac{1}{2}(g'H)^{\frac{1}{2}}\\ u_2&=u_b\sin\omega t+\tfrac{1}{2}(g'H)^{\frac{1}{2}}\\ H_1&=H_2=\tfrac{1}{2}H\end{aligned}\right\},\quad u_1^*\leqslant 0,\ u_2^*\geqslant 0,$$

$$\left.\begin{aligned} u_1&=H_1=0\\ u_2&=u_b\sin\omega t\\ H_2&=H\end{aligned}\right\},\quad u_1^*>0,\ u_2^*>0, \tag{5}$$

$$\left.\begin{aligned} u_1&=u_b\sin\omega t\\ u_2&=H_2=0\\ H_1&=H\end{aligned}\right\},\quad u_1^*<0,\ u_2^*<0.$$

Layer thickness and current velocity are thus supposed to change more or less discontinuously when u_1^* and u_2^* pass zero. This happens for $\omega t=\phi_1, \phi_2$, where ϕ_1 and ϕ_2 are given by

$$\left.\begin{aligned}\phi_1&=\arcsin F^{-1}, & u_1^*&=0\\ \phi_2&=-\arcsin F^{-1}, & u_2^*&=0\end{aligned}\right\}, \tag{6}$$

where $F^{-1}=(g'H)^{\frac{1}{2}}/2u_b$, an inverse densimetric Froude number. The course of events described by (5) has been represented graphically in Fig. 1 for $F^{-1}=\frac{1}{2}$. The abrupt changes which are thought to take place are compatible with the effect of wandering fronts. The time mean values of the transports through the constriction are

$$\bar{q}_i=\frac{1}{T}\int_0^T BH_iu_i dt,\quad i=1, 2, \tag{7}$$

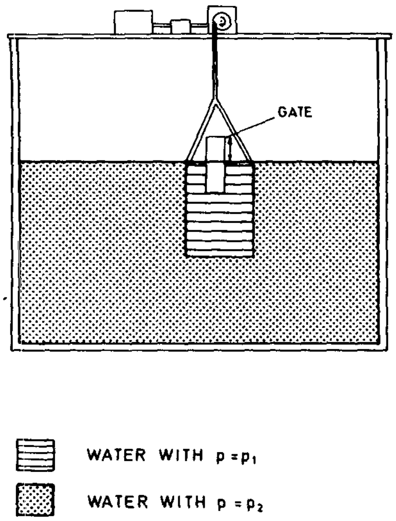

FIG. 3. The laboratory arrangement.

where T is a multiple of the barotropic current period and B is the width of the constriction. For the case $F^{-1} \geq 1$, we find that

$$\bar{q}_i = \tfrac{1}{4} BH (g'H)^{\frac{1}{2}}, \quad F^{-1} \geq 1, i=1, 2, \tag{8}$$

and the barotropic current fluctuations have no effect on the transports. For the case $F^{-1}<1$ the current is alternating barotropic plus baroclinic or just barotropic. By integration of (7) for this case we find

$$\bar{q}_i = \tfrac{1}{4} BH (g'H)^{\frac{1}{2}} \left(\frac{\phi_1}{90} + \frac{2}{\pi} F \cos\phi_1 \right), \quad F^{-1}<1, i=1, 2. \tag{9}$$

The transport through the constriction as a function of F^{-1} is shown in Fig. 2. The transport has been normalized by the transport for the stationary case [Eq. (8)]. We see that the transport capacity can be expected to increase substantially when the amplitude of the barotropic current is such that F^{-1} is materially lower than 1. The amplitude of the barotropic current at the constriction can be expressed as $u_b = \omega a Y/BH$, where a is the amplitude in water level in the fjord for the fluctuation with frequency ω.

This estimation of the effect of barotropic current fluctuations on the transport capacity of the constriction should be valid under the conditions that 1) the interface separating water of density ρ_1 and heavier water is situated at a depth essentially lower than H inside the fjord; 2) a good sink for the light fjord water exists outside the constriction; and 3) the transition between the fjord and the sea is short compared to the excursion length of the fluctuating current. If these conditions are fulfilled recirculation of water between the two basins is supposed to be of minor significance.

3. Laboratory experiments

A fjord model with area $Y = 0.180 \times 0.184$ m^2 containing water of density ρ_1 was submerged in a larger basin with area 1.0×1.5 m^2 and containing water with density ρ_2. At the beginning of the experiment the opening between the basins was closed with a gate. A motor-driven sheave forced the fjord model to oscillate vertically (see Fig. 3). In that way an oscillating barotropic current was established over the sill when the gate was opened at time $t=0$. At time $t=t_1$ the gate was closed again. Before and after the experiment the mean salinity and temperature were measured in the fjord model. In this way it was possible to calculate the volume of heavy water that had entered the fjord during time t_1. The experiment was repeated with tides of different periods (different values of F^{-1}). The results are plotted in Fig. 4 (crosses) and the curve through the crosses (broken line) is the theoretical curve (continuous line) times a factor of 0.75. The flow rates determined experimentally were thus lower than the theoretically expected. This departure is believed to be caused by the effect one expects in a contraction with sharp edges. A contraction coefficient of 0.75 seems to be reasonable

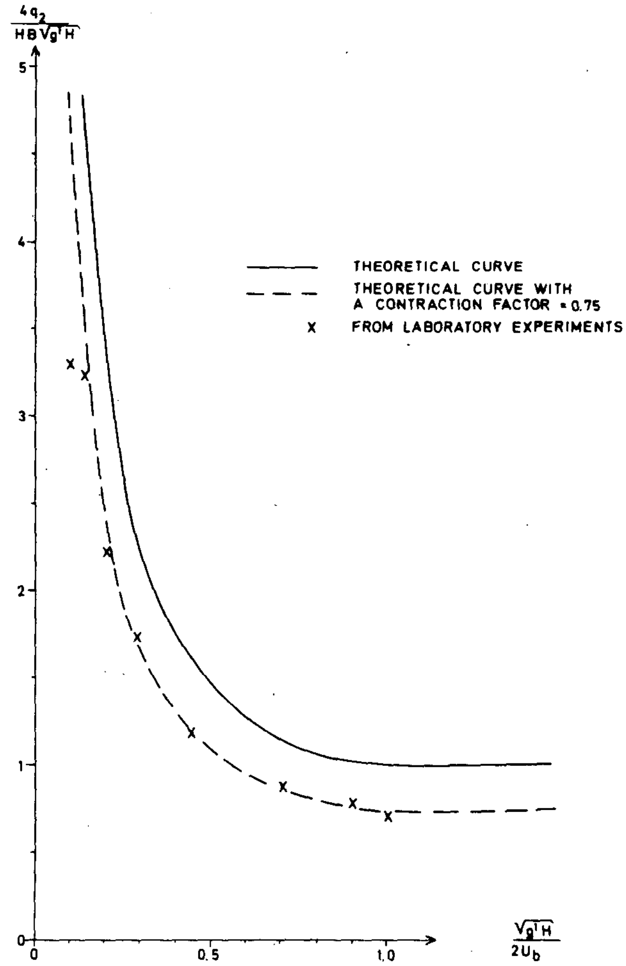

FIG. 4. Theoretically and experimentally determined transport capacity (normalized) as a function of F^{-1}. Not shown in the figure is the value 0.74 which was obtained for $u_b=0$.

(see Batchelor, 1970). For the smallest observed value of F^{-1} the transport was lower than what should be expected from the empirical curve. The reason for this is certainly that recirculation is beginning to be a serious problem in the experiment at such low values of F^{-1}.

4. Application to the Oslofjord

We shall apply the theory to the exchange of deep water in the Oslofjord. The deep water is, as a rule, more or less completely renewed once every year. This normally happens in the spring, when unusually dense water appears outside the sill. The dense water reaches a high level just for a short period (a few weeks) and it is only during this period an effective renewal of bottom water can take place. To make sure that the bottom water will be completely exchanged every year it has long been argued that the sill should be made wider by taking away the jetty that is blocking the western part of the narrows at Drøbak (see map, Fig. 5). According to (8) one finds that the transport capacity of the constriction is proportional to the width and this fact lay behind the old idea.

However, a broadening of the entrance will render the barotropic effect induced by tides less efficient and for a given density difference between the old fjord water and the new deep water F^{-1} will increase with the width of the sill. In Fig. 6 the transport capacity of the Drøbak constriction as a function of the width is shown for a moderate density difference ($\Delta\rho=2$ kg m^{-3}) according to (8) (without barotropic effect, broken line) and according to (9) (with barotropic effect, continuous line). The Oslofjord parameters given in Table 1 have been used.

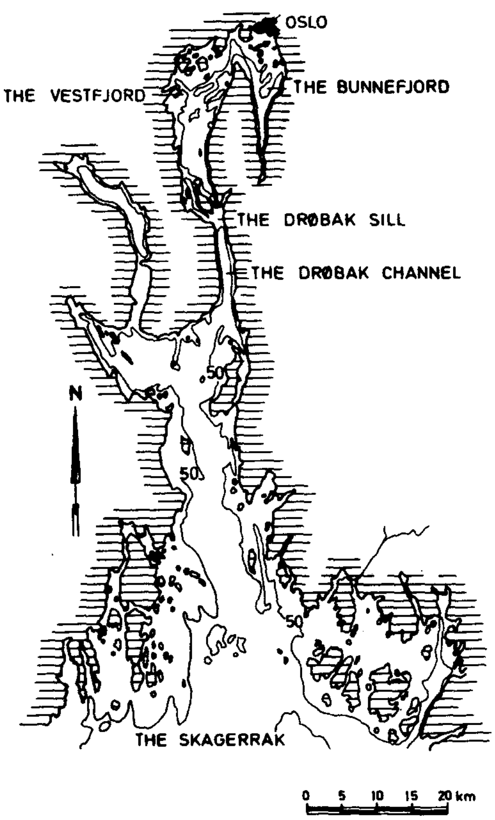

FIG. 5. Map of the Oslofjord with the 50 m isobath shown.

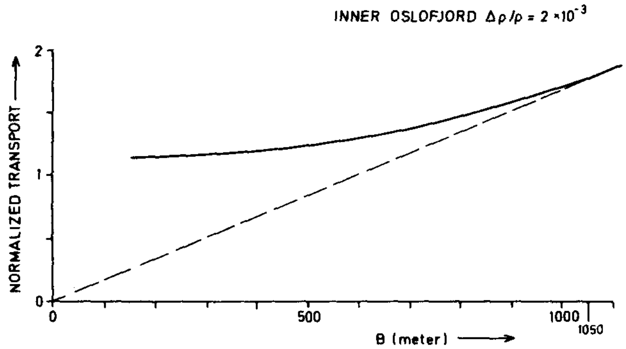

FIG. 6. Transport capacity (normalized) of the Drøbak constriction. The width of the constriction is today 600 m.

As can be seen from Fig. 6 the effect of a broadening of the sill is much less for this moderate density difference than earlier presumed. The physical explanation for this unexpected behavior of the transport capacity is thus that an increase in width B leads to an increase in F^{-1} so that the barotropic effect diminishes (see Fig. 2).

It can in this connection be mentioned that it has been shown that internal waves are generated at the sill in some fjords (e.g., the Oslofjord), during most of the year (Stigebrandt, 1976). These waves carry energy into the fjord where they break against sloping bottoms, thereby creating small-scale turbulence. These internal waves seem to constitute the main energy source for the turbulence in the deep water (below sill depth) in the Oslofjord and probably in some other fjords, too. The amplitude of the internal waves is critically dependent on the width of the sill and it turns out that the Oslofjord sill has a nearly optimal width. A broadening of the sill should therefore lead to a slower mixing rate in the deep water and, as we have already seen from Fig. 6, only a minute increase in the transport capacity of the constriction (for a given density difference). The result of a widening of the Drøbak narrows would most probably be a less efficient renewal of the deep water.

With a given outer fluctuating density field it should now be possible to manipulate the sill in some fjords to minimize the residence time of deep water. Work

TABLE 1. Oslofjord parameters used to compute transport capacity of the Drøbak constriction.

$a=0.15$ m	$H=15$ m
$T=4.5\times10^4$ s	$g=10$ m s^{-2}
$Y=200\times10^6$ m^2	$\rho_1\approx\rho_2\approx10^3$ kg m^{-3}

with special reference to the Oslofjord, is in progress at the River and Harbour Laboratory in Trondheim and will be reported elsewhere.

5. Concluding comments

In the case we have investigated we found that the effect of barotropic current fluctuations on the two-layer transport capacity increased with a Froude number $F = 2u_b/(g'H)^{\frac{1}{2}}$. In the limit when $F \to \infty$ ($F^{-1} \to 0$) we find, from (9), that $\overline{q_i} \to aY\omega/\pi$, which is the expected transport rate when the "tidal prism" is not recirculated between the basins. This transport rate is also the lowest one possible in this idealized system with barotropic current fluctuations.

The prerequisites for the theory are poorly fulfilled in some natural transitions, especially for those with a considerable length. In these, high-frequency barotropic current fluctuations and mixing will induce recirculation, whose influence on the transport capacity is a subject for further study.

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REFERENCES

Assaf, G., and A. Hecht, 1974: Sea straits: A dynamical model. *Deep-Sea Res.*, **21**, 947–58.
Batchelor, K. G., 1970: *An Introduction to Fluid Dynamics.* Cambridge University Press, 615 pp.
Stigebrandt, A., 1976: Vertical diffusion driven by internal waves in a sill fjord. *J. Phys. Oceanogr.*, **6**, 486–495.
Stommel, H., and H. G. Farmer, 1952: Abrupt change in width in two-layer open channel flow. *J. Mar. Res.*, **11**, 205–214.
——, and ——, (1953): Control of salinity in an estuary by a transition. *J. Mar. Res.*, **12**, 13–20.
Welander, P., 1974: Two-layer exchange in an estuary basin with special reference to the Baltic Sea. *J. Phys. Oceanogr.*, **4**, 542–556.